

Conservation laws of magnetohydrodynamics and their symmetry transformation properties

by

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Abstract

All kinematic conservation laws along with their symmetry transformation properties are derived for the system of magnetohydrodynamic equations governing incompressible viscous plasmas (or any other conducting fluid) in which the dynamic and magnetic viscosities are constant. Reductions of this system under translation symmetries, axial rotation symmetries, and helical symmetries are considered. For each reduced system, all kinematic conservation laws and point symmetries are obtained. The results yields several new conservation laws which are expected to be relevant for physical applications of magnetohydrodynamics.

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Chapter 1

Introduction

Conservation laws and symmetries are well-established and widely used methods [1, 2, 3, 4, 5, 6] for studying different properties of differential equations. As a consequence, corresponding mathematical techniques and computer packages are well developed. However, the results of these methods require a physical interpretation for them to be useful, and this fact has prevented them from becoming standard instruments for a physicist. One domain of physics where these methods are in demand is physics of continua in general, and magnetohydrodynamics (MHD) in particular.

MHD systems describe plasmas and conducting fluids, whose main dynamical variables consist of the density, the velocity, the magnetic field, the pressure, and the temperature or entropy [7, 8, 9, 10, 11]. These systems are a generalization of the Navier-Stokes equations of fluid flow [12]. Symmetries and conservation laws have been studied in [13, 14, 15, 16, 17, 18, 19, 20] for the Navier-Stokes equations, including the case of ideal fluid flow governed by the Euler equations.

The MHD system of the most interest in connection with plasma physics is the compressible non-barotropic ideal MHD system [7, 8, 9, 10, 11, 21, 22, 23] in which the pressure depends on the density as well as the temperature or entropy, and no viscosity is present in the flow. This system has a Hamiltonian formulation [24] and a canonical Poisson bracket [22, 25, 26, 27, 28, 29, 30] using Clebsch variables. The basic local conservation laws of the compressible non-barotropic ideal MHD system are well-known (see, for example [31, 32]) and comprise mass, momentum, angular momentum, energy, entropy, and magnetic helicity. If the pressure depends only on the density or the magnetic field is orthogonal to the gradient of entropy, then the MHD system possesses an additional helicity-type conservation law known as cross-helicity [33, 34]. Magnetic helicity and cross-helicity are quantities that describe the topology of magnetic field lines in the fluid flow [35, 36, 37, 38, 39, 40, 41, 42]. In particular, magnetic helicity can be interpreted as the linking number of the magnetic field lines, while cross-helicity has a similar meaning as the mutual linking number of magnetic field lines and lines (integral curves) of vorticity. Both helicities are used, for instance, in the description of MHD turbulence [43, 44] and in the study of dynamo theories

[42, 45, 46]. The helicity of vorticity lines, which is an important conserved quantity in fluid mechanics, is not a conserved quantity in MHD systems [42].

Apart from plasma physics, MHD systems arise in other physical applications. When the fluid velocity is small compared to the speed of sound in the plasma or conducting fluid, whether or not there is viscosity in the flow, the fluid can be approximated to be incompressible [12] and the fluid density then obeys a transport equation. (The ratio of the velocity to the speed of sound is the Mach number [47].) If in addition the fluid density is initially close to homogeneous, then the fluid can be approximated to have constant density [48]. There are many physical applications of such constant-density MHD systems: flow of liquid metals, melting of salts, electrolytes [49, 50, 51], and planetary dynamo theories [42, 43, 44, 45, 52].

Another situation of physical interest is steady-state MHD flow (equilibria) [53, 54, 55] where all of the dynamical variables are time independent. Equilibrium structures are usually considered in connection with applications to controlled nuclear fusion and space plasma. Steady-state MHD systems have a rich structure of symmetries and conservation laws [54, 56, 57, 58, 59, 60] which are important in the analysis of equilibria. For rotationally invariant ideal equilibria, the reduced steady-state MHD system is known as the Grad-Shafranov equation [7, 61, 62, 64], which describes magnetic surfaces. Invariance of the steady-state MHD system under helical symmetry gives the Johnson-Frieman-Kruskal-Oberman (JFKO) equation [63, 64].

In this thesis, conservation laws and their symmetry transformation properties are derived for the MHD system describing constant-density viscous conducting fluids, with the dynamic and magnetic viscosities being constant. The equations in this system connect a magnetic field \mathbf{B} , a velocity field \mathbf{v} , a pressure P , and their derivatives:

$$k \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla (m P + \frac{1}{2} n \mathbf{B} \cdot \mathbf{B}) - n (\mathbf{B} \cdot \nabla) \mathbf{B} - a \Delta \mathbf{v} = 0, \quad (1.0.1a)$$

$$k \partial_t \mathbf{B} + (\mathbf{v} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{v} - b \Delta \mathbf{B} = 0, \quad (1.0.1b)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (1.0.1c)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (1.0.1d)$$

Here $a = 1/\text{Re}$, $b = 1/\text{Re}_m$, $n = G^2/(\text{Re} \text{Re}_m)$, where Re is the Reynolds number, Re_m is the magnetic Reynolds number, G is the Hartmann number, $k = \text{St}$ is the Strouhal number, and $m = \text{Eu}$ is the Euler number. (See [47, 7].) Note equations (1.0.1a) and (1.0.1b) describe the evolution of \mathbf{v} and \mathbf{B} ; in turn, an equation for P can be obtained by applying the divergence operator to equation (1.0.1a) and

substituting equations (1.0.1c) and (1.0.1d):

$$m \Delta P = \nabla^2 \cdot (n \mathbf{B} \otimes \mathbf{B} - \mathbf{v} \otimes \mathbf{v}) - n (\nabla \otimes \mathbf{B}) \cdot (\nabla \otimes \mathbf{B})^T - n \mathbf{B} \cdot (\Delta \mathbf{B}), \quad (1.0.1e)$$

where $\nabla^2 = \nabla \otimes \nabla$, with \otimes denoting the tensor product. These equations (1.0.1a)–(1.0.1e) will be studied in three-dimensional Euclidean space, without boundary conditions.

A scaling of the time $t/k \rightarrow t$, the pressure $mP \rightarrow P$, and the magnetic field $\mathbf{B}\sqrt{n} \rightarrow \mathbf{B}$ can be done to get a two-parameter system, which is given by

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla (P + \frac{1}{2} \mathbf{B} \cdot \mathbf{B}) - (\mathbf{B} \cdot \nabla) \mathbf{B} - a \Delta \mathbf{v} = 0, \quad (1.0.2a)$$

$$\partial_t \mathbf{B} + (\mathbf{v} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{v} - b \Delta \mathbf{B} = 0, \quad (1.0.2b)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (1.0.2c)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (1.0.2d)$$

This scaling excludes models in which k, n and m have an infinitely small or infinitely large value. In the scaled system, the independent variables t, x and the dependent variables $\mathbf{v}, \mathbf{B}, P$ are unitless.

This system (1.0.2) possesses an equivalence transformation

$$\tilde{t} = \alpha^{-1} t, \quad \tilde{\mathbf{v}} = \alpha \mathbf{v}, \quad \tilde{\mathbf{B}} = \alpha \mathbf{B}, \quad \tilde{P} = \alpha^2 P, \quad \tilde{a} = \alpha a, \quad \tilde{b} = \alpha b, \quad (1.0.3)$$

where $\alpha \neq 0$ is a constant scaling parameter. Under this transformation, one of the two constitutive parameters a and b can be scaled to unity, which gives a one-parameter system. From a mathematical point of view, since the ratio a/b is invariant, this would be the remaining constitutive parameter. From a physical point of view, this invariant is the ratio of viscosity and magnetic diffusivity, which is called the magnetic Prandtl number [9], $\text{Pr}_m = a/b$. (See [10].)

1.1 Physical aspects

The general two-parameter model (1.0.2) under consideration physically describes incompressible flows of a viscous conducting liquid in which the influence of density gradients, heat flows, and gravitation force on both the flow pattern and the magnetic field is negligibly small, and in which the flow has a small Mach number. This MHD model is widely applicable, from liquid metal flow to laboratory plasma and space plasma and stellar matter, under the previous conditions. Analysis of different boundary problems and applications of this PDE system is considered in [48, 65].

Four physically important subcases of the model will be considered.

I. *Viscous, finite conducting fluid: $a \neq 0, b \neq 0$.* The equivalence transformation (1.0.3) allows putting $ab = 1$, with a/b being the constitutive parameter, without loss of generality. This model is commonly used especially for the simulation of liquid metal flow, where the values of Reynolds number and the magnetic Reynolds number are approximately $\text{Re} \approx 1, \text{Re}_m \approx 1$. In space, solar and atmospheric plasma physics, this model is used for studying processes of magnetic reconnection and formulation of tearing modes [21], where the Mach number is small.

II. *Viscous, infinitely conducting fluid: $a \neq 0, b = 0$.* The equivalence transformation (1.0.3) in this case allows putting $a = 1$ in the MHD system (1.0.2) without loss of generality. From a physical point of view, this corresponds to a model of the large magnetic Prandtl number, which arises, for instance, in the theory of dynamos [44].

III. *Non-viscous, finite conducting fluid: $a = 0, b \neq 0$.* The equivalence transformation (1.0.3) in this case allows putting $b = 1$ in the MHD system (1.0.2) without loss of generality. From a physical point of view, this model includes the small magnetic Prandtl number approximation [10], which is used for some simulations of liquid metal flow where $\text{Pr}_m = 10^{-1} - 10^{-5}$ [43, 45].

IV. *Ideal fluid: $a = 0, b = 0$.* This is one of the most commonly used MHD models, especially when space plasmas and fusion plasmas are considered. In the absence of viscous terms, the nonlinear terms dominate the system. Consequently, geometrical methods are effective in understanding the behaviour of solutions. In particular, the system has a Hamiltonian formulation [24, 26, 25], and the topology of magnetic field lines is conserved [10, 11, 27].

Other models. There are a lot of well known models that correspond to infinitely large values of a or b . For instance, the linear Stokes and Oseen models of the evolution equations for the velocity [12], and the non-inductive Braginskii

approximation of the evolution equations for the magnetic field [7, 66]. From a mathematical point of view, these models can be derived as approximations when the dynamical variables are decomposed in a series in powers of small parameters a^{-1} and b^{-1} . Such models have very different physical behaviour, but since they are derived from the MHD system (1.0.2) under consideration, they can inherit many of its properties and features.

1.2 Outline of main results

The rest of the thesis is organized as follows.

In chapter 2, the preliminaries necessary for carrying out an explicitly classification of conservation laws of the constant-density MHD system are explained. A useful technical contribution is the introduction of a solved form for the MHD equations in terms of a set of leading derivatives of the fluid velocity, magnetic field, and pressure. This solved form allows a direct construction method for conservation laws to be applied to the constant-density MHD system.

In chapter 3, first the literature on known conservation laws of the constant-density MHD system is reviewed. Next the motivation and definition of kinematic conservation laws is given, and then the classification results for kinematic conservation laws are presented. This is the first-ever systematic classification of conservation laws for the MHD system under consideration. The main result of the classification is to show that the well-know conservation law for cross-helicity in the absence of viscosity holds for a special viscous case in which the sum of the dynamic and magnetic viscosities is zero. Finally, the physical meaning of all of the conservation laws is discussed.

In chapter 4, first the definition of point symmetries is reviewed, followed by a summary of literature on the point symmetries of the constant-density MHD system. Next a classification of point symmetries is presented and their physical meaning is discussed. For later use, the class of geometrical point symmetries generated by Killing vectors (in Euclidean space) is summarized.

In chapter 5, the action of the point symmetries on the kinematic conservation laws is derived. This allows identifying a subset of kinematic conservation laws that generate the whole kinematic class, under the symmetry action.

In chapter 6, reductions of the constant-density MHD system under three physically important geometrical point symmetries are studied: space translations, axial rotations, helical motion. For each reduction, a classification of kinematic conservation laws and point symmetries is presented. The results yield new conservation laws and new point symmetries, which are not inherited from the full MHD system.

In chapter 7, some conclusions and directions for future work are discussed.

Chapter 2

Preliminaries

A review of the basic formulation of conservation laws for PDE systems can be found in [2, 3, 4, 5, 6].

For computations of conservation laws and symmetries of the MHD system (1.0.2), it will be convenient to work in index notation adapted to Cartesian coordinates:

$$\begin{aligned}
 \mathbf{x} &= (x^1, x^2, x^3) \leftrightarrow x^i, \\
 \mathbf{v} &= (v^1, v^2, v^3) \leftrightarrow v^i, \\
 \mathbf{B} &= (B^1, B^2, B^3) \leftrightarrow B^i, \\
 \nabla f &\leftrightarrow f_{,i}, \\
 \nabla \cdot \mathbf{f} &\leftrightarrow f^j_{,j}, \\
 \nabla \times \mathbf{f} &\leftrightarrow \epsilon^{ijk} \delta_{kl} f^l_{,j}, \\
 \Delta f &\leftrightarrow \delta^{ij} f_{,ij},
 \end{aligned} \tag{2.0.1}$$

where

$$\begin{aligned}
 \delta^{ij} = \delta_{ij} &= \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \\
 \epsilon^{ijk} = \epsilon_{ijk} &= \begin{cases} 1, & (i, j, k) = \text{even permutation of } (1, 2, 3) \\ -1, & (i, j, k) = \text{odd permutation of } (1, 2, 3) \\ 0, & \text{otherwise} \end{cases}
 \end{aligned} \tag{2.0.2}$$

respectively denote the Kronecker symbol and the Levi-Civita symbol. Hereafter, indices will be raised and lowered by using the Kronecker symbol.

The MHD system (1.0.2) in index notation is given by

$$v^i_t = -v^j v^i_{,j} - P_{,i} - \delta_{jk} B^j B^k_{,i} + B^j B^i_{,j} + a \Delta v^i, \tag{2.0.3a}$$

$$B^i_t = B^j v^i_{,j} - v^j B^i_{,j} + b \Delta B^i, \tag{2.0.3b}$$

$$v^i_{,i} = 0, \tag{2.0.3c}$$

$$B^i_{,i} = 0, \quad (2.0.3d)$$

$$\Delta P = B^j_{,i} B^i_{,j} - v^j_{,i} v^i_{,j} - \delta_{ik} (\delta^{jl} B^k_{,l} B^i_{,j} + B^k \Delta B^i). \quad (2.0.3e)$$

The independent variables t, x^i , the dependent variables v^i, B^i, P , and their derivatives define a coordinate space called jet space $J = (t, x^i, v^i, B^i, P, v^i_t, B^i_t, P_t, v^i_{,j}, B^i_{,j}, P_{,j}, \dots)$. Total derivatives with respect to t, x^i are denoted by D_t, D_i . The MHD solution space $\mathcal{E} \subset J$ is defined by the system of equations (2.0.3) and their differential consequences.

A *conservation law* of the MHD system is a local continuity equation

$$(D_t T + D_i X^i)|_{\mathcal{E}} = 0 \quad (2.0.4)$$

holding on the MHD solution space \mathcal{E} , where T is a scalar function on J and X^i is a vector function on J , up to some finite order in derivatives. The integral form of a conservation law (2.0.4) is defined by a balance equation

$$\frac{d}{dt} \int_{\Omega(t)} T|_{\mathcal{E}} dV = - \oint_{\partial\Omega(t)} (X^i + T v^i)|_{\mathcal{E}} dS_i \quad (2.0.5)$$

on a spatial moving domain $\Omega(t)$ that is transported by the velocity field v^i . In particular, each point $x^i(t) \in \Omega(t)$ satisfies the streamline equation

$$\frac{dx^i(t)}{dt} = v^i(t, x^i(t)). \quad (2.0.6)$$

The scalar T is called the *conserved density*, and the vector X^i is called the *spatial flux*.

The integral quantity associated to a conservation law

$$\int_{\Omega(t)} T|_{\mathcal{E}} dV \quad (2.0.7)$$

is a constant of motion iff the net flux through the moving boundary $\partial\Omega(t)$ vanishes,

$$\oint_{\partial\Omega(t)} (X^i + T v^i)|_{\mathcal{E}} dS_i = 0. \quad (2.0.8)$$

Constants of motion can be viewed as advected invariants [67] of the fluid flow. A geometrical approach to deriving these conservation laws for non-barotropic ideal MHD systems is presented in [68] by Lie dragging differential forms and

scalars with the flow [67]. This method, however, cannot be extended to obtain conservation laws that have non-zero flux.

A conservation law is physically trivial if its integral quantity itself vanishes on the MHD solution space \mathcal{E} for all moving domains $\Omega(t)$. This happens iff T , X^i have the respective forms

$$T|_{\mathcal{E}} = D_i \Psi^i, \quad X^i|_{\mathcal{E}} = -D_t \Psi^i + D_j \Theta^{ij} \quad (2.0.9)$$

holding for some vector function Ψ^i on J and some antisymmetric tensor function $\Theta^{ij} = -\Theta^{ji}$ on J . A conservation law of this form is said to be *locally trivially*. Two conservation laws are physically equivalent if they differ by a locally trivial conservation law.

Every conservation law has a *characteristic form* given by a space-time divergence identity on J which provides the starting point for a general method of finding all conservation laws [3, 5, 6]. To set up this method, we need to coordinatize the MHD solution space $\mathcal{E} \subset J$, which can be accomplished by decomposing the spatial derivative variables $v^i_{,j}$, $B^i_{,j}$, $v^i_{,j}$, $B^i_{,j}$ etc. into parts that vanish on \mathcal{E} and parts that provide coordinates on \mathcal{E} . One way is by using trace/trace-free decompositions

$$\begin{aligned} v^i_{,j} &= (v^i_{,j})_0 + \frac{1}{3} \delta^i_j v^k_{,k}, & \delta^j_i (v^i_{,j})_0 &= 0, \\ B^i_{,j} &= (B^i_{,j})_0 + \frac{1}{3} \delta^i_j B^k_{,k}, & \delta^j_i (B^i_{,j})_0 &= 0, \end{aligned} \quad (2.0.10)$$

where $(v^i_{,j})_0$ and $(B^i_{,j})_0$ provide coordinates on \mathcal{E} . This coordinatization preserves rotational invariance but is complicated for computations involving higher-order derivatives.

Another way is by using a leading derivative method [5, 6], which is convenient for computations but breaks rotational invariance. First, we express the divergence equations (2.0.3c) and (2.0.3d) in a solved form for x^1 derivatives:

$$v^1_{,1} = -v^{i'}_{,i'}, \quad (2.0.11a)$$

$$B^1_{,1} = -B^{i'}_{,i'}, \quad (2.0.11b)$$

where $i' = 2, 3$. These two equations define a constraint subspace $\mathcal{E}_{\text{constr}} \subset \mathcal{E}$ in the MHD solution space. Next, we substitute both equations (2.0.11a) and (2.0.11b) into the evolution equations (2.0.3a) and (2.0.3b), and we express the

resulting equations in solved form for t derivatives:

$$v^1_t = -P^1 - v^j v^1_{,j'} + v^1 v^j_{,j'} + B^{j'} (B^1_{,j'} - \delta_{j'k'} B^{k'}_{,1}) + a \Delta' v^1 - a v^j_{,1j'}, \quad (2.0.11c)$$

$$v^{i'}_t = -P^{i'} - v^j v^{i'}_{,j} + B^j B^{i'}_{,j} - \delta_{jk} B^j B^{k'}_{,i'} + a \Delta v^{i'}, \quad (2.0.11d)$$

$$B^1_t = B^{j'} v^1_{,j'} - B^1 v^j_{,j'} - v^j B^1_{,j'} + v^1 B^{j'}_{,j'} + b \Delta' B^1 - b B^{j'}_{,1j'}, \quad (2.0.11e)$$

$$B^{i'}_t = B^j v^{i'}_{,j} - v^j B^{i'}_{,j} + b \Delta B^{i'}, \quad (2.0.11f)$$

where

$$\Delta' f = \delta^{i'j'} f_{,i'j'}.$$

Finally, we also substitute both equations (2.0.11a) and (2.0.11b) into the remaining equation (2.0.3e) which we express in a solved form

$$\begin{aligned} P_{,11} = & -\Delta' P - v^{i'}_{,i'} v^j_{,j'} - v^j_{,i'} v^{i'}_{,j'} - 2v^1_{,i'} v^{i'}_{,1} + B^{j'}_{,i'} B^{i'}_{,j'} \\ & + 2B^1_{,i'} B^{i'}_{,1} - \delta^{i'j'} B^1_{,i'} B^1_{,j'} - B^1 \Delta' B^1 + B^1 B^{i'}_{,i'1} \\ & - \delta_{i'k'} (\delta^{jl} B^{k'}_{,l} B^{i'}_{,j} + B^{k'} \Delta B^{i'}). \end{aligned} \quad (2.0.11g)$$

These equations (2.0.11a)–(2.0.11g) define a dynamical subspace $\mathcal{E}_{\text{dyn}} \subset \mathcal{E}$ in the MHD solution space. Note that we have $\mathcal{E} = \mathcal{E}_{\text{constr}} \cup \mathcal{E}_{\text{dyn}}$. The coordinates on \mathcal{E} then consist of the $x^{i'}$ derivatives of v^1, B^1, P , and the x^i derivatives of $v^{j'}, B^{j'}$.

It will be convenient to denote the right hand sides of equations (2.0.11a), (2.0.11b), (2.0.11c), (2.0.11d), (2.0.11e), (2.0.11f), and (2.0.11g) by $\tilde{S}_v, \tilde{S}_B, S^1_v, S^{i'}_v, S^1_B, S^{i'}_B$, and S_P , respectively.

With this coordinatization of the MHD solution space \mathcal{E} , the characteristic form of a conservation law (2.0.4) is obtained by the following steps. First, we apply Hadamard's lemma [3, 6, 69], which states that $D_t T + D_i X^i$ vanishes on \mathcal{E} if and only if it is a linear combination of the equations (2.0.11a)–(2.0.11g) and their differential consequences on J . This yields the divergence identity

$$\begin{aligned} D_t T + D_i X^i = & \mathcal{R}_i^v (v^i_t - S^i_B) + \mathcal{R}_i^B (B^i_t - S^i_B) + \mathcal{R}^v (v^1_{,1} - \tilde{S}^v) \\ & + \mathcal{R}^B (B^1_{,1} - \tilde{S}_B) + \mathcal{R}^P (P_{,11} - S_P) \end{aligned} \quad (2.0.12)$$

where $\mathcal{R}_i^v, \mathcal{R}_i^B, \mathcal{R}^v, \mathcal{R}^B$, and \mathcal{R}^P are some linear differential operators with coefficients that are non-singular functions on $\mathcal{E} \in J$. Next, we apply integration by parts to the right hand side of the conservation law identity (2.0.12) and combine the resulting total derivative terms with the ones on the left hand side. This

yields an equivalent identity

$$\begin{aligned} D_t \tilde{T} + D_i \tilde{X}^i &= Q_i^v (v_t^i - S_B^i) + Q_i^B (B_t^i - S_B^i) + \tilde{Q}^v (v^1_{,1} - \tilde{S}_v) \\ &\quad + \tilde{Q}^B (B^1_{,1} - \tilde{S}_B) + Q^P (P_{,11} - S_P) \end{aligned} \quad (2.0.13)$$

which is called the *characteristic form* of the conservation law, with

$$\tilde{T}|_{\mathcal{E}} = T, \quad \tilde{X}^i|_{\mathcal{E}} = X^i \quad (2.0.14)$$

and with Q_i^v , Q_i^B , \tilde{Q}^v , \tilde{Q}^B , Q^P being functions on J that are non-singular on \mathcal{E} . Since the right hand side of this identity (2.0.13) vanishes on the MHD solution space \mathcal{E} , note that the resulting conservation law

$$(D_t \tilde{T} + D_i \tilde{X}^i)|_{\mathcal{E}} = 0 \quad (2.0.15)$$

is equivalent to the original one. The set of functions $(Q_i^v, Q_i^B, \tilde{Q}^v, \tilde{Q}^B, Q^P)$ is referred to as the *conservation law multiplier*.

Multiplier functions are determined by the condition that the left hand side of the characteristic equation (2.0.13) is a total divergence. This condition holds iff the right hand side of equation (2.0.13) is annihilated by the Euler operators with respect to v^i , B^i , and P :

$$E_{v^i} = \frac{\partial}{\partial v^i} - D_t \frac{\partial}{\partial v_t^i} - D_j \frac{\partial}{\partial v^i_{,j}} + D_t D_k \frac{\partial}{\partial v^i_{t,j}} + D_j D_k \frac{\partial}{\partial v^i_{,jk}} + \dots, \quad (2.0.16a)$$

$$E_{B^i} = \frac{\partial}{\partial B^i} - D_t \frac{\partial}{\partial B_t^i} - D_j \frac{\partial}{\partial B^i_{,j}} + D_t D_k \frac{\partial}{\partial B^i_{t,j}} + D_j D_k \frac{\partial}{\partial B^i_{,jk}} + \dots, \quad (2.0.16b)$$

$$E_P = \frac{\partial}{\partial P} - D_t \frac{\partial}{\partial P_t} - D_j \frac{\partial}{\partial P_{,j}} + D_t D_k \frac{\partial}{\partial P_{t,k}} + D_j D_k \frac{\partial}{\partial P_{,jk}} + \dots. \quad (2.0.16c)$$

These operators (2.0.16) applied to equation (2.0.13) yield a linear system of equa-

tions on $(Q_i^v, Q_i^B, \tilde{Q}^v, \tilde{Q}^B)$, called the *multiplier determining system*:

$$E_{v^i} (Q_i^v (v_t^i - S_v^i) + Q_i^B (B_t^i - S_B^i) + \tilde{Q}^v (v^1_{,1} - \tilde{S}_v) + \tilde{Q}^B (B^1_{,1} - \tilde{S}_B) + Q^P (P_{,11} - S_P)) = 0, \quad (2.0.17a)$$

$$E_{B^i} (Q_i^v (v_t^i - S_v^i) + Q_i^B (B_t^i - S_B^i) + \tilde{Q}^v (v^1_{,1} - \tilde{S}_v) + \tilde{Q}^B (B^1_{,1} - \tilde{S}_B) + Q^P (P_{,11} - S_P)) = 0, \quad (2.0.17b)$$

$$E_P (Q_i^v (v_t^i - S_v^i) + Q_i^B (B_t^i - S_B^i) + \tilde{Q}^v (v^1_{,1} - \tilde{S}_v) + \tilde{Q}^B (B^1_{,1} - \tilde{S}_B) + Q^P (P_{,11} - S_P)) = 0. \quad (2.0.17c)$$

Each solution of this system determines a corresponding conservation law (2.0.15) through equation (2.0.13). In particular, the conserved density and spatial flux can be found by several different methods, including the direct integration of equation (2.0.13) [5], or the use of a homotopy integral formula [70, 71, 72], or the application of an algebraic scaling formula [6, 73].

Chapter 3

Classification of kinematic conservation laws

The constant-density MHD system (2.0.11a)–(2.0.11g) inherits the conservation laws known for [7, 9, 10, 11, 21, 74] compressible barotropic MHD systems. These inherited conservation laws consist of momentum and angular momentum, along with energy and cross-helicity when the dynamic viscosity vanishes, and magnetic helicity in the case when the magnetic viscosity vanishes. (Mass conservation is also inherited but holds automatically since the density is constant.) An important open problem is to determine if the constant-density MHD system (2.0.11a)–(2.0.11g) admits any additional conservation laws.

All of the known conservation laws, with the exception of magnetic helicity, belong to the physically important class comprising conserved densities

$$T(t, x^i, v^i, B^i, P) \quad (3.0.1)$$

which depend on the time and space coordinates, the velocity, magnetic field, and pressure, but not their spatial derivatives. This class is called *kinematic conservation laws* [17, 18].

Magnetic helicity [75] has the conserved density $T = \delta_{ij} A^i B^j$ where A^i is the magnetic vector potential defined by $B^i = \epsilon^i_{jk} A^k{}_{,j}$. Note that this potential has the gradient gauge freedom $A^i \rightarrow A^i + \phi_{,i}$ where ϕ is an arbitrary function of t, x^i . Hence, magnetic helicity has the gauge transformation $T \rightarrow T + D_i(\phi B^i)$ under which the conserved density changes by a locally trivial density. In contrast, kinematic conserved densities (3.0.1) are gauge invariant.

The main goal now will be to derive all kinematic conservation laws for the MHD system (2.0.11a)–(2.0.11g), including any that exist only for special values of the viscosity parameters a, b . No systematic classification has previously appeared in the literature.

The general form for a kinematic conserved density (3.0.1) determines a corresponding general form for a kinematic spatial flux through the characteristic equation (2.0.13), up to the addition of an arbitrary locally trivial flux $D_j \Theta^{ij}$

where $\Theta^{ij} = -\Theta^{ji}$ is some antisymmetric tensor function on the MHD solution space \mathcal{E} . To determine the minimal dependence on the variables needed in the spatial flux components X^i , we look at the highest derivative terms that occur in $D_t T|_{\mathcal{E}_{\text{constr}}}$ coming from the right hand sides of the MHD equations (2.0.11c), (2.0.11d), (2.0.11e), (2.0.11f) [76]. These terms are linear in the second-order spatial derivatives $\Delta' v^1$, $v^{i'}_{,i'1}$, $\Delta v^{i'}$, $\Delta' B^1$, $B^{i'}_{,i'1}$, and $\Delta B^{i'}$, all of which must be balanced by the highest derivative terms that come from $D_i X^i = D_1 X^1 + D_{i'} X^{i'}$. Hence, X^1 must have linear dependence on $v^{i'}_{,i'}$, $v^{i'}_{,1}$, $B^{i'}_{,i'}$, $B^{i'}_{,1}$, and $X^{i'}$ must have linear dependence on $v^1_{,i'}$, $v^{i'}_{,1}$, $v^{j'}_{,i'}$, $B^{i'}_{,i'}$, $B^{i'}_{,1}$, $B^{j'}_{,i'}$. Their coefficients will depend, at most, on the same variables that appear in T . This yields the general form

$$X^1(t, x^j, v^j, B^j, P; v^{i'}_{,1}, v^{i'}_{,i'}, B^{i'}_{,i'}, B^{i'}_{,1}), \quad (3.0.2)$$

$$X^{i'}(t, x^j, v^j, B^j, P; v^1_{,i'}, v^{j'}_{,1}, v^{j'}_{,k'}, B^{j'}_{,i'}, B^{j'}_{,1}, B^{j'}_{,k'}) \quad (3.0.3)$$

for kinematic fluxes with minimal dependence on highest derivative variables. Note, throughout, we will use a semicolon to separate variables that occur linearly from variables that occur nonlinearly.

The resulting characteristic form of a kinematic conservation law (3.0.1)–(3.0.3) is obtained from the divergence identity (2.0.12), which is given by

$$\begin{aligned} D_t T + D_i X^i &= Q_i^v (v^i_t - S_v^i) + Q_i^B (B^i_t - S_B^i) \\ &\quad + \tilde{Q}^v (v^1_{,1} - \tilde{S}_v) + \tilde{Q}^B (B^1_{,1} - \tilde{S}_B), \end{aligned} \quad (3.0.4)$$

with

$$Q_i^v = \partial_{v^i} T, \quad Q_i^B = \partial_{B^i} T, \quad \tilde{Q}^v = \partial_{v^1} X^1, \quad \tilde{Q}^B = \partial_{B^1} X^1, \quad Q^P = 0. \quad (3.0.5)$$

Since the right hand side of this divergence identity (3.0.4) has no derivatives of \tilde{S}_v , \tilde{S}_B , S_v^1 , $S_v^{i'}$, S_B^1 , $S_B^{i'}$, and S_P , it coincides with the characteristic equation (2.0.13). The dependence of the multiplier functions (3.0.5) on the variables in J is determined by the general kinematic form of the conserved density (3.0.1) and

the spatial flux components (3.0.2)–(3.0.3). This yields

$$Q_i^v(t, x^j, v^j, B^j, P), \quad (3.0.6a)$$

$$Q_i^B(t, x^j, v^j, B^j, P), \quad (3.0.6b)$$

$$\tilde{Q}^v(t, x^j, v^j, B^j, P; v^{i'}_{,1}, v^{i'}_{,i'}, B^{i'}_{,i'}, B^{i'}_{,1}), \quad (3.0.6c)$$

$$\tilde{Q}^B(t, x^j, v^j, B^j, P; v^{i'}_{,1}, v^{i'}_{,i'}, B^{i'}_{,i'}, B^{i'}_{,1}), \quad (3.0.6d)$$

which is the general form for a multiplier of a kinematic conservation law.

3.1 Classification results

To find all multipliers (3.0.6), we set up and solve the determining system (2.0.17) by using computer algebra, as explained in appendix .1.

This computation yields the following result.

Proposition 1 *All conservation law multipliers (3.0.6) for kinematic conserved densities (3.0.1) of the MHD system (2.0.11a)–(2.0.11g) are linear combinations of*

$$(\epsilon_{jik}\alpha^i x^k, 0, \epsilon_{jik}\alpha^i x^k v^j, -\epsilon_{jik}\alpha^i x^k B^j), \quad (3.1.1a)$$

$$(\beta_i, 0, \beta_i v^i - \beta_{i,t} B^i, -\beta_i B^i), \quad (3.1.1b)$$

$$(0, 0, \gamma, 0), \quad (3.1.1c)$$

$$(0, \chi_{,i}, B^i \chi_{,i}, -\chi_{,t} - v^i \chi_{,i} - b \Delta \chi), \quad (3.1.1d)$$

$$(\delta_{ij} B^j, \delta_{ij} v^j, v^i B_i - a B^i_{,i}, v^1 v_1 - \frac{1}{2} \delta_{ij} v^i v^j + a v^i_{,i} + P), \quad a + b = 0, \quad (3.1.1e)$$

$$(\delta_{ij} v^j, \delta_{ij} B^j, v^1 v_1 + \frac{1}{2} \delta_{ij} v^i v^j + \delta_{i,j} B^i B^j + P, -\delta_{i,j} v^i B^j), \quad a = b = 0, \quad (3.1.1f)$$

where α^i is an arbitrary constant vector, $\beta_i(t)$ is an arbitrary covector function of t , $\gamma(t)$ is an arbitrary function of t , and $\chi(t, x^i)$ is an arbitrary function of t, x^i .

This classification yields four different families of local conservation laws plus two individual local conservation laws. For each multiplier, the conserved density and spatial flux is constructed from the characteristic equation, as explained in appendix .1. These conserved densities and spatial fluxes are expressed in a rotationally symmetric form by replacing all expressions \tilde{S}_v, \tilde{S}_B with $v^1_{,1}, B^1_{,1}$, respectively. Hence we obtain the following main result.

Theorem 1 *All kinematic conservation laws (3.0.1), (3.0.2), (3.0.3) of the MHD system (2.0.3) are linear combinations of:*

(1) *angular momentum*

$$\begin{aligned} T[\alpha^j] &= \alpha^j \epsilon_{jkl} x^k v^l \\ X^i[\alpha^j] &= \epsilon_{jkl} \alpha^j x^k (v^l v^i - B^l B^i - a v^l_{,i}) + \epsilon^i_{jkl} \alpha^j (x^k (P + \frac{1}{2} \delta_{ln} B^l B^n) - a v^k) \end{aligned} \quad (3.1.2a)$$

(2) *generalized momentum*

$$\begin{aligned} T[\beta_j(t)] &= \beta_k v^k \\ X^i[\beta_j(t)] &= \beta_j (v^j v^i - B^j B^i + \delta^{ji} (P + \frac{1}{2} \delta_{nl} B^n B^l) - a \delta^{in} v^j_{,n}) - v^i x^j \beta_{j,t} \end{aligned} \quad (3.1.2b)$$

(3) *generalized incompressibility flux*

$$\begin{aligned} T[\gamma(t)] &= 0 \\ X^i[\gamma(t)] &= \gamma v^i \end{aligned} \quad (3.1.2c)$$

(4) *generalized magnetic flux*

$$\begin{aligned} T[\chi(t, x^j)] &= \chi_{,k} B^k \\ X^i[\chi(t, x^j)] &= \chi_{,j} (v^i B^j - B^i v^j + b (B^{i,j} - B^{j,i})) - \chi_{,t} B^i \end{aligned} \quad (3.1.2d)$$

(5) *cross-helicity*

$$\begin{aligned} T &= \delta_{jk} v^j B^k \\ X^i &= v^i \delta_{jk} v^k B^j - B^i \frac{1}{2} \delta_{jk} (B^j B^k + v^j v^k) \\ &\quad + B^i (P + \frac{1}{2} \delta_{jk} B^j B^k) + a \delta^{in} \delta_{jk} (v^j B^k_{,n} - B^j v^k_{,n}) \\ 0 &= a + b \end{aligned} \quad (3.1.2e)$$

(6) *energy*

$$\begin{aligned} T &= \frac{1}{2} \delta_{jk} (v^j v^k + B^j B^k) \\ X^i &= v^i \frac{1}{2} \delta_{jk} (v^j v^k + B^j B^k) - B^i \delta_{jk} v^j B^k + v^i (P + \frac{1}{2} \delta_{jk} B^j B^k) \\ 0 &= a = b \end{aligned} \quad (3.1.2f)$$

where α^i is an arbitrary constant vector, $\beta_i(t)$ is an arbitrary covector function of t , $\gamma(t)$ is an arbitrary function of t , and $\chi(t, x^i)$ is an arbitrary function of t, x^i .

The conservation laws (3.1.2a)–(3.1.2d) holding without conditions on the viscosities a, b are well-known [10, 11, 21] and widely used in applications. Likewise, both the energy conservation law (3.1.2f) holding when $a = b = 0$ and the case $a = b = 0$ of the cross-helicity conservation law (3.1.2e) are well-known when inviscid plasmas are considered [7, 9, 11]. Moreover, the conservation laws for angular momentum (3.1.2a), generalized momentum (3.1.2b), generalized incompressibility flux (3.1.2c), and energy (3.1.2f) are straightforward generalizations of ones [4, 15, 16, 32] holding for the Navier-Stokes equations (viscous flow) and the Euler equations (ideal flow).

The extension of conservation of cross-helicity to the viscous case $a + b = 0$ is a new result. It requires that one of the viscosities is negative, $a < 0$ or $b < 0$.

This situation arises for a physical MHD system that describes an active media or that uses effective parameters to model viscosities [77, 78, 79]. For such systems, $a - b$ becomes the parameter that controls the rate of cross-helicity dissipation.

Cross-helicity is an important quantity because it gives qualitative topological information about the flow [34, 35, 36, 41, 42, 80, 81]. It can be interpreted as the mutual linking number of magnetic field lines (integral curves of B^i) and lines of vorticity (integral curves of $\epsilon_{ik}^i v^k, j$). It is also locally equivalent to the mutual linking number of streamlines (integral curves of v^i) and lines of magnetic vorticity (integral curves of $\epsilon_{ik}^i B^k, j$, which is proportional to the current density, according to the Ampère's law [8]). Cross-helicity is used in the description of MHD turbulence [44, 43] and in the study of dynamo theories [42, 45, 46]. In particular, an estimation of the sign or the value of the cross-helicity can be used to assess the stability of both the magnetic field and the flow configuration and to determine the existence and speed of dynamo mechanisms [42].

Chapter 4

Classification of point symmetries

A review of the basic formulation of symmetries for PDE systems can be found in [2, 3, 4, 5, 6].

A point symmetry is defined by an infinitesimal generator

$$X = \tau \partial / \partial t + \zeta^i \partial / \partial x^i + \eta_v^i \partial / \partial v^i + \eta_B^i \partial / \partial B^i + \eta_P \partial / \partial P, \quad (4.0.1)$$

where τ, η_P are scalar functions of t, x^i, v^i, B^i, P , and $\zeta^i, \eta_v^i, \eta_B^i$ are vector functions of t, x^i, v^i, B^i, P , such that the MHD system (2.0.3) is invariant under the prolongation of the generator.

The point symmetries of the MHD system (2.0.3) were computed in [82] (see also [83, 84, 85, 86, 87]) for the case $a \neq 0, b \neq 0$, with these two parameters considered as fixed constants. To obtain a complete classification, including any special values of a, b for which extra point symmetries exist, we use computer algebra to set up and solve the symmetry determining system

$$(\text{pr}X^{(2)}(v_t^i - S_v^i))|_{\mathcal{E}} = 0, \quad (4.0.2a)$$

$$(\text{pr}X^{(2)}(B_t^i - S_B^i))|_{\mathcal{E}} = 0, \quad (4.0.2b)$$

$$(\text{pr}X^{(2)}(v_{,1}^1 - \tilde{S}_v))|_{\mathcal{E}} = 0, \quad (4.0.2c)$$

$$(\text{pr}X^{(2)}(B_{,1}^1 - \tilde{S}_B))|_{\mathcal{E}} = 0, \quad (4.0.2d)$$

$$(\text{pr}X^{(2)}(P_{,11} - S_P))|_{\mathcal{E}} = 0, \quad (4.0.2e)$$

for the MHD system in the form (2.0.11a)–(2.0.11g), where $\text{pr}X^{(2)}$ denotes the second prolongation of the operator (4.0.1). This computation is explained in appendix .5, which yields the following result.

Proposition 2 *The point symmetries of the MHD system (2.0.3) are generated by:*

(1) *time translation*

$$X_{\text{trans.}} = \partial / \partial t \quad (4.0.3a)$$

(2) rotations

$$X[\mu^i] = \epsilon^m{}_{ik}\mu^i(x^k\partial/\partial x^m + v^k\partial/\partial v^m + B^k\partial/\partial B^m) \quad (4.0.3b)$$

(3) generalized Galilean boosts

$$X[v^i(t)] = v^i\partial/\partial x^i + (v^i)'\partial/\partial v^i - \delta_{ij}(v^i)''x^j\partial/\partial P \quad (4.0.3c)$$

(4) pressure shift

$$X[\sigma(t)] = \sigma\partial/\partial P \quad (4.0.3d)$$

(5) scaling

$$X_{\text{scal.}} = 2t\partial/\partial t + x^i\partial/\partial x^i - v^i\partial/\partial v^i - B^i\partial/\partial B^i - 2P\partial/\partial P \quad (4.0.3e)$$

(6) dilation

$$X_{\text{dil.}} = t\partial/\partial t + x^i\partial/\partial x^i, \quad a = b = 0 \quad (4.0.3f)$$

where μ^i is an arbitrary constant vector, $v^i(t)$ is an arbitrary vector function of t , and $\sigma(t)$ is an arbitrary function of t .

The non-trivial commutators of these point symmetries (4.0.3) are given by

$$[X_{\text{trans.}}, X[v^i(t)]] = X[\tilde{v}^i(t)], \quad \tilde{v}^i = (v^i)', \quad (4.0.4a)$$

$$[X_{\text{trans.}}, X[\sigma(t)]] = X[\tilde{\sigma}(t)], \quad \tilde{\sigma} = \sigma', \quad (4.0.4b)$$

$$[X_{\text{trans.}}, X_{\text{scal.}}] = 2X_{\text{trans.}}, \quad (4.0.4c)$$

$$[X[\mu_1^i], X[\mu_2^j]] = X[\tilde{\mu}^i], \quad \tilde{\mu}^i = -\epsilon^i{}_{jk}\mu_1^j\mu_2^k, \quad (4.0.4d)$$

$$[X[\mu^i], X[v^j(t)]] = X[\tilde{v}^i(t)], \quad \tilde{v}^i = -\epsilon^i{}_{jk}\mu^j v^k, \quad (4.0.4e)$$

$$[X[v_1^i(t)], X[v_2^j(t)]] = X[\tilde{\sigma}(t)], \quad \tilde{\sigma} = \delta_{ij}(v_1^i(v_2^j)'' - v_2^i(v_1^j)''), \quad (4.0.4f)$$

$$[X[v^i(t)], X_{\text{scal.}}] = X[\tilde{v}^i(t)], \quad \tilde{v}^i = v^i - 2t(v^i)', \quad (4.0.4g)$$

$$[X_{\text{scal.}}, X[\sigma(t)]] = X[\tilde{\sigma}(t)], \quad \tilde{\sigma}(t) = 2(\sigma + t(\sigma)'), \quad (4.0.4h)$$

$$[X[v^i(t)], X_{\text{dil.}}] = X[\tilde{v}^i(t)], \quad \tilde{v}^i = v^i - t(v^i)', \quad a = b = 0, \quad (4.0.4i)$$

$$[X_{\text{dil.}}, X[\sigma(t)]] = X[\tilde{\sigma}(t)], \quad \tilde{\sigma}(t) = t(\sigma)', \quad a = b = 0. \quad (4.0.4j)$$

We remark that the results stated in Ref.[82] omit the non-trivial commutator (4.0.4f) for the generalized Galilean boosts and the dilation symmetry (4.0.3f) for the case $a = b = 0$.

All of the point symmetries (4.0.3) are straightforward generalizations of ones [4, 13, 16] holding for the Navier-Stokes equations and the inviscid Euler equa-

tions.

Every point symmetry generates a group of point transformations on t, x^i, v^i, B^i, P under which the MHD solution space \mathcal{E} is mapped into itself. In physical applications, the generalized Galilean boosts can be interpreted as a transformation between two different reference frames that are in relative motion. When two inertial reference frames are considered, all transformations between them are generated by Galilean boosts with $v^i(t) = x_0^i + v_0^i t$ which is a linear function of time, where v_0^i is the constant relativity velocity of the two inertial frames, and x_0^i is a constant space translation. In this case, the commutator (4.0.4f) vanishes. More interesting is case of a transformation from an inertial reference frame to an accelerating reference frame. Two commonly considered examples are Galilean boosts with

$$v^i(t) = \frac{1}{2}a_0^i t^2 \quad (4.0.5)$$

and with

$$\begin{aligned} v^1(t) &= x_0 \cos(\omega t) - y_0 \sin(\omega t) \\ v^2(t) &= x_0 \sin(\omega t) + y_0 \cos(\omega t) \\ v^3(t) &= 0, \end{aligned} \quad (4.0.6)$$

where a_0^i is a constant acceleration vector, ω is a constant angular speed, and x_0, y_0 are components of a constant position vector in the x^1, x^2 -plane. These two Galilean boosts respectively generate a transformation to a reference frame that has a constant acceleration in a fixed direction and a reference frame that is undergoing uniform circular motion in a fixed plane.

Another situation where the commutator (4.0.4f) is relevant arises when boundary/initial conditions are considered. In this situation, if the posed problem is invariant under two generalized Galilean boosts, then the problem is also invariant under a pressure shift connected with the commutator of these boosts.

4.1 Geometric symmetries

The symmetry algebra (4.0.3) has a subalgebra comprising spatial geometric symmetries that consist of space translations and rotations:

$$X[v^i] = v^i \partial / \partial x^i, \quad (4.1.1a)$$

$$X[\mu^i] = \epsilon^m{}_{ik} \mu^i (x^k \partial / \partial x^m + v^k \partial / \partial v^m + B^k \partial / \partial B^m), \quad (4.1.1b)$$

where μ^i, v^i are arbitrary constant vectors. This subalgebra is isomorphic to $SO(3) \times \mathbb{R}^3$ which is the Lie algebra of the isometry group of Euclidean space. These symmetry generators can be written in a geometrical form in terms of Killing vectors. A Killing vector $\zeta^i(x^j)$ in Euclidean space is a vector function that satisfies the equation

$$\mathfrak{L}_\zeta \delta_{ij} = \frac{1}{2} (\delta_{jk} \partial_i \zeta^k + \delta_{ik} \partial_j \zeta^k) = 0, \quad (4.1.2)$$

where \mathfrak{L}_ζ denotes the Lie derivative. The set of solutions is given by

$$\zeta^i = v^i + \epsilon^i{}_{jk} \mu^j x^k, \quad \mu^i, v^i = \text{const.} \quad (4.1.3)$$

Each Killing vector geometrically defines a family of integral curves that generate an isometry of the Euclidean metric.

Then the symmetry generators (4.1.1a) and (4.1.1b) take the combined form

$$X[\zeta^i] = \zeta^i \partial / \partial x^i + v^j \partial_j \zeta^i \partial / \partial v^i + B^j \partial_j \zeta^i \partial / \partial B^i. \quad (4.1.4)$$

This formulation will be useful when we consider reductions of the MHD system in the next section.

Chapter 5

Action of symmetries on conservation laws

The well-known formula for the action of a point symmetry generator (4.0.1) on a conservation law (2.0.4) is given by [3]

$$\tilde{T} = \text{pr}X T - (X^j D_j) \tau + T(D_j \tilde{\zeta}^j), \quad (5.0.1a)$$

$$\tilde{X}^i = \text{pr}X X^i - T D_t \tilde{\zeta}^i - (X^j D_j) \tilde{\zeta}^i + X^i (D_t \tau + D_j \tilde{\zeta}^j) \quad (5.0.1b)$$

where T, X^i are the conserved density and spatial flux in a conservation law, and \tilde{T}, \tilde{X}^i are the transformed conserved density and spatial flux, modulo a locally trivial density and flux (2.0.9).

The infinitesimal action (5.0.1) leads to a corresponding formula for the action of the Lie group generated by the set of all point symmetries applied to the set of all conservation laws. This group action maps the set of locally trivial conservation laws (2.0.9) into itself and therefore the group action is well-defined on equivalence classes of conservation laws. A generating set of conservation laws is defined to be a subset of non-trivial conservation laws whose image under the point symmetry group is the entire set of non-trivial conservation laws up to equivalence. A *minimal generating set* is defined to be a generating set with the minimal number of linearly independent non-trivial conservation laws (up to equivalence).

From the classifications of kinematic conservation laws in Theorem 1 and point symmetries in Proposition 2, we compute the action of the point symmetry generators (4.0.3) on the kinematic conservation laws (3.1.2). This yields the results shown in Table 5.1.

We see that the entire set of angular momenta, generalized momenta, and incompressibility fluxes can be obtained from any angular momentum (for any fixed vector α^i). by the action of the symmetry subgroup comprising rotations and generalized Galilean boosts. We also see that the family of generalized magnetic fluxes is closed under the action of the full group of point symmetries.

	angl. mom. α^j	gen. mom. $\beta_j(t)$	gen. mag. flux $\chi(t, x^j)$	incomp. flux $\gamma(t)$	cross-hel. ($a + b = 0$)	energy ($a = b = 0$)
time translation	0	gen. mom. β'_j	gen. mag. flux χ_t	incomp. flux h'	0	0
rotations μ^i	ang. mom. $\epsilon^k{}_{ij}\alpha^j\mu^i$	gen. mom. $\epsilon^j{}_{ik}\beta_j\mu^i$	gen. mag. flux $\epsilon^j{}_{ik}\chi_{,i}\mu^j x^k$	0	0	0
generalized Galilean boosts $v^i(t)$	gen. mom. $\epsilon_{kji}\alpha^j v^i$	incomp. flux $(v_i\beta'_j - \beta_j v'_i)\delta^{ij}$	gen. mag. flux $v^i\chi_{,i}$	0	gen. mag. fluxes $\delta^j{}_i x^i v'_j$	gen. mom. $\delta^{ij}v'_j$
pressure shift $\sigma(t)$	0	0	0	0	incomp. flux $-\int \sigma dt$	gen. mag. flux σ
scaling	angl. mom. $3\alpha^j$	gen. mom. $2\beta_j + 2t\beta'_j$	gen. mag. flux $\chi + 2t\chi_{,t} + x^i\chi_{,i}$	incomp. flux $3\gamma + 2t\gamma'$	energy	cross-hel.
dilation ($a = b = 0$)	angl. mom. $4\alpha^j$	gen. mom. $3\beta_j + t\beta'_j$	gen. mag. flux $2\chi + t\chi_{,t} + x^i\chi_{,i}$	incomp. flux $\gamma + t\gamma'$	$3 \times$ cross-hel.	$3 \times$ energy

Table 5.1: Point symmetry action on kinematic conservation laws

Thus, we can formulate the following result.

Theorem 2 *The minimal generating set for the set of all kinematic conservation laws of the MHD system (2.0.3a) – (2.0.3d) consists of:*

- (1) *a single angular momentum (with respect to any fixed vector) together with the infinite family of generalized magnetic fluxes when $a + b \neq 0$;*
- (2) *additionally the cross-helicity when $a + b = 0$;*
- (3) *additionally the energy (and the cross-helicity) when $a = b = 0$.*

Chapter 6

Symmetry reductions

For each point symmetry (4.0.3) of the constant-density MHD system (2.0.3), group invariant solutions can be considered. Such solutions satisfy a PDE system in fewer variables, and these PDE systems can often be solved analytically while their numerical solution is always easier than the numerical solution of the unreduced MHD system. A review of symmetry reduction for PDE systems can be found in [1, 3, 5].

Symmetries of the most physical interest for the constant-density MHD system, other than time translations, are spatial translations, rotations, and their linear combinations. These symmetries are generated by Killing vectors (4.1.2) in Euclidean space. When acting on solutions of the MHD system, a Killing vector symmetry has the form

$$\widehat{X}[\xi^i] = \mathfrak{L}_\xi v^i \partial / \partial v^i + \mathfrak{L}_\xi B^i \partial / \partial B^i + \mathfrak{L}_\xi P \partial / \partial P. \quad (6.0.1)$$

The conditions for a MHD solution $(v^i(t, x^j), B^i(t, x^j), P(t, x^j))$ to be invariant under a symmetry (6.0.1) are given by

$$\mathfrak{L}_\xi v^i = 0, \quad \mathfrak{L}_\xi B^i = 0, \quad \mathfrak{L}_\xi P = 0. \quad (6.0.2)$$

Hence, with respect to a Killing vector symmetry, group-invariant solutions of the MHD system are determined by the invariance equations (6.0.2) together with the PDEs (2.0.3).

We consider three physically important Killing vector symmetries: space translations, axial rotations, helical motion. For each symmetry, we will first write out the resulting reduction of the MHD system (2.0.3) and the form for group-invariant solutions (6.0.2), and then we will present a classification of kinematic conservation laws and point symmetries for the reduced system.

A similar analysis has been carried out for the Navier-Stokes equations and the Euler equations of fluid mechanics in [88, 89]. They have introduced the alternative helical coordinates that depend on two scalar parameters. Two limiting cases of this coordinate system correspond to the 2D-Cartesian coordinates and

rotation-invariant cylindrical coordinates. The space of reduced coordinates coincides with (6.3.2) after reduction, except that one parametric helical coordinates (6.3.2) exclude the limiting case - the rotation-invariant cylindrical coordinates.

The method of direct construction of local conservation laws have been used. All classical conservation laws of the Navier-Stokes equations and the Euler equations have been found. The new conservation law, that they called the conservation law of the generalized momenta/angular momenta, is defined by an arbitrary function of the velocity component in invariant direction.

6.1 Spatial translational reduction

Spatial translations are represented by Killing vectors $\zeta_{\text{trans.}}^i = v^i$, where v^i is a constant vector. Without loss of generality, we can choose $v^i = \delta^{i3}$, so then $\zeta_{\text{trans.}}^i$ is a translation in the $x^3 = z$ coordinate direction. The resulting Killing vector $\zeta_{\text{trans.}}^i = \delta^{i3}$ geometrically represents the generator of a z -translation isometry of the Euclidean metric.

The invariance condition (6.0.2) on the MHD variables v^i, B^i, P becomes

$$v^i{}_{,3} = 0, B^i{}_{,3} = 0, P_{,3} = 0. \quad (6.1.1)$$

This is equivalent to having v^i, B^i, P be functions only of $t, x^1 = x, x^2 = y$. We will use the following index notation: $i'' = 1, 2; i = i'', 3$.

The MHD system (2.0.3) combined with the invariance equations (6.1.1) yields the reduced system

$$v^{i''}{}_t = -v^{j''} v^{i''}{}_{,j''} - P_{,i''} - \delta_{jk} B^j B^k{}_{,i''} + B^{j''} B^{i''}{}_{,j''} + a \Delta'' v^{i''}, \quad (6.1.2a)$$

$$v^3{}_t = -v^{j''} v^3{}_{,j''} + B^{j''} B^3{}_{,j''} + a \Delta'' v^3, \quad (6.1.2b)$$

$$B^i{}_t = -v^{j''} B^i{}_{,j''} + B^{j''} v^i{}_{,j''} + b \Delta'' B^i, \quad (6.1.2c)$$

$$v^{i''}{}_{,i''} = 0, \quad (6.1.2d)$$

$$B^{i''}{}_{,i''} = 0, \quad (6.1.2e)$$

$$\Delta'' P = B^{j''}{}_{,i''} B^{i''}{}_{,j''} - v^{j''}{}_{,i''} v^{i''}{}_{,j''} - \delta_{ik} (\delta^{j''l''} B^k{}_{,l''} B^i{}_{,j''} + B^k \Delta'' B^i). \quad (6.1.2f)$$

The solution space $\mathcal{E}_{\text{trans.}}$ of this translation-invariant system (6.1.2) is a subset of the unreduced MHD solution space, $\mathcal{E}_{\text{trans.}} \subset \mathcal{E}$.

A conservation law of the translation-invariant MHD system (6.1.2) is a local continuity equation

$$(D_t T + D_{i''} X^{i''})|_{\mathcal{E}_{\text{trans.}}} = 0 \quad (6.1.3)$$

holding on the translation-invariant MHD solution space $\mathcal{E}_{\text{trans.}}$, where T is a scalar function on J and $X^{i''}$ are vector functions on J , up to some finite order in derivatives. The form of a locally trivial conservation law consists of

$$T|_{\mathcal{E}_{\text{trans.}}} = D_{i''} \Psi^{i''}, \quad X^{i''}|_{\mathcal{E}_{\text{trans.}}} = -D_t \Psi^{i''} + D_{j''} \Theta^{i''j''} \quad (6.1.4)$$

holding for some vector function $\Psi^{i''}$ on J and some antisymmetric tensor func-

tion $\Theta^{i''j''} = -\Theta^{j''i''}$ on J .

To derive the characteristic form for a conservation law (6.1.3), we will coordinatize the translation-invariant MHD solution space $\mathcal{E}_{\text{trans.}} \subset J$ by using the same leading derivatives considered for coordinatizing the unreduced MHD solution space \mathcal{E} . First, we express the divergence equations (6.1.2d) and (6.1.2e) in a solved form for x^1 derivatives:

$$v^1_{,1} = -v^2_{,2}, \quad (6.1.5a)$$

$$B^1_{,1} = -B^2_{,2}. \quad (6.1.5b)$$

Next, we substitute both equations (6.1.5a) and (6.1.5b) into the evolution equations (6.1.2a)–(6.1.2c), and we express the resulting equations in solved form for t derivatives:

$$v^1_t = -P_{,1} - v^{j''} v^1_{,j''} + v^1 v^{j''}_{,j''} + B^{j''} (B^1_{,j''} - \delta_{j''k''} B^{k''}_{,1}) + a \Delta' v^1 - a v^j_{,1j'}, \quad (6.1.5c)$$

$$v^2_t = -P_{,2} - v^j v^2_{,j} + B^j B^2_{,j} - \delta_{jk} B^j B^k_{,2} + a \Delta'' v^2, \quad (6.1.5d)$$

$$v^3_t = -v^j v^3_{,j} + B^j B^3_{,j} - \delta_{jk} B^j B^k_{,3} + a \Delta'' v^3, \quad (6.1.5e)$$

$$B^1_t = B^{j''} v^1_{,j''} - B^1 v^{j''}_{,j''} - v^{j''} B^1_{,j''} + v^1 B^{j''}_{,j''} + b \Delta' B^1 - b B^{j''}_{,1j''}, \quad (6.1.5f)$$

$$B^{i''}_t = B^j v^{i''}_{,j} - v^j B^{i''}_{,j} + b \Delta'' B^{i''}, \quad (6.1.5g)$$

where

$$\Delta'' f = \delta^{i''j''} f_{,i''j''}.$$

Finally, we also substitute both equations (6.1.5a) and (6.1.5b) into the remaining equation (6.1.2f) which we express in a solved form

$$\begin{aligned} P_{,11} = & -P_{,22} - v^{i''}_{,i''} v^{j''}_{,j''} - v^{j''}_{,i''} v^{i''}_{,j''} - 2v^1_{,i''} v^{i''}_{,1} + B^{j''}_{,i''} B^{i''}_{,j''} \\ & + 2B^1_{,i''} B^{i''}_{,1} - \delta^{i''j''} B^1_{,i''} B^1_{,j''} - B^1 \Delta'' B^1 + B^1 B^{i''}_{,i''1} \\ & - \delta_{i''k''} (\delta^{jl} B^{k''}_{,l} B^{i''}_{,j} + B^{k''} \Delta B^{i''}). \end{aligned} \quad (6.1.5h)$$

Then the coordinates on $\mathcal{E}_{\text{trans.}}$ consist of the x^2 derivatives of v^1, B^1, P , and the $x^{i''}$ derivatives of $v^{j''}, B^{j''}$.

In this coordinatization, the characteristic form of a conservation law (6.1.3)

is given by a divergence identity

$$D_t \tilde{T} + D_{i''} \tilde{X}^{i''} = Q_i^v (v_t^i - S_B^i) + Q_i^B (B_t^i - S_B^i) + \tilde{Q}^v (v^1_{,1} - \tilde{S}_v) \\ + \tilde{Q}^B (B^1_{,1} - \tilde{S}_B) + Q^P (P_{,11} - S_P) \quad (6.1.6)$$

with

$$\tilde{T}|_{\mathcal{E}_{\text{trans.}}} = T, \quad \tilde{X}^{i''}|_{\mathcal{E}_{\text{trans.}}} = X^{i''} \quad (6.1.7)$$

and with $Q_i^v, Q_i^B, \tilde{Q}^v, \tilde{Q}^B, Q^P$ being multiplier functions on J that are non-singular on $\mathcal{E}_{\text{trans.}}$.

Multiplier functions ($Q_i^v, Q_i^B, \tilde{Q}^v, \tilde{Q}^B, Q^P$) are determined by the condition that the Euler operators (2.0.16) with respect to v^i, B^i, P annihilate the right hand side of equation (6.1.6). This yields the multiplier determining system (2.0.17). Each solution of this system determines a corresponding conservation law (6.1.3) through equation (6.1.6).

We will classify all conservation laws of kinematic form for the translation-invariant MHD system (6.1.2). The general form for a kinematic conserved density is given by

$$T(t, x^{j''}, v^j, B^j, P). \quad (6.1.8a)$$

The corresponding general form for a kinematic spatial flux can be determined by the same method used for the unreduced MHD system (2.0.3), which leads to

$$X^1(t, x^{j''}, v^j, B^j, P; v^2_{,1}, v^3_{,1}, v^2_{,2}, B^2_{,1}, B^3_{,1}, B^2_{,2}), \quad (6.1.8b)$$

$$X^2(t, x^{j''}, v^j, B^j, P; v^2_{,1}, v^3_{,1}, v^j_{,2}, B^2_{,1}, B^3_{,1}, B^j_{,2}). \quad (6.1.8c)$$

Then, from the general expressions (3.0.5) relating the form of the conserved density and the flux components to the form of the multiplier functions, we have

$$Q_i^v(t, x^{j''}, v^j, B^j, P) = \partial_{v^i} T, \quad (6.1.9a)$$

$$Q_i^B(t, x^{j''}, v^j, B^j, P) = \partial_{B^i} T, \quad (6.1.9b)$$

$$\tilde{Q}^v(t, x^{j''}, v^j, B^j, P; v^2_{,1}, v^3_{,1}, v^2_{,2}, B^2_{,1}, B^3_{,1}, B^2_{,2}) = \partial_{v^1} X^1, \quad (6.1.9c)$$

$$\tilde{Q}^B(t, x^{j''}, v^j, B^j, P; v^2_{,1}, v^3_{,1}, v^2_{,2}, B^2_{,1}, B^3_{,1}, B^2_{,2}) = \partial_{B^1} X^1, \quad (6.1.9d)$$

$$Q^P = 0. \quad (6.1.9e)$$

Using computer algebra, as explained in appendix .2, we set up and solve the

multiplier determining system (2.0.17) to find all multiplier functions (6.1.9), and then we obtain the resulting conserved densities and spatial fluxes (6.1.8).

This yields the following classification result.

Theorem 3 *All kinematic conservation laws (6.1.8) of the translation-invariant MHD system (6.1.2) are linear combinations of:*

(1) *transverse angular momentum*

$$\begin{aligned} T &= \epsilon_{j''k''} x^{j''} v^{k''} \\ X^{i''} &= \epsilon_{j''k''} (x^{j''} (v^{i''} v^{k''} - B^{i''} B^{k''}) + \delta^{i''k''} x^{j''} (P + \frac{1}{2} \delta_{jk} B^j B^k) \\ &\quad + a (x^{k''} v^{j'',i''} - \delta^{i''k''} v^{j''})) \end{aligned} \quad (6.1.10a)$$

(2) *generalized momentum*

$$\begin{aligned} T[\beta_{j''}(t)] &= \beta_{j''} v^{j''}, \\ X^{i''}[\beta_{j''}(t)] &= \beta_{j''} (v^{j''} v^{i''} - B^{j''} B^{i''} + \delta^{j''i''} (P + \frac{1}{2} \delta_{kl} B^k B^l) - a \delta^{i''k} v^{j''}_{,k}) \\ &\quad - x^{j''} \beta_{j'',t} v^{i''} \end{aligned} \quad (6.1.10b)$$

(3) *generalized incompressibility flux*

$$\begin{aligned} T[\gamma(t)] &= 0 \\ X^{i''}[\gamma(t)] &= \gamma v^{i''} \end{aligned} \quad (6.1.10c)$$

(4) *generalized magnetic flux*

$$\begin{aligned} T[\chi(t, x^{j''})] &= \chi_{,j''} B^{j''} \\ X^{i''}[\chi(t, x^{j''})] &= \chi_{,j''} (v^{i''} B^{j''} - B^{i''} v^{j''} + b (B^{i'',j''} - B^{j'',i''})) - \chi_{,t} B^{i''} \end{aligned} \quad (6.1.10d)$$

(5) *cross-helicity*

$$\begin{aligned} T &= \delta_{jk} v^j B^k \\ X^{i''} &= v^{i''} \delta_{jk} v^j B^k - B^{i''} \frac{1}{2} \delta_{jk} (v^j v^k + B^j B^k) + B^{i''} (P + \frac{1}{2} \delta_{jk} B^j B^k) \\ &\quad + a \delta^{i''n''} \delta_{jk} (v^j B^k_{,n''} - B^j v^k_{,n''}) \\ 0 &= a + b \end{aligned} \quad (6.1.10e)$$

(6) energy

$$\begin{aligned} T &= \frac{1}{2} \delta_{jk} (v^j v^k + B^j B^k) \\ X^{i''} &= v^{i''} \frac{1}{2} \delta_{jk} (v^j v^k + B^j B^k) - B^{i''} \delta_{jk} v^j B^k + v^{i''} (P + \frac{1}{2} \delta_{jk} B^j B^k) \\ 0 &= a = b \end{aligned} \quad (6.1.10f)$$

(7) transverse-field Riemann (coupling) invariant

$$\begin{aligned} T &= \partial_{v^3} \kappa \\ X^{i''} &= v^{i''} \partial_{v^3} \kappa - B^{i''} \partial_{B^3} \kappa - a v^3,^{i''} \partial_{v^3}^2 \kappa - b B^3,^{i''} \partial_{v^3} \partial_{B^3} \kappa \end{aligned} \quad (6.1.10g)$$

where $\epsilon_{i''j''} = \epsilon_{i''j''3}$ is the reduced Levi-Civita tensor, $\beta_{i''}(t)$ is an arbitrary covector function of t , $\gamma(t)$ is an arbitrary function of t , $\chi(t, x^{i''})$ is an arbitrary function of $t, x^{i''}$, and $\kappa(v^3, B^3)$ is an arbitrary solution of the linear equations

$$\partial_{v^3}^2 \kappa - \partial_{B^3}^2 \kappa = 0, \quad a = b = 0 \quad (6.1.11a)$$

$$\partial_{v^3}^2 \kappa - \partial_{B^3}^2 \kappa = 0, \quad \partial_{v^3}^3 \kappa = 0, \quad a + b = 0 \quad (6.1.11b)$$

$$\partial_{v^3}^2 \kappa - \partial_{B^3}^2 \kappa = 0, \quad \partial_{v^3}^3 \kappa = 0, \quad \partial_{B^3}^3 \kappa = 0, \quad a + b \neq 0. \quad (6.1.11c)$$

The conservation laws (6.1.10a) – (6.1.10f) are inherited from the unreduced MHD system. They are widely used in applications involving physical problems [10, 21] that have a translational invariance.

The conservation law (6.1.10g) is new, although some special cases of it coincide with well-known conservation laws. In particular, this conservation law involves a function, κ , satisfying a wave equation (6.1.11a) in the variables v^3, B^3 . In the case of inviscid plasmas, there is no other condition on this function, and so it yields two infinite families of conservation laws, corresponding to

$$\kappa = \kappa_1(v^3 + B^3) + \kappa_2(v^3 - B^3) \quad (6.1.12)$$

where κ_1, κ_2 are arbitrary functions of their arguments. In case of viscous plasmas, the function κ satisfies extra equations (6.1.11b) and (6.1.11c). This yields, respectively, families of four and five conservation laws, corresponding to

$$\kappa = C_4(v^3 v^3 + B^3 B^3) + C_3 v^3 B^3 + C_2 v^3 + C_1 B^3 \quad (6.1.13a)$$

and

$$\kappa = C_5 B^3 (3v^3 v^3 + B^3 B^3) + C_4 (v^3 v^3 + B^3 B^3) + C_3 v^3 B^3 + C_2 v^3 + C_1 B^3 \quad (6.1.13b)$$

where C_1, \dots, C_5 are arbitrary constants.

These conservation laws (6.1.10g) unify and generalize several known physical conservation laws, which are inherited from the unreduced MHD system:

transverse momentum

$$\begin{aligned} T &= v^3 \\ X^{i''} &= v^{i''} v^3 - B^{i''} B^3 - a v^{3,i''} \end{aligned} \quad (6.1.14a)$$

transverse magnetic flux

$$\begin{aligned} T &= B^3 \\ X^{i''} &= v^{i''} B^3 - B^{i''} v^3 - b B^{3,i''} \end{aligned} \quad (6.1.14b)$$

incompressibility flux

$$\begin{aligned} T &= 0 \\ X^{i''} &= v^{i''} \end{aligned} \quad (6.1.14c)$$

solenoidal flux

$$\begin{aligned} T &= 0 \\ X^{i''} &= B^{i''} \end{aligned} \quad (6.1.14d)$$

transverse cross-helicity

$$\begin{aligned} T &= v^3 B^3 \\ X^{i''} &= v^{i''} v^3 B^3 - B^{i''} \frac{1}{2} (v^3 v^3 + B^3 B^3) + a v^3 B^{3,i''} - a B^3 v^{3,i''} \\ 0 &= a + b \end{aligned} \quad (6.1.14e)$$

transverse energy

$$\begin{aligned} T &= \frac{1}{2} (v^3 v^3 + B^3 B^3) \\ X^{i''} &= v^{i''} \frac{1}{2} (v^3 v^3 + B^3 B^3) - B^{i''} v^3 B^3 \\ 0 &= a = b \end{aligned} \quad (6.1.14f)$$

A point symmetry of the translation-invariant MHD system (6.1.2) is an infinitesimal generator of the form (4.0.1) under which the solution space $\mathcal{E}_{\text{trans.}}$ is mapped into itself. We classify all point symmetries by using computer algebra to set up and solve the symmetry determining system (4.0.2) for the translation-

invariant MHD system in the form (6.1.5a)–(6.1.5h). This computation is explained in appendix .6, which yields the following result.

Proposition 3 *The point symmetries of the translation-invariant MHD system (6.1.2) are generated by:*

(1) *time translation*

$$X = \partial/\partial t \quad (6.1.15a)$$

(2) *rotation*

$$X_{\text{rot.}} = \epsilon^{i''j''3}(x^{j''}\partial/\partial x^{i''} + v^{j''}\partial/\partial v^{i''} + B^{j''}\partial/\partial B^{i''}) \quad (6.1.15b)$$

(3) *generalized Galilean boosts*

$$X[v^{i''}(t)] = v^{i''}\partial/\partial x^{i''} + (v^{i''})'\partial/\partial v^{i''} - \delta_{i''j''}(v^{i''})''x^{j''}\partial/\partial P \quad (6.1.15c)$$

(4) *pressure shift*

$$X[\sigma(t)] = \sigma \partial/\partial P \quad (6.1.15d)$$

(5) *scaling*

$$X_{\text{scal.}} = 2t\partial/\partial t + x^{i''}\partial/\partial x^{i''} - v^i\partial/\partial v^i - B^i\partial/\partial B^i - 2P\partial/\partial P \quad (6.1.15e)$$

(6) *dilation*

$$X_{\text{dil.}} = t\partial/\partial t + x^{i''}\partial/\partial x^{i''}, \quad a = b = 0 \quad (6.1.15f)$$

(7) *transverse-field generalized dilation/shift*

$$X[\lambda(v^3, B^3)] = \partial_{B^3}\lambda\partial/\partial B^3 + \partial_{v^3}\lambda\partial/\partial v^3 - B^3\partial_{B^3}\lambda\partial/\partial P, \quad (6.1.15g)$$

where $v^{i''}(t)$ is an arbitrary vector function of t , $\sigma(t)$ is an arbitrary function of t , and $\lambda(v^3, B^3)$ is an arbitrary solution of the linear equations

$$\partial_{v^3}^2\lambda - \partial_{B^3}^2\lambda = 0, \quad a = b = 0 \quad (6.1.16a)$$

$$\partial_{v^3}^2\lambda - \partial_{B^3}^2\lambda = 0, \quad \partial_{v^3}^2\partial_{B^3}\lambda = \partial_{v^3}\partial_{B^3}^2\lambda = 0, \quad a = b \neq 0 \quad (6.1.16b)$$

$$\partial_{v^3}^2\lambda - \partial_{B^3}^2\lambda = 0, \quad \partial_{v^3}\partial_{B^3}\lambda = 0, \quad a - b \neq 0 \quad (6.1.16c)$$

The point symmetries (6.1.15a)–(6.1.15f) are inherited from the unreduced MHD system. A complete classification of these symmetries has not previously appeared in the literature.

The symmetry (6.1.15g) is new. It involves a function, λ , satisfying a wave equation (6.1.16a) in the variables v^3, B^3 . In the case of inviscid plasmas, there is no other condition on this function, and so it yields two infinite families of point symmetries, corresponding to

$$\lambda = \lambda_1(v^3 + B^3) + \lambda_2(v^3 - B^3) \quad (6.1.17)$$

where λ_1, λ_2 are arbitrary functions of their arguments. In case of viscous plasmas, the function λ satisfies extra equations (6.1.16b) and (6.1.16c). This yields, respectively, families of four and three and point symmetries, corresponding to

$$\lambda = C_4 v^3 B^3 + C_3(v^3 v^3 + B^3 B^3) + C_2 B^3 + C_1 v^3 \quad (6.1.18a)$$

and

$$\lambda = C_3(v^3 v^3 + B^3 B^3) + C_2 B^3 + C_1 v^3 \quad (6.1.18b)$$

where C_1, \dots, C_4 are arbitrary constants. In particular, the family (6.1.18b) is given by:

transverse velocity shift

$$\mathcal{X} = \partial / \partial v^3, \quad (6.1.19a)$$

transverse magnetic shift

$$\mathcal{X} = \partial / \partial B^3 - B^3 \partial / \partial P, \quad (6.1.19b)$$

transverse scaling

$$\mathcal{X} = B^3 \partial / \partial B^3 + v^3 \partial / \partial v^3 - (B^3)^2 \partial / \partial P \quad (6.1.19c)$$

In the larger family (6.1.18a), the additional symmetry is given by

$$\mathcal{X} = v^3 \partial / \partial B^3 + B^3 \partial / \partial v^3 - v^3 B^3 \partial / \partial P \quad (6.1.20)$$

which generates the transformation

$$\tilde{v}^3 = v^3 \cosh \vartheta + B^3 \sinh \vartheta, \quad \tilde{B}^3 = v^3 \sinh \vartheta + B^3 \cosh \vartheta, \quad (6.1.21a)$$

$$\tilde{P} = P - \frac{1}{2}((v^3)^2 + (B^3)^2) \frac{1}{2} \sinh 2\vartheta + v^3 B^3 \cosh 2\vartheta \quad (6.1.21b)$$

where

$$4\tilde{P} + (\tilde{v}^3)^2 + (\tilde{B}^3)^2 = 4P + (v^3)^2 + (B^3)^2, \quad (\tilde{v}^3)^2 - (\tilde{B}^3)^2 = (v^3)^2 - (B^3)^2 \quad (6.1.22)$$

are the invariants. This transformation (6.1.21) is similar to a transformation presented in Ref.[5, 59] for equilibrium solutions of an ideal plasma.

6.2 Axial rotational reduction

Rotations are represented by Killing vectors $\xi_{\text{rot.}}^i = \epsilon^i_{jk} \mu^j x^k$, where μ^j is a constant vector which defines the axis of rotation (namely, the rotation acts in the plane perpendicular to this vector). Without loss of generality, we can choose $\mu^j = \delta^{j3}$, so that $\xi_{\text{rot.}}^i$ geometrically represents a rotation in the x^1, x^2 -plane. The invariance condition (6.0.2) on the MHD variables v^i, B^i, P is then given by

$$x^1 v^1_{,2} - x^2 v^1_{,1} + v^2 = 0, \quad (6.2.1a)$$

$$x^1 v^2_{,2} - x^2 v^2_{,1} - v^1 = 0, \quad (6.2.1b)$$

$$x^1 v^3_{,2} - x^2 v^3_{,1} = 0, \quad (6.2.1c)$$

$$x^1 B^1_{,2} - x^2 B^1_{,1} + B^2 = 0, \quad (6.2.1d)$$

$$x^1 B^2_{,2} - x^2 B^2_{,1} - B^1 = 0, \quad (6.2.1e)$$

$$x^1 B^3_{,2} - x^2 B^3_{,1} = 0, \quad (6.2.1f)$$

$$x^1 P_{,2} - x^2 P_{,1} = 0. \quad (6.2.1g)$$

These equations (6.2.1) have a simpler form in cylindrical coordinates (ρ, ϕ, z) adapted to the x^1, x^2 -plane:

$$x^1 = \rho \cos \phi, \quad x^2 = \rho \sin \phi, \quad x^3 = z, \quad (6.2.2)$$

with

$$x^1 \partial / \partial x^2 - x^2 \partial / \partial x^1 = \partial_\phi. \quad (6.2.3)$$

In these coordinates, the corresponding cylindrical components of the velocity v^ρ, v^ϕ, v^z and the magnetic field B^ρ, B^ϕ, B^z are given in terms of the standard Cartesian components by the transformations

$$v^1 = v^\rho \cos \phi - v^\phi \rho \sin \phi, \quad v^2 = v^\rho \sin \phi + v^\phi \rho \cos \phi, \quad v^3 = v^z, \quad (6.2.4a)$$

$$B^1 = B^\rho \cos \phi - B^\phi \rho \sin \phi, \quad B^2 = B^\rho \sin \phi + B^\phi \rho \cos \phi, \quad B^3 = B^z. \quad (6.2.4b)$$

The invariance equations (6.2.1) become

$$\begin{aligned} v^\rho_{,\phi} = 0, \quad v^\phi_{,\phi} = 0, \quad v^z_{,\phi} = 0, \\ B^\rho_{,\phi} = 0, \quad B^\phi_{,\phi} = 0, \quad B^z_{,\phi} = 0, \quad P_{,\phi} = 0, \end{aligned} \quad (6.2.5)$$

which is equivalent to having $v^\rho, v^\phi, v^z, B^\rho, B^\phi, B^z, P$ be functions only of t, ρ, z .

However, it is more common in fluid mechanics to use a different set of components, called the physical representation, as defined by

$$v^\rho = v^\rho, \quad v^\phi = \rho v^\phi, \quad v^z = v^z, \quad (6.2.6a)$$

$$B^\rho = B^\rho, \quad B^\phi = \rho B^\phi, \quad B^z = B^z. \quad (6.2.6b)$$

Since the invariance equations (6.2.5) only involve ϕ derivatives, they have the same form in terms of the physical representation:

$$\begin{aligned} v^\rho_{,\phi} &= 0, & v^\phi_{,\phi} &= 0, & v^z_{,\phi} &= 0, \\ B^\rho_{,\phi} &= 0, & B^\phi_{,\phi} &= 0, & B^z_{,\phi} &= 0, & P_{,\phi} &= 0. \end{aligned} \quad (6.2.7)$$

Thus, these components $v^\rho, B^\rho, v^\phi, B^\phi, v^z, B^z$, and P also depend only on t, ρ, z .

The MHD system (2.0.3) combined with invariance equations (6.2.7) yields the reduced system

$$\begin{aligned} \partial_t v^\rho &= -v^\rho v^\rho_{,\rho} - v^z v^\rho_{,z} + \rho^{-1} (v^\phi)^2 - P_{,\rho} - B^\phi B^\phi_{,\rho} - B^z B^z_{,\rho} + B^z B^\rho_{,z} \\ &\quad - \rho^{-1} (B^\phi)^2 + a (v^\rho_{,\rho\rho} + \rho^{-1} v^\rho_{,\rho} + v^\rho_{,zz} - \rho^{-2} v^\rho) \end{aligned} \quad (6.2.8a)$$

$$\begin{aligned} \partial_t v^\phi &= -v^\rho v^\phi_{,\rho} - v^z v^\phi_{,z} - \rho^{-1} v^\rho v^\phi + B^\rho B^\phi_{,\rho} + B^z B^\phi_{,z} + \rho^{-1} B^\rho B^\phi \\ &\quad + a (v^\phi_{,\rho\rho} + \rho^{-1} v^\phi_{,\rho} + v^\phi_{,zz} - \rho^{-2} v^\phi), \end{aligned} \quad (6.2.8b)$$

$$\begin{aligned} \partial_t v^z &= -v^\rho v^z_{,\rho} - v^z v^z_{,z} - P_{,z} - B^\phi B^\phi_{,z} + B^\rho B^z_{,\rho} - B^\rho B^\rho_{,z} \\ &\quad + a (v^z_{,\rho\rho} + \rho^{-1} v^z_{,\rho} + v^z_{,zz}), \end{aligned} \quad (6.2.8c)$$

$$\begin{aligned} \partial_t B^\rho &= -v^\rho B^\rho_{,\rho} - v^z B^\rho_{,z} + B^\rho v^\rho_{,\rho} + B^z v^\rho_{,z} \\ &\quad + b (B^\rho_{,\rho\rho} + \rho^{-1} B^\rho_{,\rho} + B^\rho_{,zz} - \rho^{-2} B^\rho), \end{aligned} \quad (6.2.8d)$$

$$\begin{aligned} \partial_t B^\phi &= -v^\rho B^\phi_{,\rho} - v^z B^\phi_{,z} - \rho^{-1} B^\rho v^\phi + B^\rho v^\phi_{,\rho} + B^z v^\phi_{,z} + \rho^{-1} B^\rho B^\phi \\ &\quad + b (B^\phi_{,\rho\rho} + \rho^{-1} B^\phi_{,\rho} + B^\phi_{,zz} - \rho^{-2} B^\phi), \end{aligned} \quad (6.2.8e)$$

$$\partial_t B^z = -v^\rho B^z_{,\rho} - v^z B^z_{,z} + B^\rho v^z_{,\rho} + B^z v^z_{,z} + b (B^z_{,\rho\rho} + \rho^{-1} B^z_{,\rho} + B^z_{,zz}), \quad (6.2.8f)$$

$$v^\rho_{,\rho} + \rho^{-1} v^\rho + v^z_{,z} = 0, \quad (6.2.8g)$$

$$B^\rho_{,\rho} + \rho^{-1} B^\rho + B^z_{,z} = 0, \quad (6.2.8h)$$

$$\begin{aligned} \Delta P &= -2v^\rho_{,z} v^z_{,\rho} + 2\rho^{-1} v^\phi v^\phi_{,\rho} - 2\rho^{-1} v^\rho v^z_{,z} - 2\rho^{-2} (v^\rho)^2 - 2(v^z_{,z})^2 \\ &\quad - (B^\phi_{,\rho})^2 - B^\phi B^\phi_{,\rho\rho} - B^\phi B^\phi_{,zz} - (B^z_{,\rho})^2 + 2B^z_{,\rho} B^\rho_{,z} - (B^\rho_{,z})^2 \\ &\quad - B^z B^z_{,\rho\rho} - B^z B^z_{,zz} - B^\rho B^\rho_{,zz} + B^\rho B^z_{,z\rho} - (B^\phi_{,z})^2 \\ &\quad - 3\rho^{-1} B^\phi B^\phi_{,\rho} - \rho^{-1} B^z B^z_{,\rho}. \end{aligned} \quad (6.2.8i)$$

The solution space $\mathcal{E}_{\text{rot.}}$ of this system (6.2.8) is a subset of the unreduced MHD solution space, $\mathcal{E}_{\text{rot.}} \subset \mathcal{E}$.

A conservation law of the axially rotation-invariant MHD system (6.2.8) is a local continuity equation

$$(D_t T + D_\rho X^\rho + D_z X^z)|_{\mathcal{E}_{\text{rot.}}} = 0 \quad (6.2.9)$$

holding on the rotation-invariant MHD solution space $\mathcal{E}_{\text{rot.}}$, where T, X^ρ, X^z are scalar functions on J , up to some finite order in derivatives. The form of a locally trivial conservation law consists of

$$\begin{aligned} T|_{\mathcal{E}_{\text{rot.}}} &= D_\rho \Psi^\rho + D_z \Psi^z, & X^\rho|_{\mathcal{E}_{\text{rot.}}} &= -D_t \Psi^\rho + D_\rho \Theta, \\ X^z|_{\mathcal{E}_{\text{rot.}}} &= -D_t \Psi^z - D_z \Theta \end{aligned} \quad (6.2.10)$$

holding for some scalar functions $\Psi^\rho, \Psi^z, \Theta$ on J .

To derive the characteristic form for a conservation law (6.2.9), we will coordinatize the axially rotation-invariant MHD solution space $\mathcal{E}_{\text{rot.}} \subset J$ by adapting the leading derivative method used for coordinatizing the unreduced MHD solution space \mathcal{E} . First, we express the divergence equations (6.2.8g) and (6.2.8g) in a solved form for ρ derivatives:

$$v^\rho_{,\rho} = -\rho^{-1} v^\rho - v^z_{,z}, \quad (6.2.11a)$$

$$B^\rho_{,\rho} = -\rho^{-1} B^\rho - B^z_{,z}. \quad (6.2.11b)$$

Next, we substitute both equations (6.2.11a), (6.2.11b) and their differential consequences into the evolution equations (6.2.8a)–(6.2.8f), and we express the resulting equations in solved form for t derivatives:

$$\begin{aligned} \partial_t v^\rho &= v^\rho v^z_{,z} - v^z v^\rho_{,z} + \rho^{-1} (v^\rho)^2 + \rho^{-1} (v^\phi)^2 - P_{,\rho} - B^\phi B^\phi_{,\rho} - B^z B^z_{,\rho} \\ &\quad + B^z B^\rho_{,z} - \rho^{-1} (B^\phi)^2 + a (v^\rho_{,zz} - v^z_{,z\rho}), \end{aligned} \quad (6.2.11c)$$

$$\begin{aligned} \partial_t v^\phi &= -v^\rho v^\phi_{,\rho} - v^z v^\phi_{,z} - \rho^{-1} v^\rho v^\phi + B^\rho B^\phi_{,\rho} + B^z B^\phi_{,z} + \rho^{-1} B^\rho B^\phi \\ &\quad + a (v^\phi_{,\rho\rho} + \rho^{-1} v^\phi_{,\rho} + v^\phi_{,zz} - \rho^{-2} v^\phi), \end{aligned} \quad (6.2.11d)$$

$$\begin{aligned} \partial_t v^z &= -v^\rho v^z_{,\rho} - v^z v^z_{,z} - P_{,z} - B^\phi B^\phi_{,z} + B^\rho B^z_{,\rho} - B^\rho B^\rho_{,z} \\ &\quad + a (v^z_{,\rho\rho} + \rho^{-1} v^z_{,\rho} + v^z_{,zz}), \end{aligned} \quad (6.2.11e)$$

$$\begin{aligned} \partial_t B^\rho &= v^\rho B^z_{,z} + \rho^{-1} v^\rho B^\rho - v^z B^\rho_{,z} + B^\rho v^\rho_{,\rho} + B^z v^\rho_{,z} \\ &\quad + b (B^\rho_{,zz} - B^z_{,z\rho}), \end{aligned} \quad (6.2.11f)$$

$$\begin{aligned} \partial_t B^\phi &= -v^\rho B^\phi_{,\rho} - v^z B^\phi_{,z} - \rho^{-1} B^\rho v^\phi + B^\rho v^\phi_{,\rho} + B^z v^\phi_{,z} + \rho^{-1} v^\rho B^\phi \\ &\quad + b (B^\phi_{,\rho\rho} + \rho^{-1} B^\phi_{,\rho} + B^\phi_{,zz} - \rho^{-2} B^\phi), \end{aligned} \quad (6.2.11g)$$

$$\partial_t B^z = -v^\rho B^z_{,\rho} - v^z B^z_{,z} + B^\rho v^z_{,\rho} + B^z v^z_{,z} + b (B^z_{,\rho\rho} + \rho^{-1} B^z_{,\rho} + B^z_{,zz}). \quad (6.2.11h)$$

Finally, we also substitute both equations (6.2.11a) and (6.2.11b) into the remaining equation (6.2.8i) which we express in a solved form

$$\begin{aligned} P_{,\rho\rho} &= -\rho^{-1} P_{,\rho} - P_{,zz} - 2v^\rho_{,z} v^z_{,\rho} + 2\rho^{-1} v^\phi v^\phi_{,\rho} - 2\rho^{-1} v^\rho v^z_{,z} \\ &\quad - 2\rho^{-2} (v^\rho)^2 - 2(v^z_{,z})^2 - (B^\phi_{,\rho})^2 - B^\phi B^\phi_{,\rho\rho} - B^\phi B^\phi_{,zz} \\ &\quad - (B^z_{,\rho})^2 + 2B^z_{,\rho} B^\rho_{,z} - (B^\rho_{,z})^2 - B^z B^z_{,\rho\rho} - B^z B^z_{,zz} - B^\rho B^\rho_{,zz} \\ &\quad + B^\rho B^z_{,z\rho} - (B^\phi_{,z})^2 - 3\rho^{-1} B^\phi B^\phi_{,\rho} - \rho^{-1} B^z B^z_{,\rho}. \end{aligned} \quad (6.2.11i)$$

Then the coordinates on $\mathcal{E}_{\text{rot.}}$ consist of the z derivatives of v^ρ, B^ρ , and the ρ, z derivatives of $v^\phi, B^\phi, v^z, B^z, P$. Hereafter, we will use the index notation: $i = \rho, \phi, z$.

In this coordinatization, the characteristic form of a conservation law (6.2.9) is given by a divergence identity

$$\begin{aligned} D_t \tilde{T} + D_\rho \tilde{X}^\rho + D_z \tilde{X}^z &= Q_i^v (v^i_t - S_B^i) + Q_i^B (B^i_t - S_B^i) + \tilde{Q}^v (v^\rho_{,\rho} - \tilde{S}_v) \\ &\quad + \tilde{Q}^B (B^\rho_{,\rho} - \tilde{S}_B) + Q^P (P_{,\rho\rho} - S_P) \end{aligned} \quad (6.2.12)$$

with

$$\tilde{T}|_{\mathcal{E}_{\text{rot.}}} = T, \quad \tilde{X}^\rho|_{\mathcal{E}_{\text{rot.}}} = X^\rho, \quad \tilde{X}^z|_{\mathcal{E}_{\text{rot.}}} = X^z \quad (6.2.13)$$

and with $Q_i^v, Q_i^B, \tilde{Q}^v, \tilde{Q}^B, Q^P$ being multiplier functions on J that are non-singular on $\mathcal{E}_{\text{rot.}}$, where $S_v^i, S_B^i, \tilde{S}_v, \tilde{S}_B, S_P$ are the right hand sides of equations (6.2.11c)–(6.2.11h), (6.2.11a)–(6.2.11b), (6.2.11i).

Multiplier functions $(Q_i^v, Q_i^B, \tilde{Q}^v, \tilde{Q}^B, Q^P)$ are determined by the condition that the Euler operators (2.0.16) with respect to v^i, B^i, P annihilate the right hand side of equation (6.2.12). This yields the multiplier determining system (2.0.17). Each solution of this system determines a corresponding conservation law (6.2.9) through equation (6.2.12).

We will classify all conservation laws of kinematic form for the axially rotation-invariant MHD system (6.2.8). The general form for a kinematic conserved density is given by

$$T(t, \rho, z, v^i, B^i, P). \quad (6.2.14a)$$

The corresponding general form for a kinematic spatial flux can be determined

by the same method used for the unreduced MHD system (2.0.3), which leads to

$$X^\rho(t, \rho, z, v^i, B^i, P, v^\phi_{,\rho}, v^z_{,\rho}, v^z_{,z}, B^\phi_{,\rho}, B^z_{,\rho}, B^z_{,z}), \quad (6.2.14b)$$

$$X^z(t, \rho, z, v^i, B^i, P, v^\phi_{,\rho}, v^z_{,\rho}, v^i_{,z}, B^\phi_{,\rho}, B^z_{,\rho}, B^i_{,z}). \quad (6.2.14c)$$

Then, from the general expressions (3.0.5) relating the form of the conserved density and the flux components to the form of the multiplier functions, we have

$$Q_i^v(t, \rho, z, v^j, B^j, P) = \partial_{v^i} T, \quad (6.2.15a)$$

$$Q_i^B(t, \rho, z, v^j, B^j, P) = \partial_{B^i} T, \quad (6.2.15b)$$

$$\tilde{Q}^v(t, \rho, z, v^j, B^j, P, v^\phi_{,\rho}, v^z_{,\rho}, v^z_{,z}, B^\phi_{,\rho}, B^z_{,\rho}, B^z_{,z}) = \partial_{v^\rho} X^\rho, \quad (6.2.15c)$$

$$\tilde{Q}^B(t, \rho, z, v^j, B^j, P, v^\phi_{,\rho}, v^z_{,\rho}, v^z_{,z}, B^\phi_{,\rho}, B^z_{,\rho}, B^z_{,z}) = \partial_{B^\rho} X^\rho, \quad (6.2.15d)$$

$$Q^P = 0. \quad (6.2.15e)$$

where \tilde{Q}^v, \tilde{Q}^B are linear in derivative variables.

Using computer algebra, as explained in appendix .3, we set up and solve the multiplier determining system (2.0.17) to find all multiplier functions (6.2.15a), and then we obtain the resulting conserved densities and spatial fluxes (6.2.14).

This yields the following classification result.

Theorem 4 *All kinematic conservation laws (6.2.14) of the axially rotation-invariant MHD system (6.2.8) are linear combination of:*

(1) *axial angular momentum*

$$\begin{aligned} T &= \rho^2 v^\phi \\ X^\rho &= \rho^2 (v^\phi v^\rho - B^\phi B^\rho - a (v^\phi_{,\rho} - \rho^{-1} v^\phi)) \\ X^z &= \rho^2 (v^\phi v^z - B^\phi B^z - a v^\phi_{,z}) \end{aligned} \quad (6.2.16a)$$

(2) *axial generalized momentum*

$$\begin{aligned} T[\beta(t)] &= \beta \rho v^z \\ X^\rho[\beta(t)] &= \rho (-z v^\rho \partial_t \beta + \beta (v^\rho v^z - B^\rho B^z - a v^z_{,\rho} + a v^\rho_{,z})) \\ X^z[\beta(t)] &= \rho (-z v^z \partial_t \beta + \beta (v^z v^z - B^z B^z \\ &\quad + P + \frac{1}{2} (B^\rho B^\rho + B^\phi B^\phi + B^z B^z))) \end{aligned} \quad (6.2.16b)$$

(3) poloidal generalized magnetic fluxes

$$\begin{aligned}
T[\chi(t, \rho, z)] &= \rho(B^\rho \chi_{,\rho} + B^z \chi_{,z}) \\
X^\rho[\chi(t, \rho, z)] &= \rho(-B^\rho \partial_t \chi + (B^z v^\rho - B^\rho v^z) \chi_{,z} \\
&\quad + b(B^z \chi_{,\rho z} - B^\rho \chi_{,zz} - (B^\rho_{,\rho} + \rho^{-1} B^\rho) \chi_{,\rho} - B^z_{,\rho} \chi_{,z})) \quad (6.2.16c) \\
X^z[\chi(t, \rho, z)] &= \rho(-B^z \partial_t \chi - (B^z v^\rho - B^\rho v^z) \chi_{,\rho} \\
&\quad + b(B^\rho \chi_{,\rho z} - B^z \chi_{,\rho \rho} - (B^\rho_{,z} + \rho^{-1} B^z) \chi_{,\rho} - B^z_{,z} \chi_{,z}))
\end{aligned}$$

(4) circular magnetic flux

$$\begin{aligned}
T &= B^\phi \\
X^\rho &= B^\phi v^\rho - B^\rho v^\phi - b(B^\phi_{,\rho} + \rho^{-1} B^\phi) \quad (6.2.16d) \\
X^z &= B^\phi v^z - B^z v^\phi - b B^\phi_{,z}
\end{aligned}$$

(5) incompressibility flux

$$\begin{aligned}
T[\gamma(t)] &= 0 \\
X^\rho[\gamma(t)] &= \gamma \rho v^\rho \\
X^z[\gamma(t)] &= \gamma \rho v^z \quad (6.2.16e)
\end{aligned}$$

(6) cross-helicity

$$\begin{aligned}
T &= \rho(v^\rho B^\rho + v^\phi B^\phi + v^z B^z) \\
X^\rho &= \rho(B^\rho (P - \frac{1}{2}(v^\rho v^\rho + v^\phi v^\phi + v^z v^z)) \\
&\quad + v^\rho(v^\rho B^\rho + v^\phi B^\phi + v^z B^z) + a(v^\rho(B^\rho_{,\rho} + B^z_{,\phi}) \\
&\quad + v^\phi B^\phi_{,\rho} + v^z(B^z_{,\rho} - B^\rho_{,z}) - B^\rho(v^\rho_{,\rho} + v^z_{,z}) - B^\phi v^\phi_{,\rho} \\
&\quad - B^z(v^z_{,\rho} - v^\rho_{,z}))) \quad (6.2.16f) \\
X^z &= \rho(B^z (P - \frac{1}{2}(v^\rho v^\rho + v^\phi v^\phi + v^z v^z)) \\
&\quad + v^z(v^\rho B^\rho + v^\phi B^\phi + v^z B^z) + a(v^\rho(B^\rho_{,z} - B^z_{,\rho}) \\
&\quad + v^\phi B^\phi_{,z} - B^\rho(v^\rho_{,z} - v^z_{,\rho}) - B^\phi v^\phi_{,z})) \\
0 &= a + b
\end{aligned}$$

(7) energy

$$\begin{aligned}
T &= \frac{1}{2}\rho(v^\rho v^\rho + v^\phi v^\phi + v^z v^z + B^\rho B^\rho + B^\phi B^\phi + B^z B^z) \\
X^\rho &= \rho\left(v^\rho\left(P + \frac{1}{2}(v^\rho v^\rho + v^\phi v^\phi + v^z v^z) + B^\rho B^\rho + B^\phi B^\phi + B^z B^z\right) \right. \\
&\quad \left. - B^\rho(v^\rho B^\rho + v^\phi B^\phi + v^z B^z)\right) \\
X^z &= \rho\left(v^z\left(P + \frac{1}{2}(v^\rho v^\rho + v^\phi v^\phi + v^z v^z) + B^\rho B^\rho + B^\phi B^\phi + B^z B^z\right) \right. \\
&\quad \left. - B^z(v^\rho B^\rho + v^\phi B^\phi + v^z B^z)\right) \\
0 &= b = a
\end{aligned} \tag{6.2.16g}$$

where $\beta(t)$ and $\gamma(t)$ are arbitrary functions of t , and $\chi(t, \rho, z)$ is an arbitrary function of t, ρ, z .

The conservation laws (6.2.16a) – (6.2.16g) are inherited from the unreduced MHD system. They are widely used in applications involving physical problems [10, 21] that have an axial rotation invariance. However, a complete classification of these conservation laws has not previously appeared in the literature.

A point symmetry of the axially rotation-invariant MHD system (6.2.8) is an infinitesimal generator of the form (4.0.1) under which the solution space \mathcal{E}_{rot} is mapped into itself. We classify all point symmetries by using computer algebra to set up and solve the symmetry determining system (4.0.2) for the axially rotation-invariant MHD system in the form (6.2.11a)–(6.2.11i).

This computation is explained in appendix .7, which yields the following result.

Proposition 4 *The point symmetries of the axially rotation-invariant MHD system (6.2.8) are generated by:*

(1) time translation

$$X_{\text{trans.}} = \partial/\partial t \tag{6.2.17a}$$

(2) axial generalized Galilean boosts

$$X[v(t)] = v\partial/\partial z + (\partial_t v)\partial/\partial v^z - (\partial_t^2 v)z\partial/\partial P \tag{6.2.17b}$$

(3) pressure shift

$$X[\sigma(t)] = \sigma\partial/\partial P \tag{6.2.17c}$$

(4) *scaling*

$$X_{\text{scal.}} = 2t\partial/\partial t + \rho\partial/\partial\rho + z\partial/\partial z - v^\rho\partial/\partial v^\rho - v^\phi\partial/\partial v^\phi \\ - v^z\partial/\partial v^z - B^\rho\partial/\partial B^\rho - B^\phi\partial/\partial B^\phi - B^z\partial/\partial B^z - 2P\partial/\partial P \quad (6.2.17d)$$

(5) *dilation*

$$X_{\text{dil.}} = t\partial/\partial t + \rho\partial/\partial\rho + z\partial/\partial z, \quad a = b = 0 \quad (6.2.17e)$$

where $v(t)$ and $\sigma(t)$ are functions of t .

The point symmetries (6.2.17) are inherited from the unreduced MHD system. A complete classification of these symmetries has not previously appeared in the literature.

6.3 Helical reduction

Helical (screw) motion is represented by Killing vectors $\xi_{\text{hel.}}^i = v^i + m\epsilon^i{}_{jk}\mu^j x^k$, where v^i and μ^i are constant parallel vectors and $m \neq 0$ is a constant. Without loss of generality, we can choose $\mu^j = \delta^{j3}$ and $v^i = \delta^{i3}$, so then $\xi_{\text{hel.}}^i$ geometrically represents a helical motion in the x^3 direction. The invariance condition (6.0.2) on the MHD variables v^i, B^i, P is given by

$$v^1{}_{,3} + mx^1 v^1{}_{,2} - mx^2 v^1{}_{,1} + mv^2 = 0, \quad (6.3.1a)$$

$$v^2{}_{,3} + mx^1 v^2{}_{,2} - mx^2 v^2{}_{,1} - mv^1 = 0, \quad (6.3.1b)$$

$$v^3{}_{,3} + mx^1 v^3{}_{,2} - mx^2 v^3{}_{,1} = 0, \quad (6.3.1c)$$

$$B^1{}_{,3} + mx^1 B^1{}_{,2} - mx^2 B^1{}_{,1} + mB^2 = 0, \quad (6.3.1d)$$

$$B^2{}_{,3} + mx^1 B^2{}_{,2} - mx^2 B^2{}_{,1} - mB^1 = 0, \quad (6.3.1e)$$

$$B^3{}_{,3} + mx^1 B^3{}_{,2} - mx^2 B^3{}_{,1} = 0, \quad (6.3.1f)$$

$$P_{,3} + mx^1 P_{,2} - mx^2 P_{,1} = 0. \quad (6.3.1g)$$

These equations have a simpler form in helical coordinates (ρ, ψ, z) adapted to the screw axis:

$$x^1 = \rho \cos(\psi + mz), \quad x^2 = \rho \sin(\psi + mz), \quad x^3 = z \quad (6.3.2)$$

with

$$\partial/\partial x^3 + mx^1 \partial/\partial x^2 - mx^2 \partial/\partial x^1 = \partial_z. \quad (6.3.3)$$

Note the constant m is inversely proportional to the pitch of helix. If m is positive then a clockwise screwing motion moves the helix in the positive direction along z axis, and conversely, if m is negative then a clockwise screwing motion moves the helix in the negative direction along z axis. Also, if m is zero then the helical coordinate system reduces to the cylindrical coordinate system. In these coordinates [90, 91], the corresponding helical components of the velocity v^ρ, v^ψ, v^z and the magnetic field B^ρ, B^ψ, B^z are given in terms of the standard Cartesian components by the transformations

$$v^1 = v^\rho \cos(\psi + mz) - v^\psi \rho \sin(\psi + mz), \quad (6.3.4a)$$

$$v^2 = v^\rho \sin(\psi + mz) + v^\psi \rho \cos(\psi + mz), \quad (6.3.4b)$$

$$v^3 = mv^\psi + v^z, \quad (6.3.4c)$$

$$B^1 = B^\rho \cos(\psi + mz) - B^\psi \rho \sin(\psi + mz), \quad (6.3.4d)$$

$$B^2 = B^\rho \sin(\psi + mz) + B^\psi \rho \cos(\psi + mz), \quad (6.3.4e)$$

$$B^3 = mB^\psi + B^z. \quad (6.3.4f)$$

The invariance condition (6.3.1) becomes simply

$$\begin{aligned} v^{\rho, z} = 0, \quad v^{\psi, z} = 0, \quad v^{z, z} = 0, \\ B^{\rho, z} = 0, \quad B^{\psi, z} = 0, \quad B^{z, z} = 0, \quad P_{,z} = 0. \end{aligned} \quad (6.3.5)$$

This is equivalent to having $v^\rho, v^\psi, v^z, B^\rho, B^\psi, B^z, P$ be functions only of t, ρ, ψ .

In fluid mechanics, it is more common to use a different set of components, called the physical representation, which we extend to the MHD system by defining

$$v^\rho = \mathbf{v}^\rho, \quad v^\psi = \Gamma^{-1} \mathbf{v}^\psi, \quad v^z = \mathbf{v}^z, \quad (6.3.6a)$$

$$B^\rho = \mathbf{B}^\rho, \quad B^\psi = \Gamma^{-1} \mathbf{B}^\psi, \quad B^z = \mathbf{B}^z, \quad (6.3.6b)$$

where $\Gamma = 1/\sqrt{\rho^2 + m^2}$. Since the invariance equations (6.3.5) only involve z derivatives, they have the same form in terms of the physical representation:

$$\begin{aligned} v^{\rho, z} = 0, \quad v^{\psi, z} = 0, \quad v^{z, z} = 0, \\ B^{\rho, z} = 0, \quad B^{\psi, z} = 0, \quad B^{z, z} = 0, \quad P_{,z} = 0, \end{aligned} \quad (6.3.7)$$

where these components depend only on t, ρ, ψ .

The MHD system (2.0.3) combined with invariance equations (6.3.7) yields the reduced system

$$\begin{aligned} \partial_t v^\rho = -v^\rho v^{\rho, \rho} - \Gamma v^\psi v^{\rho, \psi} + \rho \Gamma^2 (v^\psi)^2 - P_{, \rho} - B^\psi B^{\psi, \rho} - B^z B^{z, \rho} \\ + \Gamma B^\psi B^{\rho, \psi} - m \Gamma B^\psi B^{z, \rho} - m \Gamma B^z B^{\psi, \rho} - \rho \Gamma^2 (B^\psi)^2 + m \rho \Gamma^3 B^\psi B^z \\ + a(v^{\rho, \rho \rho} + \rho^{-1} v^{\rho, \rho} + \rho^{-2} v^{\rho, \psi \psi} - \rho^{-2} v^\rho - 2 \Gamma \rho^{-1} v^{\psi, \psi}), \end{aligned} \quad (6.3.8a)$$

$$\begin{aligned} \partial_t v^\psi = -v^\rho v^{\psi, \rho} - \Gamma v^\psi v^{\psi, \psi} - (2\rho^{-1} - \rho \Gamma^2) v^\psi v^\rho - \rho^{-2} \Gamma^{-1} P_{, \psi} + B^\rho (B^{\psi, \rho} \\ - B^{\rho, \psi} + \rho^{-1} (1 + m^2 \Gamma^2) B^\psi) - \rho^{-2} \Gamma^{-1} (m \Gamma B^\psi + B^z) (B^{z, \psi} \\ + m \Gamma B^{\psi, \psi}) + a(v^{\psi, \rho \rho} + \rho^{-1} v^{\psi, \rho} + \rho^{-2} v^{\psi, \psi \psi} + (3\rho^2 \Gamma^4 - 4\Gamma^2) v^\psi \\ + 2\rho^{-3} \Gamma^{-1} v^{\rho, \psi} + (2\rho^{-1} - 2\rho \Gamma^2) v^{\psi, \rho}), \end{aligned} \quad (6.3.8b)$$

$$\begin{aligned}
\partial_t v^z = & -v^\rho v^z_{,\rho} - \Gamma v^\psi v^z_{,\psi} + 2m\Gamma\rho^{-1}v^\rho v^\psi + m\rho^{-2}P_{,\psi} + B^\rho(m\rho^{-2}B^\rho_{,\psi} \\
& + B^z_{,\rho} - 2m\Gamma\rho^{-1}B^\psi) + \rho^{-2}\Gamma^{-1}(B^\psi + m\Gamma B^z)(B^z_{,\psi} + m\Gamma B^\psi_{,\psi}) \\
& + a(v^z_{,\rho\rho} + \rho^{-1}v^z_{,\rho} + \rho^{-2}v^z_{,\psi\psi} + 2m\Gamma^3v^\psi - 2m\Gamma\rho^{-1}v^\psi_{,\rho} \\
& - 2m\rho^{-3}v^\rho_{,\psi}),
\end{aligned} \tag{6.3.8c}$$

$$\begin{aligned}
\partial_t B^\rho = & -v^\rho B^\rho_{,\rho} - \Gamma v^\psi B^\rho_{,\psi} + B^\rho v^\rho_{,\rho} + \Gamma B^\psi v^\rho_{,\psi} \\
& + b(B^\rho_{,\rho\rho} + \rho^{-1}B^\rho_{,\rho} + \rho^{-2}B^\rho_{,\psi\psi} - \rho^{-2}B^\rho - 2\Gamma\rho^{-1}B^\psi_{,\psi}),
\end{aligned} \tag{6.3.8d}$$

$$\begin{aligned}
\partial_t B^\psi = & -v^\rho B^\psi_{,\rho} - \Gamma v^\psi B^\psi_{,\psi} + \rho\Gamma^2 B^\psi v^\rho + B^\rho v^\psi_{,\rho} + \Gamma B^\psi v^\psi_{,\psi} - \rho\Gamma^2 v^\psi B^\rho \\
& + b(B^\psi_{,\rho\rho} + \rho^{-1}B^\psi_{,\rho} + \rho^{-2}B^\psi_{,\psi\psi} + (3\rho^2\Gamma^4 - 4\Gamma^2)B^\psi \\
& + 2\rho^{-3}\Gamma^{-1}B^\rho_{,\psi} + (2\rho^{-1} - 2\rho\Gamma^2)B^\psi_{,\rho}),
\end{aligned} \tag{6.3.8e}$$

$$\begin{aligned}
\partial_t B^z = & -v^\rho B^z_{,\rho} - \Gamma v^\psi B^z_{,\psi} + B^\rho v^z_{,\rho} + \Gamma B^\psi v^z_{,\psi} + b(B^z_{,\rho\rho} + \rho^{-1}B^z_{,\rho} \\
& + \rho^{-2}B^z_{,\psi\psi} + 2m\Gamma^3 B^\psi - 2m\Gamma\rho^{-1}B^\psi_{,\rho} - 2m\rho^{-3}B^\rho_{,\psi}),
\end{aligned} \tag{6.3.8f}$$

$$v^\rho_{,\rho} + \rho^{-1}v^\rho + \Gamma v^\psi_{,\psi} = 0, \tag{6.3.8g}$$

$$B^\rho_{,\rho} + \rho^{-1}B^\rho + \Gamma B^\psi_{,\psi} = 0, \tag{6.3.8h}$$

$$\begin{aligned}
\Delta P = & -(B^\psi + m\Gamma B^z)\rho^{-2}B^\psi_{,\psi\psi} - (B^\psi - m\Gamma B^z)B^\psi_{,\rho\rho} - \rho^{-2}B^\rho B^\rho_{,\psi\psi} \\
& - (m\Gamma B^\psi + B^z)\rho^{-2}B^z_{,\psi\psi} + \Gamma B^\rho B^\psi_{,\rho\psi} + m\Gamma B^\psi B^z_{,\rho\rho} - (B^\psi_{,\rho})^2 \\
& - \rho^{-2}(B^\rho_{,\psi})^2 - \rho^{-2}(B^z_{,\psi})^2 + 2m^2\Gamma^3\rho^{-1}B^\psi B^\rho_{,\psi} + 2(m\Gamma B^z_{,\rho} \\
& + \Gamma B^\rho_{,\psi} - \Gamma^2\rho B^\psi + \frac{1}{2}m\Gamma\rho^{-1}B^z - \frac{1}{2}\rho^{-1}B^\psi)B^\psi_{,\rho} - 2m^2\Gamma^4(B^\psi)^2 \\
& - m^2\Gamma^2\rho^{-2}(B^\psi_{,\psi})^2 + m\Gamma\rho^{-1}B^\psi B^z_{,\rho} + (3m^2\Gamma^2 - 1)B^\psi B^z \\
& - 2m\Gamma\rho^{-2}B^\psi_{,\psi}B^z_{,\psi} + (1 + m^2G^2)\Gamma\rho^{-1}B^\rho B^\psi_{,\psi} - 2\Gamma v^\rho_{,\psi}v^\psi_{,\rho} \\
& - 2m^2\Gamma^3\rho^{-1}v^\psi v^\rho_{,\psi} + 2\Gamma^2\rho v^\psi v^\psi_{,\rho} - 2\Gamma^2(v^\psi_{,\psi})^2 - 4\Gamma\rho^{-1}v^\rho v^\psi_{,\psi} \\
& + 2m^2\Gamma^4(v^\psi)^2 - 2\rho^{-2}(v^\rho)^2.
\end{aligned} \tag{6.3.8i}$$

The solution space $\mathcal{E}_{\text{hel.}}$ of this helical-invariant system (6.3.8) is a subset of the unreduced MHD solution space, $\mathcal{E}_{\text{hel.}} \subset \mathcal{E}$. Despite its obvious physical importance, this reduction has not been previously studied in the literature on MHD systems.

A conservation law of the helical-invariant MHD system (6.3.8) is a local continuity equation

$$(D_t T + D_\rho X^\rho + D_\psi X^\psi)|_{\mathcal{E}_{\text{hel.}}} = 0 \tag{6.3.9}$$

holding on the helical transformation invariant MHD solution space $\mathcal{E}_{\text{hel.}}$, where T, X^ρ, X^ψ are scalar functions on J , up to some finite order in derivatives. The

form of a locally trivial conservation law consists of

$$\begin{aligned} T|_{\mathcal{E}_{\text{hel.}}} &= D_\rho \Psi^\rho + D_\psi \Psi^\psi, \\ X^\rho|_{\mathcal{E}_{\text{hel.}}} &= -D_t \Psi^\rho + D_\psi \Theta, \quad X^\psi|_{\mathcal{E}_{\text{hel.}}} = -D_t \Psi^\psi - D_\rho \Theta \end{aligned} \quad (6.3.10)$$

holding for some scalar functions $\Psi^\rho, \Psi^\psi, \Theta$ on J .

To derive the characteristic form for a conservation law (6.3.9), we will coordinatize the helical transformation invariant MHD solution space $\mathcal{E}_{\text{hel.}} \subset J$ by using the same leading derivatives considered for coordinatizing the unreduced MHD solution space \mathcal{E} . First, we express the divergence equations (6.3.8g) and (6.3.8h) in a solved form for ρ derivatives:

$$v^\rho_{,\rho} = -\rho^{-1} v^\rho - \Gamma v^\psi_{,\psi}, \quad (6.3.11a)$$

$$B^\rho_{,\rho} = -\rho^{-1} B^\rho - \Gamma B^\psi_{,\psi}. \quad (6.3.11b)$$

Next, we substitute both equations (6.3.11a), (6.3.11b) and their differential consequences into the evolution equations (6.3.8a)–(6.3.8f), and we express the resulting equations in solved form for t derivatives:

$$\begin{aligned} \partial_t v^\rho &= \Gamma v^\rho v^\psi_{,\psi} + \rho^{-1} (v^\rho)^2 - \Gamma v^\psi v^\rho_{,\psi} + \rho \Gamma^2 (v^\psi)^2 - P_{,\rho} - B^\psi B^\psi_{,\rho} \\ &\quad - B^z B^z_{,\rho} + \Gamma B^\psi B^\rho_{,\psi} - m \Gamma B^\psi B^z_{,\rho} - m \Gamma B^z B^\psi_{,\rho} + m \rho \Gamma^3 B^\psi B^z \\ &\quad - \rho \Gamma^2 (B^\psi)^2 + a (\rho^{-2} v^\rho_{,\psi\psi} - \Gamma v^\psi_{,\psi\rho} - (\rho^{-1} \Gamma \\ &\quad + 2\rho^{-1} m^2 \Gamma^3) v^\psi_{,\psi}), \end{aligned} \quad (6.3.11c)$$

$$\begin{aligned} \partial_t v^\psi &= -v^\rho v^\psi_{,\rho} - \Gamma v^\psi v^\psi_{,\psi} - (2\rho^{-1} - \rho \Gamma^2) v^\psi v^\rho - \rho^{-2} \Gamma^{-1} P_{,\psi} + B^\rho (B^\psi_{,\rho} \\ &\quad - B^\rho_{,\psi} + \rho^{-1} (1 + m^2 \Gamma^2) B^\psi) - \rho^{-2} \Gamma^{-1} (m \Gamma B^\psi + B^z) (B^z_{,\psi} \\ &\quad + m \Gamma B^\psi_{,\psi}) + a (v^\psi_{,\rho\rho} + \rho^{-1} v^\psi_{,\rho} + \rho^{-2} v^\psi_{,\psi\psi} + (3\rho^2 \Gamma^4 \\ &\quad - 4\Gamma^2) v^\psi + 2\rho^{-3} \Gamma^{-1} v^\rho_{,\psi}), \end{aligned} \quad (6.3.11d)$$

$$\begin{aligned} \partial_t v^z &= -v^\rho v^z_{,\rho} - \Gamma v^\psi v^z_{,\psi} + 2m \Gamma \rho^{-1} v^\rho v^\psi + m \rho^{-2} P_{,\psi} + B^\rho (m \rho^{-2} B^\rho_{,\psi} \\ &\quad + B^z_{,\rho} - 2m \Gamma \rho^{-1} B^\psi) + \rho^{-2} \Gamma^{-1} (B^\psi + m \Gamma B^z) (B^z_{,\psi} + m \Gamma B^\psi_{,\psi}) \\ &\quad + a (v^z_{,\rho\rho} + \rho^{-1} v^z_{,\rho} + \rho^{-2} v^z_{,\psi\psi} + 2m \Gamma^3 v^\psi - 2m \Gamma \rho^{-1} v^\psi_{,\rho} \\ &\quad - 2m \rho^{-3} v^\rho_{,\psi}), \end{aligned} \quad (6.3.11e)$$

$$\begin{aligned} \partial_t B^\rho &= \Gamma v^\rho B^\psi_{,\psi} + \rho^{-1} v^\rho B^\psi - \Gamma v^\psi B^\rho_{,\psi} + B^\rho v^\rho_{,\rho} + \Gamma B^\psi v^\rho_{,\psi} \\ &\quad + b (\rho^{-2} B^\rho_{,\psi\psi} - \Gamma B^\psi_{,\psi\rho} - (\rho^{-1} \Gamma + 2\rho^{-1} m^2 \Gamma^3) B^\psi_{,\psi}), \end{aligned} \quad (6.3.11f)$$

$$\begin{aligned} \partial_t B^\psi &= -v^\rho B^\psi_{,\rho} - \Gamma v^\psi B^\psi_{,\psi} + \rho \Gamma^2 B^\psi v^\rho + B^\rho v^\psi_{,\rho} + \Gamma B^\psi v^\psi_{,\psi} - \rho \Gamma^2 v^\psi B^\rho \\ &\quad + b(B^\psi_{,\rho\rho} + \rho^{-1} B^\psi_{,\rho} + \rho^{-2} B^\psi_{,\psi\psi} + (3\rho^2 \Gamma^4 - 4\Gamma^2) B^\psi \\ &\quad + 2\rho^{-3} \Gamma^{-1} B^\rho_{,\psi} + (2\rho^{-1} - 2\rho \Gamma^2) B^\psi_{,\rho}), \end{aligned} \quad (6.3.11g)$$

$$\begin{aligned} \partial_t B^z &= -v^\rho B^z_{,\rho} - \Gamma v^\psi B^z_{,\psi} + B^\rho v^z_{,\rho} + \Gamma B^\psi v^z_{,\psi} + b(B^z_{,\rho\rho} + \rho^{-1} B^z_{,\rho} \\ &\quad + \rho^{-2} B^z_{,\psi\psi} + 2m\Gamma^3 B^\psi - 2m\Gamma\rho^{-1} B^\psi_{,\rho} - 2m\rho^{-3} B^\rho_{,\psi}). \end{aligned} \quad (6.3.11h)$$

Finally, we also substitute both equations (6.3.11a) and (6.3.11b) into the remaining equation (6.3.8i) which we express in a solved form

$$\begin{aligned} P_{,\rho\rho} &= -\rho^{-1} P_{,\rho} - \rho^{-2} P_{,\psi\psi} - (B^\psi + m\Gamma B^z) \rho^{-2} B^\psi_{,\psi\psi} - \rho^{-2} B^\rho B^\rho_{,\psi\psi} \\ &\quad - (B^\psi - m\Gamma B^z) B^\psi_{,\rho\rho} - (m\Gamma B^\psi + B^z) \rho^{-2} B^z_{,\psi\psi} + \Gamma B^\rho B^\psi_{,\rho\psi} \\ &\quad + m\Gamma B^\psi B^z_{,\rho\rho} - (B^\psi_{,\rho})^2 - \rho^{-2} (B^\rho_{,\psi})^2 - \rho^{-2} (B^z_{,\psi})^2 \\ &\quad + 2(m\Gamma B^z_{,\rho} + \Gamma B^\rho_{,\psi} - \Gamma^2 \rho B^\psi + \frac{1}{2} m\Gamma \rho^{-1} B^z - \frac{1}{2} \rho^{-1} B^\psi) B^\psi_{,\rho} \\ &\quad + 2m^2 \Gamma^3 \rho^{-1} B^\psi B^\rho_{,\psi} - m^2 \Gamma^2 \rho^{-2} (B^\psi_{,\psi})^2 + m\Gamma \rho^{-1} B^\psi B^z_{,\rho} \\ &\quad - 2m^2 \Gamma^4 (B^\psi)^2 - 2m\Gamma \rho^{-2} B^\psi_{,\psi} B^z_{,\psi} + (3m^2 \Gamma^2 - 1) B^\psi B^z \\ &\quad + (1 + m^2 \Gamma^2) \Gamma \rho^{-1} B^\rho B^\psi_{,\psi} - 2\Gamma v^\rho_{,\psi} v^\psi_{,\rho} - 2m^2 \Gamma^3 \rho^{-1} v^\psi v^\rho_{,\psi} \\ &\quad + 2\Gamma^2 \rho v^\psi v^\rho_{,\psi} - 2\Gamma^2 (v^\psi_{,\psi})^2 - 4\Gamma \rho^{-1} v^\rho v^\psi_{,\psi} + 2m^2 \Gamma^4 (v^\psi)^2 \\ &\quad - 2\rho^{-2} (v^\rho)^2. \end{aligned} \quad (6.3.11i)$$

Then the coordinates on \mathcal{E}_{hel} consist of the ψ derivatives of v^ρ, B^ρ , and the ρ, ψ derivatives of $v^\psi, B^\psi, v^z, B^z, P$.

In this coordinatization, the characteristic form of a conservation law (6.3.9) is given by a divergence identity

$$\begin{aligned} D_t \tilde{T} + D_\rho \tilde{X}^\rho + D_\psi \tilde{X}^\psi &= Q_i^v (v^i_t - S_B^i) + Q_i^B (B^i_t - S_B^i) + \tilde{Q}^v (v^\rho_{,\rho} - \tilde{S}^v) \\ &\quad + \tilde{Q}^B (B^\rho_{,\rho} - \tilde{S}^B) + Q^P (P_{,\rho\rho} - S^P) \end{aligned} \quad (6.3.12)$$

with

$$\tilde{T}|_{\mathcal{E}_{\text{hel}}} = T, \quad \tilde{X}^\rho|_{\mathcal{E}_{\text{hel}}} = X^\rho, \quad \tilde{X}^\psi|_{\mathcal{E}_{\text{hel}}} = X^\psi \quad (6.3.13)$$

and with $Q_i^v, Q_i^B, \tilde{Q}^v, \tilde{Q}^B, Q^P$ being multiplier functions on J that are non-singular on \mathcal{E}_{hel} , where $S_v^i, S_B^i, \tilde{S}^v, \tilde{S}^B, S^P$ are the right hand sides of equations (6.3.11c)–(6.3.11h), (6.3.11a)–(6.3.11b), (6.3.11i). Hereafter, we will use the index notation: $i = \rho, \psi, z$.

Multiplier functions $(Q_i^v, Q_i^B, \tilde{Q}^v, \tilde{Q}^B, Q^P)$ are determined by the condition

that the Euler operators (2.0.16) with respect to v^i, B^i, P annihilate the right hand side of equation (6.3.12). This yields the multiplier determining system (2.0.17). Each solution of this system determines a corresponding conservation law (6.3.9) through equation (6.3.12).

We will classify all conservation laws of kinematic form for the helical transformation invariant MHD system (6.3.8). The general form for a kinematic conserved density is given by

$$T(t, \rho, \psi, v^i, B^i, P). \quad (6.3.14a)$$

The corresponding general form for a kinematic spatial flux can be determined by the same method used for the unreduced MHD system (2.0.3), which leads to

$$X^\rho(t, \rho, \psi, v^i, B^i, P, v^\psi_{,\rho}, v^z_{,\rho}, v^\psi_{,\psi}, B^\psi_{,\rho}, B^z_{,\rho}, B^\psi_{,\psi}), \quad (6.3.14b)$$

$$X^\psi(t, \rho, \psi, v^i, B^i, P, v^\psi_{,\rho}, v^z_{,\rho}, v^j_{,\psi}, B^\psi_{,\rho}, B^z_{,\rho}, B^j_{,\psi}). \quad (6.3.14c)$$

Then, from the general expressions (3.0.5) relating the form of the conserved density and the flux components to the form of the multiplier functions, we have

$$Q_i^v(t, \rho, \psi, v^j, B^j, P) = \partial_{v^i} T, \quad (6.3.15a)$$

$$Q_i^B(t, \rho, \psi, v^j, B^j, P) = \partial_{B^i} T, \quad (6.3.15b)$$

$$\tilde{Q}^v(t, \rho, \psi, v^j, B^j, P, v^\psi_{,\rho}, v^z_{,\rho}, v^\psi_{,\psi}, B^\psi_{,\rho}, B^z_{,\rho}, B^\psi_{,\psi}) = \partial_{v^\rho} X^\rho, \quad (6.3.15c)$$

$$\tilde{Q}^B(t, \rho, \psi, v^j, B^j, P, v^\psi_{,\rho}, v^z_{,\rho}, v^\psi_{,\psi}, B^\psi_{,\rho}, B^z_{,\rho}, B^\psi_{,\psi}) = \partial_{B^\rho} X^\rho, \quad (6.3.15d)$$

$$Q^P = 0. \quad (6.3.15e)$$

Using computer algebra, we set up and solve the multiplier determining system (2.0.17) to find all multiplier functions (6.3.15a), and then we obtain the resulting conserved densities and spatial fluxes (6.3.14). As explained in appendix .4, the case $a = b = 0$ is computationally very hard and will be left for future work. Also, the case $m = 0$ corresponds to the translation-invariant reduction for which a classification is given in Theorem 3.

This leads to the following classification result.

Theorem 1 *All kinematic conservation laws (3.0.1), (3.0.2), (3.0.3) of MHD system (6.3.8) for $m \neq 0$ are linear combination of:*

(1) *helical angular momentum*

$$\begin{aligned}
T &= \rho^3 \Gamma v^\psi \\
X^\rho &= \rho^3 \Gamma (v^\rho v^\psi - B^\rho B^\psi + \frac{a}{2} (\Gamma^{-1} \rho^{-2} v^{\rho, \psi} - 2 \rho^{-2} v^{\psi, \rho} + 2 \Gamma^2 \rho v^\psi)) \\
X^\psi &= \rho^3 \Gamma (\Gamma^{-1} \rho^{-2} (P + \frac{1}{2} B^\rho B^\rho + \frac{1}{2} B^\psi B^\psi + \frac{1}{2} B^z B^z) \\
&\quad + \Gamma (v^\psi v^\psi + B^\psi B^\psi) + \rho^{-2} B^\psi B^z + \frac{a}{2} (\Gamma^{-1} \rho^{-2} v^{\rho, \rho} - 3 \Gamma^{-1} \rho^{-3} v^\rho))
\end{aligned} \tag{6.3.16a}$$

(2) *1st-component of generalized momentum*

$$\begin{aligned}
T[\beta_1(t)] &= \rho \beta_1 (v^\rho \sin \psi + \rho \Gamma v^\psi \cos \psi), \\
X^\rho[\beta_1(t)] &= \rho \beta_1 (v^\rho (v^\rho \sin \psi + \rho \Gamma v^\psi \cos \psi) - B^\rho (B^\rho \sin \psi \\
&\quad + \rho \Gamma B^\psi \cos \psi) + (P + \frac{1}{2} (B^\rho B^\rho + B^\psi B^\psi + 2m \Gamma B^\psi B^z \\
&\quad + B^z B^z)) \sin \psi + a ((\rho^{-1} v^{\rho, \psi} - \Gamma \rho v^{\psi, \rho} - m^2 \Gamma^3 v^\psi) \cos \psi \\
&\quad + \Gamma v^\psi \sin \psi)) - \beta_{1,t} \rho^2 v^\rho \sin \psi \\
X^\psi[\beta_1(t)] &= \rho \beta_1 (\Gamma v^\psi (v^\rho \sin \psi + \rho \Gamma v^\psi \cos \psi) - \Gamma B^\psi (B^\rho \sin \psi \\
&\quad + \rho \Gamma B^\psi \cos \psi) + \rho^{-1} (P + \frac{1}{2} (B^\rho B^\rho + B^\psi B^\psi \\
&\quad + 2m \Gamma B^\psi B^z + B^z B^z)) \cos \psi + a (\Gamma \rho^{-1} v^\psi \\
&\quad - \rho^{-2} v^{\rho, \psi}) \sin \psi) - \beta_{1,t} \Gamma \rho^2 v^\psi \sin \psi
\end{aligned} \tag{6.3.16b}$$

(3) *2nd-component of generalized momentum*

$$\begin{aligned}
T[\beta_2(t)] &= \rho \beta_2 (v^\rho \cos \psi - \rho \Gamma v^\psi \sin \psi) \\
X^\rho[\beta_2(t)] &= \rho \beta_2 (v^\rho (v^\rho \cos \psi - \rho \Gamma v^\psi \sin \psi) - B^\rho (B^\rho \cos \psi \\
&\quad - \rho \Gamma B^\psi \sin \psi) + (P + \frac{1}{2} (B^\rho B^\rho + B^\psi B^\psi + 2m \Gamma B^\psi B^z \\
&\quad + B^z B^z)) \cos \psi + a (\Gamma v^{\psi, \psi} \cos \psi - (\rho^{-1} v^{\rho, \psi} - \Gamma \rho v^{\psi, \rho} \\
&\quad - m^2 \Gamma^3 v^\psi) \sin \psi)) - \beta_{2,t} \rho^2 v^\rho \cos \psi \\
X^\psi[\beta_2(t)] &= \rho \beta_2 (\Gamma v^\psi (v^\rho \cos \psi - \rho \Gamma v^\psi \sin \psi) - \Gamma B^\psi (B^\rho \cos \psi \\
&\quad - \rho \Gamma B^\psi \sin \psi) - \rho^{-1} (P + \frac{1}{2} (B^\rho B^\rho + B^\psi B^\psi + 2m \Gamma B^\psi B^z \\
&\quad + B^z B^z)) \sin \psi + a (\Gamma \rho^{-1} v^\psi - \rho^{-2} v^{\rho, \psi}) \cos \psi) \\
&\quad - \beta_{2,t} \Gamma \rho^2 v^\psi \cos \psi
\end{aligned} \tag{6.3.16c}$$

(4) *incompressibility flux*

$$\begin{aligned}
T[\gamma(t)] &= 0 \\
X^\rho[\gamma(t)] &= \gamma\rho v^\rho \\
X^\psi[\gamma(t)] &= \gamma\rho\Gamma v^\psi
\end{aligned} \tag{6.3.16d}$$

(5) *poloidal generalized magnetic flux*

$$\begin{aligned}
T[\chi(t, \rho, \psi)] &= \rho(\chi_{,\rho} B^\rho + \Gamma\chi_{,\psi} B^\psi) \\
X^\rho[\chi(t, \rho, \psi)] &= \rho(-\chi_{,t} B^\rho + \Gamma\chi_{,\psi}(v^\rho B^\psi - B^\rho v^\psi) \\
&\quad - b(\chi_{,\rho} B^\rho_{,\rho} - \frac{1}{2}\rho^{-2}\chi_{,\psi} B^\rho_{,\psi} + \rho^{-1}\chi_{,\rho} B^\rho + \Gamma\chi_{,\psi} B^\psi_{,\rho} \\
&\quad - \Gamma\chi_{,\rho\psi} B^\psi + \rho^{-1}\Gamma\chi_{,\psi} B^\psi + m^2\rho^{-1}\Gamma^3\chi_{,\psi} B^\psi \\
&\quad + \frac{1}{2}\rho^{-2}\chi_{,\psi\psi} B^\rho)) \\
X^\psi[\chi(t, \rho, \psi)] &= \rho\Gamma(-\chi_{,t} B^\psi + \chi_{,\rho}(v^\psi B^\rho - B^\psi v^\rho) \\
&\quad - b(\rho^{-2}\Gamma^{-1}\chi_{,\rho} B^\rho_{,\psi} + \frac{1}{2}\Gamma^{-1}\rho^{-2}\chi_{,\psi} B^\psi_{,\psi} \\
&\quad + \rho^{-3}\Gamma^{-1}\chi_{,\psi} B^\rho - \rho^{-1}\chi_{,\rho} B^\psi + \chi_{,\rho\rho} B^\psi \\
&\quad - \frac{1}{2}\rho^{-2}\Gamma^{-1}\chi_{,\rho\psi} B^\rho))
\end{aligned} \tag{6.3.16e}$$

(6) *axial magnetic flux*

$$\begin{aligned}
T &= \rho B^z \\
X^\rho &= \rho(v^\rho B^z - B^\rho v^z - b(B^z_{,\rho} - 2m\Gamma\rho^{-1}B^\psi)) \\
X^\psi &= \rho\Gamma(v^\psi B^z - B^\psi v^z - b(\Gamma^{-1}\rho^{-2}B^z_{,\psi} - 2m\Gamma^{-1}\rho^{-3}B^\rho))
\end{aligned} \tag{6.3.16f}$$

(7) *screw cross-helicity*

$$\begin{aligned}
T &= \rho(v^\rho B^\rho + \rho^2 \Gamma^2 v^\psi B^\psi) \\
X^\rho &= \rho(v^\rho (v^\rho B^\rho + \rho^2 \Gamma^2 v^\psi B^\psi) \\
&\quad - B^\rho \frac{1}{2}(v^\rho v^\rho + \rho^2 \Gamma^2 v^\psi v^\psi + B^\rho B^\rho + \rho^2 \Gamma^2 B^\psi B^\psi) \\
&\quad + B^\rho (P + \frac{1}{2}(B^\rho B^\rho + B^\psi B^\psi + 2m\Gamma B^\psi B^z + B^z B^z)) \\
&\quad + a(v^\rho B^\rho_{,\rho} - B^\rho v^\rho_{,\rho} + \Gamma(v^\rho B^\psi_{,\psi} - B^\rho v^\psi_{,\psi} \\
&\quad - v^\psi B^\rho_{,\psi} + B^\psi v^\rho_{,\psi}) + \rho^2 \Gamma^2 (v^\psi B^\psi_{,\rho} - B^\psi v^\psi_{,\rho})) \\
X^\psi &= \rho\Gamma(v^\psi (v^\rho B^\rho + \rho^2 \Gamma^2 v^\psi B^\psi) \\
&\quad - B^\psi \frac{1}{2}(v^\rho v^\rho + \rho^2 \Gamma^2 v^\psi v^\psi + B^\rho B^\rho + \rho^2 \Gamma^2 B^\psi B^\psi) \\
&\quad + B^\psi (P + \frac{1}{2}(B^\rho B^\rho + B^\psi B^\psi + 2m\Gamma B^\psi B^z + B^z B^z)) \\
&\quad + a(\rho^{-2} \Gamma^{-1} (v^\rho B^\rho_{,\psi} - B^\rho v^\rho_{,\psi}) + (B^\rho v^\psi_{,\rho} - v^\rho v^\psi_{,\rho}) \\
&\quad + \rho^{-1} (1 + m^2 \Gamma^2) (B^\rho v^\psi - B^\psi v^\rho)) \\
0 &= a + b
\end{aligned} \tag{6.3.16g}$$

(8) *energy*

$$\begin{aligned}
T &= \rho \frac{1}{2} ((v^\rho)^2 + \rho^2 \Gamma^2 (v^\psi)^2 + (v^z + m\Gamma v^\psi)^2 \\
&\quad + (B^\rho)^2 + \rho^2 \Gamma^2 (B^\psi)^2 + (B^z + m\Gamma B^\psi)^2) \\
X^\rho &= \rho (v^\rho \frac{1}{2} (v^\rho v^\rho + \rho^2 \Gamma^2 v^\psi v^\psi + (v^z + m\Gamma v^\psi)^2 + B^\rho B^\rho + \rho^2 \Gamma^2 B^\psi B^\psi) \\
&\quad + (B^z + m\Gamma B^\psi)^2) - B^\rho ((v^z + m\Gamma v^\psi)(B^z + m\Gamma B^\psi) \\
&\quad + v^\rho B^\rho + \rho^2 \Gamma^2 v^\psi B^\psi) \\
&\quad + v^\rho (P + \frac{1}{2} (B^\rho B^\rho + B^\psi B^\psi + 2m\Gamma B^\psi B^z + B^z B^z)) \\
X^\psi &= \rho\Gamma (v^\psi \frac{1}{2} (v^\rho v^\rho + \rho^2 \Gamma^2 v^\psi v^\psi + (v^z + m\Gamma v^\psi)^2 + B^\rho B^\rho + \rho^2 \Gamma^2 B^\psi B^\psi) \\
&\quad + (B^z + m\Gamma B^\psi)^2) - B^\psi ((v^z + m\Gamma v^\psi)(B^z + m\Gamma B^\psi) \\
&\quad + v^\rho B^\rho + \rho^2 \Gamma^2 v^\psi B^\psi) \\
&\quad + v^\psi (P + \frac{1}{2} (B^\rho B^\rho + B^\psi B^\psi + 2m\Gamma B^\psi B^z + B^z B^z)) \\
0 &= a = b
\end{aligned} \tag{6.3.16h}$$

(9) *transverse-field Riemann (coupling) invariant*

$$\begin{aligned}
T &= \rho \partial_{\tilde{v}} \kappa \\
X^\rho &= \rho (v^\rho \partial_{\tilde{v}} \kappa - B^\rho \partial_{\tilde{B}} \kappa - a \partial_{\tilde{v}}^2 \kappa (v^z{}_{,\rho} - \frac{1}{2} m \rho^{-2} v^\rho{}_{,\psi} + m \Gamma v^\psi{}_{,\rho} \\
&\quad - m \Gamma^3 \rho v^\psi) - b \partial_{\tilde{v}} \partial_{\tilde{B}} \kappa (B^z{}_{,\rho} - \frac{1}{2} m \rho^{-2} B^\rho{}_{,\psi} + m \Gamma B^\psi{}_{,\rho} \\
&\quad - m \Gamma^3 \rho B^\psi)) \\
X^\psi &= \rho \Gamma (v^\psi \partial_{\tilde{v}} \kappa - B^\psi \partial_{\tilde{B}} \kappa - a \partial_{\tilde{v}}^2 \kappa (\Gamma^{-1} \rho^{-2} v^z{}_{,\psi} + \frac{1}{2} m \Gamma^{-1} \rho^{-2} v^\psi{}_{,\psi} \\
&\quad - m \Gamma^{-1} \rho^{-3} v^\rho) - b \partial_{\tilde{v}} \partial_{\tilde{B}} \kappa (\Gamma^{-1} \rho^{-2} B^z{}_{,\psi} + \frac{1}{2} m \Gamma^{-1} \rho^{-2} B^\psi{}_{,\psi} \\
&\quad - m \Gamma^{-1} \rho^{-3} B^\rho))
\end{aligned} \tag{6.3.16i}$$

where $\beta_1(t), \beta_2(t), \gamma(t)$ are arbitrary functions of t , $\tilde{v} = m \Gamma v^\psi + v^z$, $\tilde{B} = m \Gamma B^\psi + B^z$, and $\kappa(\tilde{v}, \tilde{B})$ is an arbitrary solution of the linear equations

$$\partial_{\tilde{v}}^2 \kappa - \partial_{\tilde{B}}^2 \kappa = 0, \quad a = b = 0 \tag{6.3.17a}$$

$$\partial_{\tilde{v}}^2 \kappa - \partial_{\tilde{B}}^2 \kappa = 0, \quad \partial_{\tilde{v}}^3 \kappa = 0, \quad a + b = 0 \tag{6.3.17b}$$

$$\partial_{\tilde{v}}^2 \kappa - \partial_{\tilde{B}}^2 \kappa = 0, \quad \partial_{\tilde{v}}^3 \kappa = 0, \quad \partial_{\tilde{B}}^3 \kappa = 0, \quad a + b \neq 0. \tag{6.3.17c}$$

The conservation law (6.3.16i) involves a function, κ , satisfying a wave equation (6.3.17a) in the variables \tilde{v}, \tilde{B} . In the case of inviscid plasmas, there is no other condition on this function, and so it yields two infinite families of conservation laws, corresponding to

$$\kappa = \kappa_1(\tilde{v} + \tilde{B}) + \kappa_2(\tilde{v} - \tilde{B}) \tag{6.3.18}$$

where κ_1, κ_2 are arbitrary functions of their arguments. In case of viscous plasmas, the function κ satisfies extra equations (6.3.17b) and (6.3.17c). This yields, respectively, families of four and five conservation laws, corresponding to

$$\kappa = C_4(\tilde{v}\tilde{v} + \tilde{B}\tilde{B}) + C_3\tilde{v}\tilde{B} + C_2\tilde{v} + C_1\tilde{B} \tag{6.3.19a}$$

and

$$\kappa = C_5\tilde{B}(3\tilde{v}\tilde{v} + \tilde{B}\tilde{B}) + C_4(\tilde{v}\tilde{v} + \tilde{B}\tilde{B}) + C_3\tilde{v}\tilde{B} + C_2\tilde{v} + C_1\tilde{B} \tag{6.3.19b}$$

where C_1, \dots, C_5 are arbitrary constants.

These conservation laws (6.3.16i) unify and generalize several known physical conservation laws, which are inherited from the unreduced MHD system:

x^3 -component of momentum

$$\begin{aligned}
T &= \rho(v^z + m\Gamma v^\psi) \\
X^\rho &= \rho(v^\rho(v^z + m\Gamma v^\psi) - B^\rho(B^z + m\Gamma B^\psi) - a(v^z_{,\rho} - \frac{1}{2}m\rho^{-2}v^\rho_{,\psi} \\
&\quad + m\Gamma v^\psi_{,\rho} - m\Gamma^3\rho v^\psi)) \\
X^\psi &= \rho\Gamma(v^\psi(v^z + m\Gamma v^\psi) - B^\psi(B^z + m\Gamma B^\psi) - a(\Gamma^{-1}\rho^{-2}v^z_{,\psi} \\
&\quad + \frac{1}{2}m\Gamma^{-1}\rho^{-2}v^\psi_{,\psi} - m\Gamma^{-1}\rho^{-3}v^\rho))
\end{aligned} \tag{6.3.20a}$$

x^3 -component of magnetic flux

$$\begin{aligned}
T &= \rho(B^z + m\Gamma B^\psi) \\
X^\rho &= \rho(v^\rho(B^z + m\Gamma B^\psi) - B^\rho(v^z + m\Gamma v^\psi) - b(B^z_{,\rho} - \frac{1}{2}m\rho^{-2}B^\rho_{,\psi} \\
&\quad + m\Gamma B^\psi_{,\rho} - m\Gamma^3\rho B^\psi)) \\
X^\psi &= \rho\Gamma(v^\psi(B^z + m\Gamma B^\psi) - B^\psi(v^z + m\Gamma v^\psi) - b(\Gamma^{-1}\rho^{-2}B^z_{,\psi} \\
&\quad + \frac{1}{2}m\Gamma^{-1}\rho^{-2}B^\psi_{,\psi} - m\Gamma^{-1}\rho^{-3}B^\rho))
\end{aligned} \tag{6.3.20b}$$

incompressibility flux

$$\begin{aligned}
T &= 0 \\
X^\rho &= \rho v^\rho \\
X^\psi &= \rho\Gamma v^\psi
\end{aligned} \tag{6.3.20c}$$

solenoidal flux

$$\begin{aligned}
T &= 0 \\
X^\rho &= \rho B^\rho \\
X^\psi &= \rho\Gamma B^\psi
\end{aligned} \tag{6.3.20d}$$

transverse cross-helicity

$$\begin{aligned}
T &= \rho(v^z B^z + m\Gamma(v^z B^\psi + v^\psi B^z + m\Gamma v^\psi B^\psi)) \\
X^\rho &= \rho \left(v^\rho (v^z B^z + m\Gamma(v^z B^\psi + v^\psi B^z + m\Gamma v^\psi B^\psi)) \right. \\
&\quad - B^\rho \frac{1}{2} ((v^z + m\Gamma v^\psi)^2 + (B^z + m\Gamma B^\psi)^2) \\
&\quad - a(B^z + m\Gamma B^\psi)(v^z{}_{,\rho} - \frac{1}{2}m\rho^{-2}v^\rho{}_{,\psi} + m\Gamma v^\psi{}_{,\rho} - m\Gamma^3 \rho v^\psi) \\
&\quad \left. + a(v^z + m\Gamma v^\psi)(B^z{}_{,\rho} - \frac{1}{2}m\rho^{-2}B^\rho{}_{,\psi} + m\Gamma B^\psi{}_{,\rho} - m\Gamma^3 \rho B^\psi) \right) \\
X^\psi &= \rho\Gamma \left(v^\psi (v^z B^z + m\Gamma(v^z B^\psi + v^\psi B^z + m\Gamma v^\psi B^\psi)) \right. \\
&\quad - B^\psi \frac{1}{2} ((v^z + m\Gamma v^\psi)^2 + (B^z + m\Gamma B^\psi)^2) \\
&\quad - a(B^z + m\Gamma B^\psi)(\Gamma^{-1}\rho^{-2}v^z{}_{,\psi} + \frac{1}{2}m\Gamma^{-1}\rho^{-2}v^\psi{}_{,\psi} - m\Gamma^{-1}\rho^{-3}v^\rho) \\
&\quad \left. + a(v^z + m\Gamma v^\psi)(\Gamma^{-1}\rho^{-2}B^z{}_{,\psi} + \frac{1}{2}m\Gamma^{-1}\rho^{-2}B^\psi{}_{,\psi} - m\Gamma^{-1}\rho^{-3}B^\rho) \right) \\
0 &= a + b
\end{aligned} \tag{6.3.20e}$$

transverse energy

$$\begin{aligned}
T &= \frac{1}{2}\rho((v^z + m\Gamma v^\psi)^2 + (B^z + m\Gamma B^\psi)^2) \\
X^\rho &= \rho \left(v^\rho \frac{1}{2} ((v^z + m\Gamma v^\psi)^2 + (B^z + m\Gamma B^\psi)^2) \right. \\
&\quad \left. - B^\rho (v^z + m\Gamma v^\psi)(B^z + m\Gamma B^\psi) \right) \\
X^\psi &= \rho\Gamma \left(v^\psi \frac{1}{2} ((v^z + m\Gamma v^\psi)^2 + (B^z + m\Gamma B^\psi)^2) \right. \\
&\quad \left. - B^\psi (v^z + m\Gamma v^\psi)(B^z + m\Gamma B^\psi) \right) \\
0 &= a = b
\end{aligned} \tag{6.3.20f}$$

The conservation laws (6.3.16a) – (6.3.16f) are inherited from the unreduced MHD system. The remaining two conservation laws (6.3.16g) – (6.3.16i) come from a decoupling of components of the unreduced helicity conservation law (3.1.2e) when it is expressed in helical coordinates.

None of these conservation laws (6.3.16a) – (6.3.16i) have previously appeared in the literature. The conservation law (6.3.16i) is new, although some special cases of it coincide with well-known conservation laws.

A point symmetry of the helical-invariant MHD system (6.3.8) is an infinitesi-

mal generator of the form (4.0.1) under which the solution space $\mathcal{E}_{\text{hel.}}$ is mapped into itself. We classify all point symmetries by using computer algebra to set up and solve the symmetry determining system (4.0.2) for the helical-invariant MHD system in the form (6.3.11a)–(6.3.11i). As explained in appendix .8, the case $a = b = 0$ is computationally very hard and will be left for future work. Also, the case $m = 0$ corresponds to the translation-invariant reduction for which a classification is given in Proposition 3.

Proposition 1 *The point symmetries of the helical-invariant MHD system (6.3.8) for $m \neq 0$ are generated by:*

(1) *time translation*

$$X_{\text{trans.}} = \partial/\partial t \quad (6.3.21a)$$

(2) *helical translation*

$$X = \partial/\partial \psi \quad (6.3.21b)$$

(3) *pressure shift*

$$X[\sigma(t)] = \sigma \partial/\partial P \quad (6.3.21c)$$

(4) *1st Generalized screw boost*

$$\begin{aligned} X[v_1(t)] = & v_1 \cos \psi \partial/\partial \rho - v_1 \rho^{-1} \sin \psi \partial/\partial \psi + (v_1' \cos \psi \\ & - v_1 \Gamma v^\psi \sin \psi) \partial/\partial v^\rho + (v_1 \rho^{-2} \Gamma^{-1} v^\rho \sin \psi \\ & - v_1 \rho^{-1} m^2 \Gamma^2 v^\psi \cos \psi - v_1' \rho^{-1} \Gamma^{-1} \sin \psi) \partial/\partial v^\psi \\ & + (v_1 \rho^{-1} m \Gamma v^\psi \cos \psi - v_1 \rho^{-2} m v^\rho \sin \psi \\ & + v_1' \rho^{-1} m \sin \psi) \partial/\partial v^z - v_1 \Gamma B^\psi \sin \psi \partial/\partial B^\rho \\ & + (v_1 \rho^{-2} \Gamma^{-1} B^\rho \sin \psi - v_1 \rho^{-1} m^2 \Gamma^2 B^\psi \cos \psi) \partial/\partial B^\psi \\ & + (v_1 \rho^{-1} m \Gamma B^\psi \cos \psi - v_1 \rho^{-2} m B^\rho \sin \psi) \partial/\partial B^z \\ & - v_1'' \rho \cos \psi \partial/\partial P \end{aligned} \quad (6.3.21d)$$

(5) *2nd Generalized screw boost*

$$\begin{aligned}
X[v_2(t)] = & v_2 \sin \psi \partial / \partial \rho + v_2 \rho^{-1} \cos \psi \partial / \partial \psi + (v_2' \sin \psi \\
& + v_2 \Gamma v^\psi \cos \psi) \partial / \partial v^\rho + (v_2' \rho^{-1} \Gamma^{-1} \cos \psi \\
& - v_2 \rho^{-2} \Gamma^{-1} v^\rho \cos \psi - v_2 \rho^{-1} m^2 \Gamma^2 v^\psi \sin \psi) \partial / \partial v^\psi \\
& + (v_2 \rho^{-1} m \Gamma v^\psi \sin \psi + v_2 \rho^{-2} m v^\rho \cos \psi \\
& - v_2' \rho^{-1} m \cos \psi) \partial / \partial v^z + v_2 \Gamma B^\psi \cos \psi \partial / \partial B^\rho \\
& - (v_2 \rho^{-2} \Gamma^{-1} B^\rho \cos \psi + v_2 \rho^{-1} m^2 \Gamma^2 B^\psi \sin \psi) \partial / \partial B^\psi \\
& + (v_2 \rho^{-1} m \Gamma B^\psi \sin \psi + v_2 \rho^{-2} m B^\rho \cos \psi) \partial / \partial B^z \\
& - v_2'' \rho \sin \psi \partial / \partial P
\end{aligned} \tag{6.3.21e}$$

(6) *scaling*

$$\begin{aligned}
X_{\text{scal.}} = & t \partial / \partial t - v^\rho \partial / \partial v^\rho - v^\psi \partial / \partial v^\psi - v^z \partial / \partial v^z \\
& - B^\rho \partial / \partial B^\rho - B^\psi \partial / \partial B^\psi - B^z \partial / \partial B^z - 2P \partial / \partial P
\end{aligned} \tag{6.3.21f}$$

(7) *dilation*

$$\begin{aligned}
X_{\text{dil.}} = & t \partial / \partial t + \rho \partial / \partial \rho - m^2 \Gamma^2 v^\psi \partial / \partial v^\psi - m^2 \Gamma^2 B^\psi \partial / \partial B^\psi \\
& + m \Gamma v^\psi \partial / \partial v^z + m \Gamma B^\psi \partial / \partial B^z, \quad a = b = 0
\end{aligned} \tag{6.3.21g}$$

(8) *transverse-field generalized dilation/shift*

$$X[\lambda(\tilde{v}, \tilde{B})] = \partial_{\tilde{B}} \lambda \partial / \partial B^z + \partial_{\tilde{v}} \lambda \partial / \partial v^z - \tilde{B} \partial_{\tilde{B}} \lambda \partial / \partial P, \tag{6.3.21h}$$

where $v_1(t)$, $v_2(t)$ and $\sigma(t)$ are arbitrary functions of t , $\tilde{v} = m \Gamma v^\psi + v^z$, $\tilde{B} = m \Gamma B^\psi + B^z$, and $\lambda(\tilde{v}, \tilde{B})$ is an arbitrary solution of the linear equations

$$\partial_{\tilde{v}}^2 \lambda - \partial_{\tilde{B}}^2 \lambda = 0, \quad a = b = 0 \tag{6.3.22a}$$

$$\partial_{\tilde{v}}^2 \lambda - \partial_{\tilde{B}}^2 \lambda = 0, \quad \partial_{\tilde{v}}^2 \partial_{\tilde{B}} \lambda = \partial_{\tilde{v}} \partial_{\tilde{B}}^2 \lambda = 0, \quad a = b \neq 0 \tag{6.3.22b}$$

$$\partial_{\tilde{v}}^2 \lambda - \partial_{\tilde{B}}^2 \lambda = 0, \quad \partial_{\tilde{v}} \partial_{\tilde{B}} \lambda = 0, \quad a - b \neq 0 \tag{6.3.22c}$$

None of these symmetries (6.3.21) have previously appeared in the literature.

The symmetry (6.3.21h) appears as a generalization of the symmetry (6.1.15g) considered in the section 6.1. It involves a function, λ , satisfying a wave equation (6.3.22a) in the variables \tilde{v}, \tilde{B} . In the case of inviscid plasmas, there is no other condition on this function, and so it yields two infinite families of point

symmetries, corresponding to

$$\lambda = \lambda_1(\tilde{v} + \tilde{B}) + \lambda_2(\tilde{v} - \tilde{B}) \quad (6.3.23)$$

where λ_1, λ_2 are arbitrary functions of their arguments. In case of viscous plasmas, the function λ satisfies extra equations (6.3.22b) and (6.3.22c). This yields, respectively, families of four and three and point symmetries, corresponding to

$$\lambda = C_4 \tilde{v} \tilde{B} + C_3(\tilde{v} \tilde{v} + \tilde{B} \tilde{B}) + C_2 \tilde{B} + C_1 \tilde{v} \quad (6.3.24a)$$

and

$$\lambda = C_3(\tilde{v} \tilde{v} + \tilde{B} \tilde{B}) + C_2 \tilde{B} + C_1 \tilde{v} \quad (6.3.24b)$$

where C_1, \dots, C_4 are arbitrary constants. In particular, the family (6.3.24b) is given by:

screw velocity shift

$$X = \partial / \partial v^\psi, \quad (6.3.25a)$$

axial magnetic shift

$$X = \partial / \partial B^z - (m\Gamma B^\psi + B^z) \partial / \partial P, \quad (6.3.25b)$$

axial scaling

$$X = (m\Gamma B^\psi + B^z) \partial / \partial B^z + (m\Gamma v^\psi + v^z) \partial / \partial v^z - (m\Gamma B^\psi + B^z)^2 \partial / \partial P \quad (6.3.25c)$$

In the larger family (6.3.24a), the additional symmetry is given by

$$\begin{aligned} X = & (m\Gamma v^\psi + v^z) \partial / \partial B^z + (m\Gamma B^\psi + B^z) \partial / \partial v^z \\ & - (m\Gamma v^\psi + v^z)(m\Gamma B^\psi + B^z) \partial / \partial P, \end{aligned} \quad (6.3.26)$$

which generates the transformation

$$\tilde{v}^z = \tilde{v} \cosh \vartheta + \tilde{B} \sinh \vartheta, \quad \tilde{B}^z = \tilde{v} \sinh \vartheta + \tilde{B} \cosh \vartheta, \quad (6.3.27a)$$

$$\tilde{P} = P - \frac{1}{2}((\tilde{v})^2 + (\tilde{B})^2) \frac{1}{2} \sinh 2\vartheta + \tilde{v} \tilde{B} \cosh 2\vartheta \quad (6.3.27b)$$

where

$$4\tilde{P} + (\tilde{v}^z)^2 + (\tilde{B}^z)^2 = 4P + (\tilde{v})^2 + (\tilde{B})^2, \quad (\tilde{v}^z)^2 - (\tilde{B}^z)^2 = (\tilde{v})^2 - (\tilde{B})^2 \quad (6.3.28)$$

are the invariants. This transformation (6.3.27) is similar to a transformation presented in Ref.[5, 59] for equilibrium solutions of an ideal plasma.

Chapter 7

Conclusion

This main results in this thesis provide the first-ever complete classification of all kinematic conservation laws for the system of magnetohydrodynamic equations (1.0.2) governing constant-density viscous plasmas (or any other conducting fluid) in which the dynamic and magnetic viscosities are constant. Reductions of this system under translation symmetries, axial rotation symmetries, and helical symmetries are considered, which are important in physical applications. For each reduced system, complete classifications of all kinematic conservation laws and point symmetries are obtained.

These classifications have yielded several new conservation laws which can be expected to be relevant for physical applications of magnetohydrodynamics.

The same methods used in this thesis can be extended to search for potential conservation laws, similar to magnetic helicity, which depend explicitly on the magnetic vector potential. By use of the vorticity formulation of the magnetohydrodynamic equations, the method can also be applied to search for conservation laws in which the conserved density depends on spatial derivatives of the fluid velocity, magnetic field, and pressure.

Chapter 8

Future work

The multiplier ansatz considered in this research can be generalized to take into account nonzero multipliers of (2.0.11g). As we show in [92], this gives a conservation law of generalized streamline flux, that depends on an arbitrary function of time and space coordinates. The same generalization will be done for three reductions: translation, axial rotation and helical motion.

The physical meaning of new families of conservation laws and point symmetries is not obvious, but the structure of wave solution can show on some their connection with Alfvén waves. It is a type of magnetohydrodynamic wave resulting from the frizzing-in of the magnetic field in plasma flow [10, 11]. The speed of propagation of the Alfvén waves is proportional to the magnetic field.

There exist a lot of physically important MHD system [8, 10], for which the MHD system (1.0.1) is just an approximation. I'm going to generalize the results and methods, used in this research, to some of these systems: compressible MHD system, Hall effect, thermal effects, different parameter expansions; two fluid systems, higher order systems.

The conserved density and fluxes of helicity-type conservation laws can depend also on either magnetic potential or curl of magnetic/velocity fields. This fact noticeably complicates computations, but usage of inspection of multipliers dependency and definition of their ansatz, potentially allow to do systematic search for helicity-type conservation laws.

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Appendices

.1 Appendix I

```

%the procedure conlaw4 solves the multiplier determining equations (2.0.17)
%with respect to conservation law multipliers using the Crack routines [93]
%and computes corresponding conserved densities and fluxes using
%the direct integration or the formulas described in [5]

load "~/crack/crfas1"$
load "~/crack/conlaw0"$
load "~/crack/conlaw4"$

lisp(depl!*:nil)$      % clearing of all dependencies
setcrackflags()$      % standart flags
lisp(print_:=nil)$     % no output of the calculation

%x1 = x1, x2 = x2, x3 = x3
%v1 = v1, v2 = v2, v3 = v3
%H1 = B1, H2 = B2, H3 = B3
array x(3), % space coordinates
      v(3), % velocity
      H(3), % magnetic field
      Cvp(3),
      CH(3)$

%definition of the pressure p as a function of t, x1, x2, x3
depend p,t,x1,x2,x3$

%jet variable list
jvl:={p}$

for n:=1:3 do x(n):=mkid(x,n)$
for n:=1:3 do <<H(n):=mkid(H,n); jvl:=cons(H(n),jvl)$ depend H(n),t,x1,x2,x3$
              v(n):=mkid(v,n); jvl:=cons(v(n),jvl)$ depend v(n),t,x1,x2,x3
              >>$

%solved form of the divergence free equations of investigated MHD system
%where df(v(1),x(1)) is v1,1, df(H(1),x(1)) is B1,1
e78:= {df(v(1),x(1)) = - df(v(2),x(2)) - df(v(3),x(3)),

```

$$df(H(1),x(1)) = - df(H(2),x(2)) - df(H(3),x(3)) \}$$$

%eqn0 is MHD system under consideration

eqn0:=e78\$

%differential consequences of the divergence free conditions

```
e78:=append({df(v(1),x(1),x(1)) = df(- df(v(2),x(2)) - df(v(3),x(3)),x(1)),
            df(v(1),x(1),x(2)) = df(- df(v(2),x(2)) - df(v(3),x(3)),x(2)),
            df(v(1),x(1),x(3)) = df(- df(v(2),x(2)) - df(v(3),x(3)),x(3)),
            df(H(1),x(1),x(1)) = df(- df(H(2),x(2)) - df(H(3),x(3)),x(1)),
            df(H(1),x(1),x(2)) = df(- df(H(2),x(2)) - df(H(3),x(3)),x(2)),
            df(H(1),x(1),x(3)) = df(- df(H(2),x(2)) - df(H(3),x(3)),x(3)) }
            ,e78)$
```

%conservation law multipliers corresponding to the evolution equations

array Q(6)\$

f1:={}\$ *%functions to be calculated*

vdefli:={t,x1,x2,x3,v1,v2,v3,H1,H2,H3}\$

f1:=cons(W,f1); *%W is a conserved density*

for each z in vdefli do depend W,z\$

for i:=1:3 do <<Q(i):= df(W,v(i))>>\$

Q_1 := Q(1)\$

Q_2 := Q(2)\$

Q_3 := Q(3)\$

for i:=1:3 do <<Q(i + 3):= df(W,H(i))>>\$

Q_4 := Q(4)\$

Q_5 := Q(5)\$

Q_6 := Q(6)\$

f1:=cons(R,f1);

for each z in vdefli do depend R,z\$

```

fl:=cons(M,fl);
for each z in vdefli do depend Mf,z$

for n:=1:3 do <<Cvp(n):=mkid(Cvp,n);
    fl:=cons(Cvp(n),fl);
    for each z in vdefli do depend Cvp(n),z
        >>$

for n:=1:3 do <<CH(n):=mkid(CH,n);
    fl:=cons(CH(n),fl);
    for each z in vdefli do depend CH(n),z
        >>$

J := Mf + R*p + df(v(2),x(1))*Cvp(1) + df(v(3),x(1))*Cvp(2)
    + (df(v(2),x(2)) + df(v(3),x(3)))*Cvp(3)
    + df(H(2),x(1))*CH(1) + df(H(3),x(1))*CH(2)
    + (df(H(2),x(2)) + df(H(3),x(3)))*CH(3)
    $

%the conservation law multipliers
%corresponding to the divergence free conditions
Q_7 := df(J,v(1)) $
Q_8 := df(J,H(1)) $

fl:=cons(a,fl);
fl:=cons(b,fl);

%the evolution equations of magnetic field components
%substitutions of e78
for n:=3 step -1 until 1 do eqn0:=cons(
df(H(n),t) = sub(e78, - (for m:=1:3 sum (v(m)*df(H(n),x(m))))
    + b*(for m:=1:3 sum df(H(n),x(m),2))
    + (for m:=1:3 sum (H(m)*df(v(n),x(m))))),eqn0)$

%the evolution equations of velocity vector components
%substitutions of e78
for n:=3 step -1 until 1 do eqn0:=cons(
df(v(n),t) = sub(e78, - df(p,x(n))

```

```

- (for m:=1:3 sum (v(m)*df(v(n),x(m))))
+ a*(for m:=1:3 sum df(v(n),x(m),2))
- (for m:=1:3 sum (H(m)*df(H(m),x(n))))
+ (for m:=1:3 sum (H(m)*df(H(n),x(m))))),eqn0)$

%Calculation of the conservation laws multipliers,
%where the list of functions to be calculated includes of three functions
%W,Mf,R depended on t,x1,x2,x3,v1,v2,v3,H1,H2,H3
%and the list of nonzero relations includes J
conlaw4({eqn0,jv1,{t,x1,x2,x3}},{0, 0, t, f1, {J}})$

end$

```

.2 Appendix II

```

%the procedure conlaw4 solves the multiplier determining equations
%for the translation-invariant MHD system (6.1.2)
%with respect to conservation law multipliers using the Crack routines [93]
%and computes corresponding conserved densities and fluxes using
%the direct integration or the formulas described in [5]

load "~/crack/crfasl"$
load "~/crack/conlaw0"$
load "~/crack/conlaw4"$

lisp(depl!*:nil)$      % clearing of all dependencies
setcrackflags()$      % standart flags
lisp(print_:=nil)$    % no output of the calculation

off batch_mode$

%x1 = x1, x2 = x2
%v1 = v1, v2 = v2, v3 = v3
%H1 = B1, H2 = B2, H3 = B3
array x(2), % space coordinates
      v(3), % velocity
      H(3), % magnetic field

```

```

    Cvp(3),
    CH(3)$

%definition of the pressure p as a function of t, x1, x2
depend p,t,x1,x2$

%jet variable list
jvl:={p}$

for n:=1:2 do x(n):=mkid(x,n)$
for n:=1:3 do <<H(n):=mkid(H,n); jvl:=cons(H(n),jvl)$ depend H(n),t,x1,x2$
    v(n):=mkid(v,n); jvl:=cons(v(n),jvl)$ depend v(n),t,x1,x2
    >>$

%solved form of the divergence free equations of investigated MHD system
%where df(v(1),x(1)) is  $v^1_{,1}$ , df(H(1),x(1)) is  $B^1_{,1}$ 
e78:= {df(v(1),x(1)) = - df(v(2),x(2)),
    df(H(1),x(1)) = - df(H(2),x(2)) }$

%eqn0 is MHD system under consideration
eqn0:=e78$

%differential consequences of the divergence free conditions
e78:=append({df(v(1),x(1),x(1)) = - df(v(2),x(2),x(1)),
    df(v(1),x(1),x(2)) = - df(v(2),x(2),x(2)),
    df(H(1),x(1),x(1)) = - df(H(2),x(2),x(1)),
    df(H(1),x(1),x(2)) = - df(H(2),x(2),x(2)) }
,e78)$

%conservation law multipliers corresponding to the evolution equations
array Q(6)$

fl:={} $ %functions to be calculated
vdefli:={t,x1,x2,v1,v2,v3,H1,H2,H3}$

fl:=cons(W,fl); % W is a conserved density
for each z in vdefli do depend W,z$

```

```

for i:=1:3 do <<Q(i):= df(W,v(i))>>$

Q_1 := Q(1)$
Q_2 := Q(2)$
Q_3 := Q(3)$

for i:=1:3 do <<Q(i + 3):= df(W,H(i))>>$

Q_4 := Q(4)$
Q_5 := Q(5)$
Q_6 := Q(6)$

fl:=cons(Mf,fl);
for each z in vdefli do depend Mf,z$

fl:=cons(R,fl);
for each z in vdefli do depend R,z$

for n:=1:3 do <<Cvp(n):=mkid(Cvp,n);
                fl:=cons(Cvp(n),fl);
                for each z in vdefli do depend Cvp(n),z
                >>$

for n:=1:3 do <<CH(n):=mkid(CH,n);
                fl:=cons(CH(n),fl);
                for each z in vdefli do depend CH(n),z
                >>$

J := Mf + R*p + for n:=2:3 sum (df(v(n),x(1))*Cvp(n))
                        + df(v(2),x(2))*Cvp(1)
                + for n:=2:3 sum (df(H(n),x(1))*CH(n))
                        + df(H(2),x(2))*CH(1))
                $

%the conservation law multipliers
%corresponding to the divergence free conditions
Q_7 := df(J,v(1)) $
Q_8 := df(J,H(1)) $

```

```

fl:=cons(a,fl);
fl:=cons(b,fl);

%the evolution equations of magnetic field components
%substitutions of e78
for n:=3 step -1 until 1 do eqn0:=cons(
df(H(n),t) = sub(e78, - (for m:=1:2 sum (v(m)*df(H(n),x(m))))
+ b*(for m:=1:2 sum df(H(n),x(m),2))
+ (for m:=1:2 sum (H(m)*df(v(n),x(m))))),eqn0)$

%the evolution equations of velocity vector components
%substitutions of e78
eqn0:=cons(
df(v(3),t) = sub(e78, - (for m:=1:2 sum (v(m)*df(v(3),x(m))))
+ a*(for m:=1:2 sum df(v(3),x(m),2))
+ (for m:=1:2 sum (H(m)*df(H(3),x(m))))),eqn0)$

for n:=2 step -1 until 1 do eqn0:=cons(
df(v(n),t) = sub(e78, - df(p,x(n))
- (for m:=1:2 sum (v(m)*df(v(n),x(m))))
+ a*(for m:=1:2 sum df(v(n),x(m),2))
- (for m:=1:3 sum (H(m)*df(H(m),x(n))))
+ (for m:=1:2 sum (H(m)*df(H(n),x(m))))),eqn0)$

%Calculation of the conservation laws multipliers,
%where the list of functions to be calculated includes of three functions
%W,Mf,R depended on t,x1,x2,v1,v2,v3,H1,H2,H3
%and the list of nonzero relations includes J
conlaw4({eqn0,jv1},{t,x1,x2},{0, 0, t, fl, {J}})$

end$

```

.3 Appendix III

```

%the procedure conlaw4 solves the multiplier determining equations
%for the rotation-invariant MHD system (6.2.8)

```

*%with respect to conservation law multipliers using the Crack routines [93]
 %and computes corresponding conserved densities and fluxes using
 %the direct integration or the formulas described in [5]*

```
load "~/crack/crfas1"$
load "~/crack/conlaw0"$
load "~/crack/conlaw4"$
```

```
lisp(depl!*:nil)$      % clearing of all dependencies
setcrackflags()$      % standart flags
lisp(print_:=nil)$     % no output of the calculation
```

```
off batch_mode$
```

```
%x1 =  $\rho$ , x2 = z
%v1 =  $v^\rho$ , v2 =  $v^\phi$ , v3 =  $v^z$ 
%H1 =  $B^\rho$ , H2 =  $B^\phi$ , H3 =  $B^z$ 
array x(2), % space coordinates
      v(3), % velocity
      H(3), % magnetic field
      Cvp(3),
      CH(3)$
```

```
%definition of the pressure p as a function of t, x1, x2
depend p,t,x1,x2$
```

```
%jet variable list
jv1:={p}$
```

```
for n:=1:2 do x(n):=mkid(x,n)$
for n:=1:3 do <<H(n):=mkid(H,n); jv1:=cons(H(n),jv1)$ depend H(n),t,x1,x2$
              v(n):=mkid(v,n); jv1:=cons(v(n),jv1)$ depend v(n),t,x1,x2
              >>$
```

```
%solved form of the divergence free equations of investigated MHD system
%where  $df(v(1),x(1))$  is  $v^\rho_{,\rho}$ ,  $df(H(1),x(1))$  is  $B^\rho_{,\rho}$ 
e78:= {df(v(1),x(1)) = -((1/x(1))*v(1)+df(v(3),x(2))),
      df(H(1),x(1)) = -((1/x(1))*H(1)+df(H(3),x(2)))}$
```

%eqn0 is MHD system under consideration

eqn0:=e78\$

%differential consequences of the divergence free equations

```
e78:=append({df(v(1),x(1),x(1)) = -df(((1/x(1))*v(1)+df(v(3),x(2))),x(1)),
            df(v(1),x(1),x(2)) = -df(((1/x(1))*v(1)+df(v(3),x(2))),x(2)),
            df(H(1),x(1),x(1)) = -df(((1/x(1))*H(1)+df(H(3),x(2))),x(1)),
            df(H(1),x(1),x(2)) = -df(((1/x(1))*H(1)+df(H(3),x(2))),x(2)) }
,e78)$
```

%conservation law multipliers corresponding to the evolution equations

array Q(6)\$

f1:={}\$ *% functions to be calculated*

vdefli:={t,x1,x2,v1,v2,v3,H1,H2,H3}\$

f1:=cons(W,f1); *% W is a conserved density*

for each z in vdefli do depend W,z\$

for i:=1:3 do <<Q(i):= df(W,v(i))>>\$

Q_1 := Q(1)\$

Q_2 := Q(2)\$

Q_3 := Q(3)\$

for i:=1:3 do <<Q(i + 3):= df(W,H(i))>>\$

Q_4 := Q(4)\$

Q_5 := Q(5)\$

Q_6 := Q(6)\$

f1:=cons(Mf,f1);

for each z in vdefli do depend Mf,z\$

f1:=cons(R,f1);

for each z in vdefli do depend R,z\$

```

for n:=1:3 do <<Cvp(n):=mkid(Cvp,n);
    fl:=cons(Cvp(n),fl);
    for each z in vdefli do depend Cvp(n),z
        >>$

for n:=1:3 do <<CH(n):=mkid(CH,n);
    fl:=cons(CH(n),fl);
    for each z in vdefli do depend CH(n),z
        >>$

J := Mf + R*p + df(v(2),x(1))*Cvp(1) + df(v(3),x(1))*Cvp(2)
    + df(v(3),x(2))*Cvp(3)
    + df(H(2),x(1))*CH(1) + df(H(3),x(1))*CH(2)
    + df(H(3),x(2))*CH(3)

$

%the conservation law multipliers
%corresponding to the divergence free conditions
Q_7 := df(J,v(1)) $
Q_8 := df(J,H(1)) $

fl:=cons(a,fl);
fl:=cons(b,fl);

%the evolution equations of magnetic field components
%substitutions of e78
eqn0:=cons(
    df(H(3),t) = sub(e78,-v(1)*df(h(3),x(1))-v(3)*df(h(3),x(2))
    +b*(df(h(3),x(1),2)+(1/x(1))*df(h(3),x(1))+df(h(3),x(2),2))
    +h(1)*df(v(3),x(1))+h(3)*df(v(3),x(2)))
    ,eqn0)$
eqn0:=cons(
    df(H(2),t) = sub(e78,-v(1)*df(h(2),x(1))-v(3)*df(h(2),x(2))
    -(1/x(1))*(h(1)*v(2))+b*(df(h(2),x(1),2)+(1/x(1))*df(h(2),x(1))
    +df(h(2),x(2),2)-(1/(x(1)^2))*h(2))+h(1)*df(v(2),x(1))
    +h(3)*df(v(2),x(2))+1/x(1))*(h(2)*v(1)))
    ,eqn0)$
eqn0:=cons(

```

```

df(H(1),t) = sub(e78,-v(1)*df(h(1),x(1))-v(3)*df(h(1),x(2))
+b*(df(h(1),x(1),2)+(1/x(1))*df(h(1),x(1))+df(h(1),x(2),2)
-(1/(x(1)^2))*h(1))+h(1)*df(v(1),x(1))+h(3)*df(v(1),x(2)))
,eqn0)$

%the evolution equations of velocity vector components
%substitutions of e78
eqn0:=cons(
df(v(3),t) = sub(e78,-v(1)*df(v(3),x(1))-v(3)*df(v(3),x(2))
-df(p,x(2))-h(2)*df(h(2),x(2))+h(1)*df(h(3),x(1))-h(1)*df(h(1),x(2))
+a*(df(v(3),x(1),2)+(1/x(1))*df(v(3),x(1))+df(v(3),x(2),2)))
,eqn0)$
eqn0:=cons(
df(v(2),t) = sub(e78,-v(1)*df(v(2),x(1))-v(3)*df(v(2),x(2))
-(1/x(1))*(v(1)*v(2))+h(1)*df(h(2),x(1))+h(3)*df(h(2),x(2))
+(1/x(1))*(h(1)*h(2))+a*(df(v(2),x(1),2)+(1/x(1))*df(v(2),x(1))
+df(v(2),x(2),2)-(1/(x(1)^2))*v(2)))
,eqn0)$
eqn0:=cons(
df(v(1),t) = sub(e78,-v(1)*df(v(1),x(1))-v(3)*df(v(1),x(2))
+(1/x(1))*(v(2)^2)-df(p,x(1))-h(2)*df(h(2),x(1))-h(3)*df(h(3),x(1))
+h(3)*df(h(1),x(2))-(1/x(1))*(h(2)^2)+a*(df(v(1),x(1),2)
+(1/x(1))*df(v(1),x(1))+df(v(1),x(2),2)-(1/(x(1)^2))*v(1)))
,eqn0)$

%Calculation of the conservation laws multipliers,
%where the list of functions to be calculated includes of three functions
%W,Mf,R depended on t,x1,x2,v1,v2,v3,H1,H2,H3
%and the list of nonzero relations includes J
conlaw4({eqn0,jv1},{t,x1,x2},{0, 0, t, f1, {J}})$

end$

```

.4 Appendix IV

```

%the procedure conlaw4 solves the multiplier determining equations
%for the helical transformation invariant MHD system (6.3.8)

```

*%with respect to conservation law multipliers using the Crack routines [93]
 %and computes corresponding conserved densities and fluxes using
 %the direct integration or the formulas described in [5]*

```
load "~/crack/crfasl"$
load "~/crack/conlaw0"$
load "~/crack/conlaw4"$
```

```
lisp(depl!*:nil)$      % clearing of all dependencies
setcrackflags()$     % standart flags
lisp(print_:=nil)$    % no output of the calculation
```

```
off batch_mode$
```

```
%x1 =  $\rho$ , x2 =  $\psi$ 
%v1 =  $v^\rho$ , v2 =  $v^\psi$ , v3 =  $v^z$ 
%H1 =  $B^\rho$ , H2 =  $B^\psi$ , H3 =  $B^z$ 
array x(2), % space coordinates
      v(3), % velocity
      H(3), % magnetic field
      Cvp(3),
      CH(3)$
```

```
for n:=1:2 do x(n):=mkid(x,n)$
```

```
%definition of the pressure p as a function of t, x1, x2
depend p,t,x1,x2$
```

```
%jet variable list
jvl:={p}$
```

```
for n:=1:3 do <<H(n):=mkid(H,n); jvl:=cons(H(n),jvl)$ depend H(n),t,x1,x2$
              v(n):=mkid(v,n); jvl:=cons(v(n),jvl)$ depend v(n),t,x1,x2
              >>$
```

```
%the Helical coordinate system Christoffel symbols
array gum(3,3,3)$
```

```

for n:=0:3 do for s:=0:3 do for k:=0:3 do gum(n,s,k):=0$
gum(1,2,2):= -x(1)$
gum(2,1,2):= gum(2,2,1):= 1/x(1)$
gum(3,1,2):= gum(3,2,1):= -m/x(1)$

G:=1/sqrt(m^2 + x(1)^2)$

%metric of the Helical coordinate system
array gm(3,3)$
for n:=0:3 do for k:=0:3 do gm(n,k):=0$
gm(1,1):= gm(3,3):= 1$
gm(2,2):= 1/G^2$
gm(2,3):= gm(3,2):= m$

%inverse metric of the Helical coordinate system
array ginv(3,3)$
for n:=0:3 do for k:=0:3 do ginv(n,k):=0$
ginv(1,1):= 1$
ginv(2,2):= 1/x(1)^2$
ginv(3,3):= 1/(x(1)*G)^2$
ginv(2,3):= ginv(3,2):= -m/(x(1)^2)$

%physical representation of the first and second order spatial derivatives
%of velocity vector components
array dv(3,3)$
for n:=0:3 do for k:=0:3 do dv(n,k):=0$
for n:=1:3 do for k:=1:2 do dv(n,k):=df(v(n)/sqrt(gm(n,n)),x(k))
+ (for i:=1:2 sum gum(n,i,k)*v(i)/sqrt(gm(i,i)))$

array ddv(3,3,3)$
for n:=0:3 do for k:=0:3 do for j:=0:3 do ddv(n,k,j):=0$
for n:=1:3 do for k:=1:2 do for j:=1:2 do ddv(n,k,j):=df(dv(n,k),x(j))
+ (for i:=1:2 sum (gum(n,i,j)*dv(i,k) - gum(i,k,j)*dv(n,i)))$

%physical representation of the first and second order spatial derivatives
%of magnetic field components
array dH(3,3)$
for n:=0:3 do for k:=0:3 do dH(n,k):=0$

```

```

for n:=1:3 do for k:=1:2 do dH(n,k):=df(H(n)/sqrt(gm(n,n)),x(k))
      + (for i:=1:2 sum gum(n,i,k)*H(i)/sqrt(gm(i,i)))$

array ddH(3,3,3)$
for n:=0:3 do for k:=0:3 do for j:=0:3 do ddH(n,k,j):=0$
for n:=1:3 do for k:=1:2 do for j:=1:2 do ddH(n,k,j):=df(dH(n,k),x(j))
      + (for i:=1:2 sum (gum(n,i,j)*dH(i,k) - gum(i,k,j)*dH(n,i)))$

%solved form of the divergence free equations of investigated MHD system
%where df(v(1),x(1)) is  $v^{\rho}_{,\rho}$ , df(H(1),x(1)) is  $B^{\rho}_{,\rho}$ 
e78:= df(v(1),x(1)) = -(1/x(1))*v(1)- G*df(v(2),x(2)),
      df(H(1),x(1)) = -(1/x(1))*H(1)- G*df(H(2),x(2))$

%eqn0 is MHD system under consideration
eqn0:=e78$

%differential consequences of the divergence free conditions
e78:=append({df(v(1),x(1),x(1)) = sub(e78,df(-(1/x(1))*v(1)
      - G*df(v(2),x(2)),x(1))),
      df(v(1),x(1),x(2)) = sub(e78,df(-(1/x(1))*v(1)
      - G*df(v(2),x(2)),x(2))),
      df(H(1),x(1),x(1)) = sub(e78,df(-(1/x(1))*H(1)
      - G*df(H(2),x(2)),x(1))),
      df(H(1),x(1),x(2)) = sub(e78,df(-(1/x(1))*H(1)
      - G*df(H(2),x(2)),x(2))) }
      ,e78)$

%conservation law multipliers corresponding to the evolution equations
array Q(6)$

fl:={} $ % functions to be calculated
vdefli:={t,x1,x2,v1,v2,v3,H1,H2,H3}$

fl:=cons(W,fl); % W is a conserved density
for each z in vdefli do depend W,z$

for i:=1:3 do <<Q(i):= df(W,v(i))>>$

```

```

Q_1 := Q(1)$
Q_2 := Q(2)$
Q_3 := Q(3)$

for i:=1:3 do <<Q(i + 3):= df(W,H(i))>>$

Q_4 := Q(4)$
Q_5 := Q(5)$
Q_6 := Q(6)$

fl:=cons(Mf,fl);
for each z in vdefli do depend Mf ,z$

fl:=cons(R,fl);
for each z in vdefli do depend R,z$

for n:=1:3 do <<Cvp(n):=mkid(Cvp,n);
                fl:=cons(Cvp(n),fl);
                for each z in vdefli do depend Cvp(n),z
                >>$

for n:=1:3 do <<CH(n):=mkid(CH,n);
                fl:=cons(CH(n),fl);
                for each z in vdefli do depend CH(n),z
                >>$

J := Mf + R*p + df(v(2),x(1))*Cvp(1) + df(v(3),x(1))*Cvp(2)
                + df(v(2),x(2))*Cvp(3)
                + df(H(2),x(1))*CH(1) + df(H(3),x(1))*CH(2)
                + df(H(1),x(2))*CH(3)

                $

%the conservation law multipliers
%corresponding to the divergence free conditions
Q_7 := df(J,v(1)) $
Q_8 := df(J,H(1)) $

fl:=cons(a,fl);

```

```

fl:=cons(b,fl);

%the evolution equations of magnetic field components
%substitutions of e78
for n:=3 step -1 until 1 do
eqn0:=cons(
df(H(n),t) = sub(e78, - sqrt(gm(n,n))*((
    for k:=1:2 sum (
        v(k)*dH(n,k)/sqrt(gm(k,k))
        - H(k)*dv(n,k)/sqrt(gm(k,k))
        - b*(for j:=1:2 sum (ginv(k,j)*ddH(n,k,j)))
    )))
    ),eqn0
)$

%the evolution equations of velocity vector components
%substitutions of e78
for n:=3 step -1 until 1 do
eqn0:=cons(
df(v(n),t) = sub(e78, - sqrt(gm(n,n))* (
    for k:=1:2 sum (
        ginv(n,k)*df(
            p + (
                for i:=1:3 sum (
                    for j:=1:3 sum (
                        (1/2)*gm(i,j)*H(i)*H(j)/sqrt(gm(i,i)*gm(j,j))
                    ))) ,x(k)
            )
        + v(k)*dv(n,k)/sqrt(gm(k,k))
        - H(k)*dH(n,k)/sqrt(gm(k,k))
        - a*(for j:=1:2 sum (ginv(k,j)*ddv(n,k,j)))
    ))
    ),eqn0
)$

%Calculation of the conservation laws multipliers,
%where the list of functions to be calculated includes of three functions
%W,Mf,R depended on t,x1,x2,v1,v2,v3,H1,H2,H3

```

```

%and the list of nonzero relations includes m, J, G
conlaw4({eqn0,jv1,{t,x1,x2}},{0, 0, t, f1, {m,J,G}})$

end$

```

.5 Appendix V

```

%the procedure liepde realizes the standard Lie symmetry algorithm [5]
%to compute the point infinitesimal symmetries of the MHD system (2.0.3)

```

```

load "~/crack/crfasl"$
load "~/crack/liepde"$

```

```

lisp(depl!*::=nil)$      % clearing of all dependencies
setcrackflags()$        % standart flags
lisp(print_:=nil)$      % no output of the calculation

```

```

%x1 =  $x^1$ , x2 =  $x^2$ , x3 =  $x^3$ 
%v1 =  $v^1$ , v2 =  $v^2$ , v3 =  $v^3$ 
%H1 =  $B^1$ , H2 =  $B^2$ , H3 =  $B^3$ 
array x(3), % space coordinates
      v(3), % velocity
      H(3)$ % magnetic field

```

```

%definition of the pressure p as a function of t, x1, x2, x3
depend p,t,x1,x2,x3$

```

```

%jet variable list
jv1:={p}$

```

```

for n:=1:3 do x(n):=mkid(x,n)$
for n:=1:3 do <<H(n):=mkid(H,n); jv1:=cons(H(n),jv1)$ depend H(n),t,x1,x2,x3$
              v(n):=mkid(v,n); jv1:=cons(v(n),jv1)$ depend v(n),t,x1,x2,x3
              >>$

```

```

%solved form of the divergence free equations of MHD system under consideration
%where  $df(v(1),x(1))$  is  $v^1_{,1}$ ,  $df(H(1),x(1))$  is  $B^1_{,1}$ 

```

```

e78:= {df(v(1),x(1)) = - df(v(2),x(2)) - df(v(3),x(3)),
      df(H(1),x(1)) = - df(H(2),x(2)) - df(H(3),x(3)) }$

%eqn0 is MHD system under consideration
eqn0:=e78$

%differential consequences of the divergence free conditions
e78:=append({df(v(1),x(1),x(1)) = df(- df(v(2),x(2)) - df(v(3),x(3)),x(1)),
            df(v(1),x(1),x(2)) = df(- df(v(2),x(2)) - df(v(3),x(3)),x(2)),
            df(v(1),x(1),x(3)) = df(- df(v(2),x(2)) - df(v(3),x(3)),x(3)),
            df(H(1),x(1),x(1)) = df(- df(H(2),x(2)) - df(H(3),x(3)),x(1)),
            df(H(1),x(1),x(2)) = df(- df(H(2),x(2)) - df(H(3),x(3)),x(2)),
            df(H(1),x(1),x(3)) = df(- df(H(2),x(2)) - df(H(3),x(3)),x(3)) }
            ,e78)$

%the evolution equations of magnetic field components
%substitutions of e78
for n:=3 step -1 until 1 do eqn0:=cons(
df(H(n),t) = sub(e78, - (for m:=1:3 sum (v(m)*df(H(n),x(m))))
                + b*(for m:=1:3 sum df(H(n),x(m),2))
                + (for m:=1:3 sum (H(m)*df(v(n),x(m))))),eqn0)$

%the evolution equations of velocity vector components
%substitutions of e78
for n:=3 step -1 until 1 do eqn0:=cons(
df(v(n),t) = sub(e78, - df(p,x(n))
                - (for m:=1:3 sum (v(m)*df(v(n),x(m))))
                + a*(for m:=1:3 sum df(v(n),x(m),2))
                - (for m:=1:3 sum (H(m)*df(H(m),x(n))))
                + (for m:=1:3 sum (H(m)*df(H(n),x(m))))),eqn0)$

%Calculation of the point symmetries,
%where the list of parameters to be calculated includes a,b
liepde({eqn0, jv1, {t,x1,x2,x3}},{ "point" },{a,b},{})$

end$

```

.6 Appendix VI

```

%the procedure liepde realizes the standard Lie symmetry algorithm [5]
%to compute the point infinitesimal symmetries of
%the translation-invariant MHD system (6.1.2)

load "~/crack/crfasl"$
load "~/crack/liepde"$

lisp(depl!*::=nil)$      % clearing of all dependencies
setcrackflags()$        % standart flags
lisp(print_:=nil)$      % no output of the calculation

off batch_mode$

%x1 = x1, x2 = x2
%v1 = v1, v2 = v2, v3 = v3
%H1 = B1, H2 = B2, H3 = B3
array x(2), % space coordinates
      v(3), % velocity
      H(3)$ % magnetic field

%definition of the pressure p as a function of t, x1, x2
depend p,t,x1,x2$

%jet variable list
jvl:={p}$

for n:=1:2 do x(n):=mkid(x,n)$
for n:=1:3 do <<H(n):=mkid(H,n); jvl:=cons(H(n),jvl)$ depend H(n),t,x1,x2$
              v(n):=mkid(v,n); jvl:=cons(v(n),jvl)$ depend v(n),t,x1,x2
              >>$

%solved form of the divergence free equations of investigated MHD system
%where df(v(1),x(1)) is v1,1, df(H(1),x(1)) is B1,1
e78:= {df(v(1),x(1)) = - df(v(2),x(2)),
      df(H(1),x(1)) = - df(H(2),x(2)) }$

```

```

%eqn0 is MHD system under consideration
eqn0:=e78$

%differential consequences of the divergence free conditions
e78:=append({df(v(1),x(1),x(1)) = - df(v(2),x(2),x(1)),
             df(v(1),x(1),x(2)) = - df(v(2),x(2),x(2)),
             df(H(1),x(1),x(1)) = - df(H(2),x(2),x(1)),
             df(H(1),x(1),x(2)) = - df(H(2),x(2),x(2)) }
            ,e78)$

%the evolution equations of magnetic field components
%substitutions of e78
for n:=3 step -1 until 1 do eqn0:=cons(
df(H(n),t) = sub(e78, - (for m:=1:2 sum (v(m)*df(H(n),x(m))))
                    + b*(for m:=1:2 sum df(H(n),x(m),2))
                    + (for m:=1:2 sum (H(m)*df(v(n),x(m))))),eqn0)$

%the evolution equations of velocity vector components
%substitutions of e78
eqn0:=cons(
df(v(3),t) = sub(e78, - (for m:=1:2 sum (v(m)*df(v(3),x(m))))
                    + a*(for m:=1:2 sum df(v(3),x(m),2))
                    + (for m:=1:2 sum (H(m)*df(H(3),x(m))))),eqn0)$

for n:=2 step -1 until 1 do eqn0:=cons(
df(v(n),t) = sub(e78, - df(p,x(n))
                    - (for m:=1:2 sum (v(m)*df(v(n),x(m))))
                    + a*(for m:=1:2 sum df(v(n),x(m),2))
                    - (for m:=1:3 sum (H(m)*df(H(m),x(n))))
                    + (for m:=1:2 sum (H(m)*df(H(n),x(m))))),eqn0)$

%Calculation of the point symmetries,
%where the list of parameters to be calculated includes a,b
liepde({eqn0, jv1, {t,x1,x2}},{ "point" },{a,b},{})$

end$

```

.7 Appendix VII

```

%the procedure liepde realizes the standard Lie symmetry algorithm [5]
%to compute the point infinitesimal symmetries of
%the rotation-invariant MHD system (6.2.8)

load "~/crack/crfasl"$
load "~/crack/liepde"$

lisp(depl!*::=nil)$      % clearing of all dependencies
setcrackflags()$       % standart flags
lisp(print_:=nil)$      % no output of the calculation

off batch_mode$

%x1 =  $\rho$ , x2 = z
%v1 =  $v^\rho$ , v2 =  $v^\phi$ , v3 =  $v^z$ 
%H1 =  $B^\rho$ , H2 =  $B^\phi$ , H3 =  $B^z$ 
array x(2), % space coordinates
      v(3), % velocity
      H(3)$ % magnetic field

%definition of the pressure p as a function of t, x1, x2
depend p,t,x1,x2$

%jet variable list
jv1:={p}$

for n:=1:2 do x(n):=mkid(x,n)$
for n:=1:3 do <<H(n):=mkid(H,n); jv1:=cons(H(n),jv1)$ depend H(n),t,x1,x2$
              v(n):=mkid(v,n); jv1:=cons(v(n),jv1)$ depend v(n),t,x1,x2
              >>$

% solved form of the divergence free equations of investigated MHD system
%where  $df(v(1),x(1))$  is  $v^\rho_{,\rho}$ ,  $df(H(1),x(1))$  is  $B^\rho_{,\rho}$ 
e78:={df(v(1),x(1)) = -((1/x(1))*v(1)+df(v(3),x(2))),
      df(H(1),x(1)) = -((1/x(1))*H(1)+df(H(3),x(2)))}$

```


%eqn0 is MHD system under consideration

eqn0:=e78\$

%differential consequences of the divergence free equations

```
e78:=append({df(v(1),x(1),x(1)) = -df(((1/x(1))*v(1)+df(v(3),x(2))),x(1)),
            df(v(1),x(1),x(2)) = -df(((1/x(1))*v(1)+df(v(3),x(2))),x(2)),
            df(H(1),x(1),x(1)) = -df(((1/x(1))*H(1)+df(H(3),x(2))),x(1)),
            df(H(1),x(1),x(2)) = -df(((1/x(1))*H(1)+df(H(3),x(2))),x(2)) }
            ,e78)$
```

%the evolution equations of magnetic field components

%substitutions of e78

eqn0:=cons(

```
df(H(3),t) = sub(e78,-v(1)*df(h(3),x(1))-v(3)*df(h(3),x(2))
+b*(df(h(3),x(1),2)+(1/x(1))*df(h(3),x(1))+df(h(3),x(2),2))
+h(1)*df(v(3),x(1))+h(3)*df(v(3),x(2)))
```

,eqn0)\$

eqn0:=cons(

```
df(H(2),t) = sub(e78,-v(1)*df(h(2),x(1))-v(3)*df(h(2),x(2))
-(1/x(1))*(h(1)*v(2))+b*(df(h(2),x(1),2)+(1/x(1))*df(h(2),x(1))
+df(h(2),x(2),2)-(1/(x(1)^2))*h(2))+h(1)*df(v(2),x(1))
+h(3)*df(v(2),x(2))+(1/x(1))*(h(2)*v(1)))
```

,eqn0)\$

eqn0:=cons(

```
df(H(1),t) = sub(e78,-v(1)*df(h(1),x(1))-v(3)*df(h(1),x(2))
+b*(df(h(1),x(1),2)+(1/x(1))*df(h(1),x(1))+df(h(1),x(2),2)
-(1/(x(1)^2))*h(1))+h(1)*df(v(1),x(1))+h(3)*df(v(1),x(2)))
```

,eqn0)\$

%the evolution equations of velocity vector components

%substitutions of e78

eqn0:=cons(

```
df(v(3),t) = sub(e78,-v(1)*df(v(3),x(1))-v(3)*df(v(3),x(2))
-df(p,x(2))-h(2)*df(h(2),x(2))+h(1)*df(h(3),x(1))-h(1)*df(h(1),x(2))
+a*(df(v(3),x(1),2)+(1/x(1))*df(v(3),x(1))+df(v(3),x(2),2)))
```

,eqn0)\$

eqn0:=cons(

```
df(v(2),t) = sub(e78,-v(1)*df(v(2),x(1))-v(3)*df(v(2),x(2))
```

```

        -(1/x(1))*(v(1)*v(2))+h(1)*df(h(2),x(1))+h(3)*df(h(2),x(2))
        +(1/x(1))*(h(1)*h(2))+a*(df(v(2),x(1),2)+(1/x(1))*df(v(2),x(1))
        +df(v(2),x(2),2)-(1/(x(1)^2))*v(2)))
    ,eqn0)$
eqn0:=cons(
    df(v(1),t) = sub(e78,-v(1)*df(v(1),x(1))-v(3)*df(v(1),x(2))
    +(1/x(1))*(v(2)^2)-df(p,x(1))-h(2)*df(h(2),x(1))-h(3)*df(h(3),x(1))
    +h(3)*df(h(1),x(2))-(1/x(1))*(h(2)^2)+a*(df(v(1),x(1),2)
    +(1/x(1))*df(v(1),x(1))+df(v(1),x(2),2)-(1/(x(1)^2))*v(1)))
    ,eqn0)$

%Calculation of the point symmetries,
%where the list of parameters to be calculated includes a,b
liepde({eqn0, jv1, {t,x1,x2}},{ "point" },{a,b},{})$

end$

```

.8 Appendix VIII

```

%the procedure liepde realizes the standard Lie symmetry algorithm [5]
%to compute the point infinitesimal symmetries of
%the helical transformation invariant MHD system (6.3.8)

```

```

load "~/crack/crfasl"$
load "~/crack/liepde"$

```

```

lisp(depl!*::=nil)$      % clearing of all dependencies
setcrackflags()$       % standart flags
lisp(print_:=nil)$      % no output of the calculation

```

```

off batch_mode$

```

```

%x1 =  $\rho$ , x2 =  $\psi$ 
%v1 =  $v^\rho$ , v2 =  $v^\psi$ , v3 =  $v^z$ 
%H1 =  $B^\rho$ , H2 =  $B^\psi$ , H3 =  $B^z$ 
array x(2), % space coordinates
      v(3), % velocity

```

```

H(3)$ % magnetic field

for n:=1:2 do x(n):=mkid(x,n)$

%definition of the pressure p as a function of t, x1, x2
depend p,t,x1,x2$

%jet variable list
jvl:={p}$

for n:=1:3 do <<H(n):=mkid(H,n); jvl:=cons(H(n),jvl)$ depend H(n),t,x1,x2$
      v(n):=mkid(v,n); jvl:=cons(v(n),jvl)$ depend v(n),t,x1,x2
    >>$

%the Helical coordinate system Christoffel symbols
array gum(3,3,3)$
for n:=0:3 do for s:=0:3 do for k:=0:3 do gum(n,s,k):=0$
gum(1,2,2):= -x(1)$
gum(2,1,2):= gum(2,2,1):= 1/x(1)$
gum(3,1,2):= gum(3,2,1):= -m/x(1)$

G:=1/sqrt(m^2 + x(1)^2)$

%metric of the Helical coordinate system
array gm(3,3)$
for n:=0:3 do for k:=0:3 do gm(n,k):=0$
gm(1,1):= gm(3,3):= 1$
gm(2,2):= 1/G^2$
gm(2,3):= gm(3,2):= m$

%inverse metric of the Helical coordinate system
array ginv(3,3)$
for n:=0:3 do for k:=0:3 do ginv(n,k):=0$
ginv(1,1):= 1$
ginv(2,2):= 1/x(1)^2$
ginv(3,3):= 1/(x(1)*G)^2$
ginv(2,3):= ginv(3,2):= -m/(x(1)^2)$

```

```

%physical representation of the first and second order spatial derivatives
%of velocity vector components
array dv(3,3)$
for n:=0:3 do for k:=0:3 do dv(n,k):=0$
for n:=1:3 do for k:=1:2 do dv(n,k):=df(v(n)/sqrt(gm(n,n)),x(k))
      + (for i:=1:2 sum gum(n,i,k)*v(i)/sqrt(gm(i,i)))$

array ddv(3,3,3)$
for n:=0:3 do for k:=0:3 do for j:=0:3 do ddv(n,k,j):=0$
for n:=1:3 do for k:=1:2 do for j:=1:2 do ddv(n,k,j):=df(dv(n,k),x(j))
      + (for i:=1:2 sum (gum(n,i,j)*dv(i,k) - gum(i,k,j)*dv(n,i)))$

%physical representation of the first and second order spatial derivatives
%of magnetic field components
array dH(3,3)$
for n:=0:3 do for k:=0:3 do dH(n,k):=0$
for n:=1:3 do for k:=1:2 do dH(n,k):=df(H(n)/sqrt(gm(n,n)),x(k))
      + (for i:=1:2 sum gum(n,i,k)*H(i)/sqrt(gm(i,i)))$

array ddH(3,3,3)$
for n:=0:3 do for k:=0:3 do for j:=0:3 do ddH(n,k,j):=0$
for n:=1:3 do for k:=1:2 do for j:=1:2 do ddH(n,k,j):=df(dH(n,k),x(j))
      + (for i:=1:2 sum (gum(n,i,j)*dH(i,k) - gum(i,k,j)*dH(n,i)))$

%solved form of the divergence free equations of investigated MHD system
%where df(v(1),x(1)) is  $v^{\rho}_{,\rho}$ , df(H(1),x(1)) is  $B^{\rho}_{,\rho}$ 
e78:= {df(v(1),x(1)) = -(1/x(1))*v(1)- G*df(v(2),x(2)),
      df(H(1),x(1)) = -(1/x(1))*H(1)- G*df(H(2),x(2))}$

%eqn0 is MHD system under consideration
eqn0:=e78$

%differential consequences of the divergence free equations
e78:=append({df(v(1),x(1),x(1)) = sub(e78,df(-(1/x(1))*v(1)
      - G*df(v(2),x(2)),x(1))),
      df(v(1),x(1),x(2)) = sub(e78,df(-(1/x(1))*v(1)
      - G*df(v(2),x(2)),x(2))),
      df(H(1),x(1),x(1)) = sub(e78,df(-(1/x(1))*H(1)

```

```

- G*df(H(2),x(2)),x(1))),
df(H(1),x(1),x(2)) = sub(e78,df(-(1/x(1))*H(1)
- G*df(H(2),x(2)),x(2)))) }
,e78)$

```

%the evolution equations of magnetic field components

%substitutions of e78

```
for n:=3 step -1 until 1 do
```

```
eqn0:=cons(
```

```
df(H(n),t) = sub(e78, - sqrt(gm(n,n))*((
for k:=1:2 sum (
v(k)*dH(n,k)/sqrt(gm(k,k))
- H(k)*dv(n,k)/sqrt(gm(k,k))
- b*(for j:=1:2 sum (ginv(k,j)*ddH(n,k,j))
)))
```

```
),eqn0
```

```
)$
```

%the evolution equations of velocity vector components

%substitutions of e78

```
for n:=3 step -1 until 1 do
```

```
eqn0:=cons(
```

```
df(v(n),t) = sub(e78, - sqrt(gm(n,n))*((
for k:=1:2 sum (
ginv(n,k)*df(
p + (
for i:=1:3 sum (
for j:=1:3 sum (
(1/2)*gm(i,j)*H(i)*H(j)/sqrt(gm(i,i)*gm(j,j))
))),x(k)
)

```

```
+ v(k)*dv(n,k)/sqrt(gm(k,k))
```

```
- H(k)*dH(n,k)/sqrt(gm(k,k))
```

```
- a*(for j:=1:2 sum (ginv(k,j)*ddv(n,k,j))
))
```

```
),eqn0
```

```
)$
```

```
%Calculation of the point symmetries,  
%where the list of parameters to be calculated includes m,a,b  
%and the list of nonzero relations includes G  
liepde({eqn0, jv1, {t,x1,x2}},{"point"},{m,a,b},{m,G})$  
  
end$
```