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# Acyclic 5-Choosability of Planar Graphs Without Adjacent Short Cycles

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## Abstract

The conjecture claiming that every planar graph is acyclic 5-choosable [Borodin et al., 2002] has been verified for several restricted classes of planar graphs. Recently, O. V. Borodin and A. O. Ivanova, [Journal of Graph Theory, 68(2), October 2011, 169-176], have shown that a planar graph is acyclically 5-choosable if it does not contain an  $i$ -cycle adjacent to a  $j$ -cycle, where  $3 \leq j \leq 5$  if  $i = 3$  and  $4 \leq j \leq 6$  if  $i = 4$ . We improve the above mentioned result and prove that every planar graph without an  $i$ -cycle adjacent to a  $j$ -cycle with  $3 \leq j \leq 5$  if  $i = 3$  and  $4 \leq j \leq 5$  if  $i = 4$  is acyclically 5-choosable.

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# Chapter 1

## Introduction

Graph theory is the study of graphs, which are structures which model items and the relations existing between these items. A graph contains a set of vertices, and a set of edges. For example, one can think of cities as vertices on a map, and an edge; i.e., a line with vertices as its endpoints, would exist from one vertex to another vertex if there was a road connecting these two cities. As in real life applications, these edges can be *directed*, as in the case of a one way street, or *undirected*, such as a street in which traffic flows in both directions.

Let  $G = (V, E)$  denote a *graph* with vertex set  $V(G)$  and edge set  $E(G)$ . Two vertices  $u$  and  $v$  in  $V(G)$  are called *adjacent* if they are endpoints of an edge in  $E(G)$ . This edge is denoted as  $uv$ . A *proper vertex colouring* of  $G$  is one in which no edge is monochromatic; i.e., adjacent vertices are assigned different colours.



A  $k$ -colouring of  $G$  is a partition  $V(G) = V_1 \cup \dots \cup V_k$  of the vertices of  $G$  into  $k$  pairwise disjoint sets. If  $G$  can be properly coloured with  $k$  colours then we say  $G$  is  $k$ -colourable. The chromatic number of  $G$ , denoted  $\chi(G)$ , is the least number of colours required to properly colour  $G$ ; i.e.,  $\chi(G) = k$  if  $G$  can be properly coloured with  $k$  colours but not with  $k - 1$  colours.

A proper vertex colouring of a graph  $G$  is *acyclic* if  $G$  contains no bi-coloured cycle; i.e., no cycle is coloured with only two colours. If the fewest number of colours required to acyclically colour a graph  $G$  is  $k$ , then the acyclic chromatic number  $\chi_a(G) = k$ .

Consider a list assignment  $L(v)$ , where by a list of admissible colours is assigned to each vertex  $v$  of  $G$ . We say  $G$  is  *$L$ -list colourable* if there exists a proper vertex colouring of  $G$  with this list assignment  $L(v)$  such that each  $v$  is coloured with one of its admissible colours, and the colouring is proper. If, regardless of how these lists are assigned to the vertices of  $G$ , there exists a proper  $L$ -list colouring for every list assignment with  $|L(v)| \geq k$  for every vertex  $v \in G$ , then  $G$  is  $k$ -choosable. That is, if any one vertex is assigned a list of  $k - 1$  admissible colours, then there exists some list assignment in which a proper colouring is not possible. The list-chromatic number  $\chi^l(G)$  is the smallest integer  $k$  such that  $|L(v)| \geq k$  for every vertex  $v \in G$  which produces such a colouring. If a graph  $G$  can be acyclically coloured for all list assignment with  $|L(v)| \geq k$ , then  $G$  is *acyclically  $k$ -choosable* and the acyclic list-chromatic number  $\chi_a^l(G) = k$ .

## 1.1 History

In 1973, Grünbaum [Grü73] defined a proper colouring of a graph to be acyclic if every cycle uses at least three colours and proved that every planar graph is acyclically 9-colourable (a *planar graph* is a graph which can be embedded in a plane such that two edges never cross). It was also, at this time, conjectured that every planar graph has an acyclic 5-colouring. He provided an example of a planar graph which is not acyclically 4-colourable (refer to Figure 1.1), making 5 the best possible bound. Borodin [Bor79] proved Grünbaums conjecture in 2002, improving earlier bounds of 8, 7, and 6, attributed to Mitchem [Mit74], Albertson and Berman [AB77], and Kostochka [Kos76], respectively. To further demonstrate that 5 is the best bound, there are bipartite 2-degenerate planar graphs that are not acyclically 4-colourable [KM76]. Note that a bipartite  $d$ -degenerate planar graph  $H$  is a planar graph such that every subgraph of  $H$  has a vertex of degree at most  $d$  and can be partitioned into two disjoint set of non-adjacent vertices.

Thomassen [Tho94] proved that every planar graph is 5-choosable. This is the best possible bound, as Voigt [Voi93] presented a non-4-choosable planar graph. Borodin [Bor79] conjectured

**Conjecture 1.** *Every planar graph is acyclically 5-choosable.*

A proof of this conjecture would improve the results of both Borodin [Bor79] and Thomassen [Tho94].

Borodin et al. [BFDFK<sup>+</sup>02] proved that every planar graph is acyclically

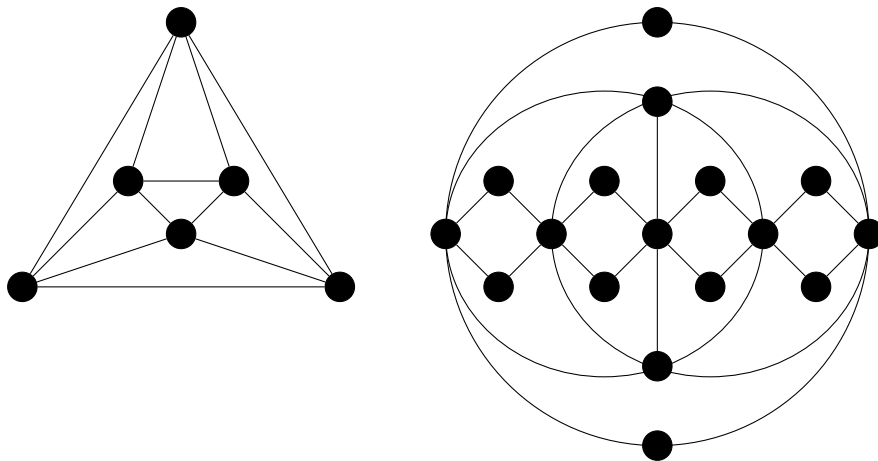


Figure 1.1: Grünbaum's example and Kostochka-Mel'nikov's example.

7-choosable. As of yet, proofs for the  $k$ -choosability,  $3 \leq k \leq 6$  of planar graphs have been verified for restricted classes of planar graphs [BCIR10] [Bor09] [CR10] [Mon07] [WC09]. Work on the 5-choosability of planar graphs, such as those graphs

- with girth at least five [MOR06],
- without 4- and 5-cycles or without 4- and 6-cycles [MRW07],
- with neither 4-cycles nor chordal 6-cycles [ZX09],
- with neither 4-cycles nor two 3-cycles at distance less than 3 [CW08],
- and without 4-cycles and intersecting 3-cycles [CR12],

have recently been absorbed by Borodin and Ivanova [BI11] which proved that a planar graph is acyclically 5-choosable if it does not contain an  $i$ -cycle

adjacent to a  $j$ -cycle, where  $3 \leq j \leq 5$  if  $i = 3$  and  $4 \leq j \leq 6$  if  $i = 4$ . The purpose of this thesis is to improve this last result by allowing the adjacency of 4- and 6-cycles which were previously forbidden.

**Theorem 1.** *Every planar graph without an  $i$ -cycle adjacent to a  $j$ -cycle with  $3 \leq j \leq 5$  if  $i = 3, 4$  is acyclically 5-choosable.*

The method of proof will be by contradiction, and shown in two parts using the discharging method. First we assume for contradiction that there exists a graph  $G$  with the necessary restrictions, i.e., contains no short cycles which have at least one edge in common, but cannot be acyclically 5-coloured with an arbitrary list assignment for  $v \in G$  such that  $|L(v)| \geq 5$ . Then we determine some notion of the structure of  $G$ . This is achieved though finding reducible configurations that would lead to a contradiction in the assumption that  $G$  is not 5-choosable. Finally we assign charges to each vertex and to each face of  $G$  such that the total charge of  $G$  is negative. We carefully redistribute those charges and show that, without adding or removing charge, the total charge of  $G$  is non-negative. Hence a contradiction, meaning the assumption of the existence of such a graph  $G$  was incorrect.

## 1.2 Overview

The purpose of Chapter 2 is to help familiarize the reader with notation that is commonly used in the study of Graph Theory and to notation used specific to Colouring Problems and this thesis. Chapter 3 is a summary reducible

configurations that has been previously proved by other authors who are examining the problem of planar graphs and there 5-choosability. Chapter 4 is comprised of three different parts. Firstly, the list of reducible configurations is expanded, as some new configurations are considered. Next, the proof of Theorem 1 is started by providing the reader a list of rules by which the discharging will follow. Lastly, an examination of the new charges carried by each face and each vertex of the minimum counterexample is calculated. Finally, this thesis will end in Chapter 5 with a look at the future of the answer to the question of the 5-choosability of planar graphs.

# Chapter 2

## Preliminaries and Notions

In this chapter, we will become familiar with the notation used in the next few chapters.

### 2.1 Vertices, Edges and Faces

A  $k$ -vertex,  $k^+$ -vertex and  $k^-$ -vertex is a vertex of degree  $k$ , at least  $k$  and at most  $k$ , respectively. The degree of a vertex refers to the number of edges incident to it.

Similarly, we define a  $k$ -face,  $k^+$ -face,  $k^-$ -face where a face  $f$  has exactly, at least, or at most  $k$  edges in its boundary, respectively.

A vertex  $v$  which is the endpoint of an edge in the boundary of a face is said to be *incident* to that face. A triangle is synonymous with a 3-face. It is a region enclosed by three successively adjacent vertices embedded in a plane. It has three boundary edges and, as expected, each triangle has three

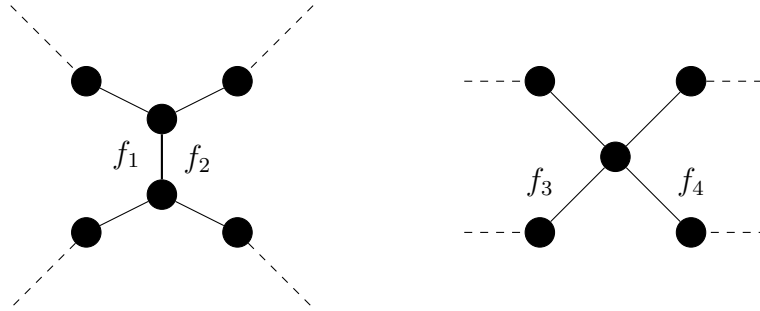


Figure 2.1: Two adjacent faces  $f_1$  and  $f_2$ . Two intersecting faces  $f_3$  and  $f_4$ .

incident vertices.

Usually, a face  $f \in F(G)$ ; the set of faces of an embedded graph  $G$ , is written as  $f = [u_1, u_2, \dots, u_n]$  if  $u_1, u_2, \dots, u_n$  are the boundary vertices of  $f$  in a cyclic order.

Two cycles are *adjacent* if they have two consecutive adjacent vertices  $v_i$  and  $v_{i+1}$  in common as boundary vertices. Otherwise, for the purposes of this thesis, two cycles *intersect* if they share common vertices but are not adjacent. If two cycles are not adjacent in a graph  $G$ , then they are not adjacent faces in the embedding of  $G$  (see Fig.2.1).

## 2.2 Neighbourhoods

For a vertex  $v$  and an integer  $i \geq 1$ , let  $t(v)$  denote the number of 3-faces incident to  $v$  and let  $n_i(v)$  denote the number of  $i$ -vertices adjacent to  $v$ .

For a face  $f \in F(G)$  and an integer  $j \geq 2$ , let  $n_j(f)$  denote the number of  $j$ -vertices incident to  $f$ .

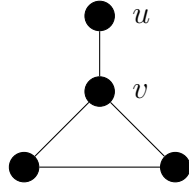


Figure 2.2: A pendant light 3-vertex  $v$  of  $u$ .

Let  $N(v)$  denote the set of neighbours of a vertex  $v$ . In other words,  $N(v)$  refers to the number of vertices adjacent to  $v$ .

A 3-face  $f = [v_1v_2v_3]$  is called an  $(a_1, a_2, a_3)$ -face if the degree of the vertex  $v_i$  is  $a_i$  for  $i = 1, 2, 3$ . Similarly we define a 4-face  $f = [v_1v_2v_3v_4]$  as being a  $(a_1, a_2, a_3, a_4)$ -face if the degree of the vertex  $v_i$  is  $a_i$  for  $i = 1, 2, 3, 4$ .

## 2.3 Pendant Light Vertices

A 3-vertex  $v$  is *pendant* if it is incident to a 3-face. If a vertex  $u$  is adjacent to a 3-vertex  $v$  such that the edge  $uv$  is not in the boundary of a 3-face, then the edge  $uv$  is called *light*. Hence, we call  $v$  a pendant light 3-vertex of  $u$  if  $uv$  is light and  $v$  is pendant (see Fig. 2.2).

If  $v$  is a pendant light 3-vertex of  $u$  which is incident to an  $(a_1, a_2, a_3)$ -face, then we call  $v$  a pendant light  $(a_1, a_2, a_3)$ -vertex of  $u$ . Let  $p_3(u)$  denote the number of pendant light 3-vertices of a vertex  $u$ .



## 2.4 Path Colouring

A path is a set of vertices  $v_1, \dots, v_k$  and edges  $e_1, \dots, e_{k-1}$  which are in sequential order  $v_1, e_1, \dots, e_{k-1}, v_k$ . Let  $\alpha$  and  $\beta$  be any two distinct colours.

An *alternating*  $(\alpha, \beta)$ -*path* in  $G$  is a path where by each vertex is coloured  $\alpha$  or  $\beta$  in alternating order. A path  $v_1, e_1, \dots, e_{k-1}, v_k$  is called a cycle if  $v_1 = v_k$ .

# Chapter 3

## Reducible Configurations - Past Work

For all figures in this thesis, a vertex is represented by a solid point  $\bullet$  when all of its incident edges are indicated; otherwise it is represented by a hollow point  $\circ$ . Letters which appear early in the alphabet, such as  $a, b, c, \dots$  will denote a colour which is assigned to a vertex. Indexed colours  $c_1, c_2, \dots$  are colours assigned to indexed vertices  $v_1, v_2, \dots$  and are distinct colours unless otherwise specified. Lastly, letters which appear late in the alphabet, such as  $u, v, \dots, y, z$  will denote vertices.

To show the acyclic 5-choosability of every planar graph which does not contain an  $i$ -cycle adjacent to a  $j$ -cycle for  $i = 3, 4$  and  $j = 3, 4, 5$ , assume for contradiction that there exists a planar graph  $G$  which does not contain the above adjacent cycles, is not acyclically 5-choosable. This graph  $G$  is a minimum planar graph; i.e., it can be embedded in a plane without any of its edges crossing, and has the fewest number of vertices such that it cannot

be acyclically *List* 5-coloured. Firstly, we assume  $G$  is a connected planar graph; that is, every two vertices in  $G$  belong to a path in  $G$ . As per the assumption that  $G$  is minimum, each component  $C_1, \dots, C_k$  of a graph which is not connected is acyclically 5-choosable. Hence  $G$  is acyclically 5-choosable which contradicts our choice of  $G$ . Next, we shall assume that  $G$  is simple. A *simple graph* is a graph with no loops or multiple edges. A *loop* is an edge  $uv$  such that  $u$  and  $v$  are the same vertex. Obviously, a loop cannot be properly coloured. *Multiple edges* are edges a set of two or more edges which share the same two endpoints. Obviously, every multiple cannot be acyclically coloured.

The following is a list of reducible configuration that are not in  $G$ . If any of these configurations where in  $G$ , then one or more vertices can be removed, the remaining graph can be acyclically *List* 5-coloured and this colouring can be extended to the vertices that were removed. Hence the graph  $G$  would be acyclically *List* 5-coloured.

**Lemma 1.** *As a consequence of  $G$  being a minimum counterexample, the following conditions hold:*

(C1) *There are no 1-vertices. [MRW07]*

(C2) *No 2-vertex is adjacent to a  $4^-$ -vertex. [MRW07]*

(C3) *Let  $v$  be a 3-vertex.*

(C3.1) *If  $v$  is adjacent to a 3-vertex, then  $v$  is not adjacent to any other  $4^-$ -vertex. [MRW07]*

(C3.2)  *$v$  has no pendant 3-vertex. [MRW07]*

(C4) *Let  $v$  be a 4-vertex, then  $v$  has no pendant 3-vertex. [BI11].*

(C5) *Let  $v$  be a 5-vertex.*

(C5.1)  *$v$  is adjacent to at most one 2-vertex. [MRW07]*

(C5.2) *If  $n_2(v) = 1$ , then  $v$  has no pendant 3-vertex. [MRW07]*

(C5.3) *If  $n_2(v) = 1$  and  $v$  is incident to a 3-face, then  $n_3(f) = 0$ . [CW08]*

(C5.4) *If  $n_2(v) = 0$ , then  $p_3(v) \leq 3$ . [CR12]*

(C5.5) *If  $n_2(v) = 0$  and  $t(v) = 1$ , then  $p_3(v) \leq 2$ . [CR12]*

(C5.6) *If  $v$  is incident to a  $(5, 3, 4)$ -face, then  $p_3(v) \leq 1$ . [CR12]*

(C6) *Let  $v$  be a 6-vertex.*

(C6.1)  *$v$  is adjacent to at most four 2-vertices. [MRW07]*

(C6.2) *If  $n_2(v) = 4$ , then  $v$  is not adjacent to any 3-vertex. [MRW07]*

(C6.3) *If  $n_2(v) = 4$ , then  $t(v) = 0$ . [CW08]*

(C6.4) *If  $n_2(v) = 3$  and  $t(v) = 1$ , then  $p_3(v) = 0$ . [CR12]*

(C6.5) If  $n_2(v) = 2$ , then  $p_3(v) \leq 2$ . [CR12]

(C6.6) If  $n_2(v) = 2$ , and  $t(v) = 1$ , then  $p_3(v) \leq 1$ . [CR12]

(C6.7) If  $n_2(v) = 0$  and  $v$  is incident to a  $(3, 3, 6)$ -face, then  $p_3(v) \leq 2$ . [CR12]

(C6.8) If  $v$  is incident to a  $(3, 4, 6)$ -face, then  $n_2(v) \leq 2$ . [CR12]

(C7) Let  $v$  be a 7-vertex.

(C7.1)  $v$  is adjacent to at most five 2-vertices. [MRW07]

(C7.2) If  $n_2(v) = 4$ , then  $n_3(v) \leq 2$ . [CW08]

(C7.3) If  $n_2(v) = 5$ , then  $n_3(v) = 0$  and  $t(v) = 0$ . [CR12]

(C7.4) If  $n_2(v) = 4$  and  $t(v) = 1$ , then  $p_3(v) = 0$ . [CR12]

(C7.5) If  $n_2(v) = 3$  and  $v$  is incident to a  $(7, 3, 3)$ -face, then  $p_3(v) \leq 1$ . [CR12]

(C8) Let  $v$  be a 8-vertex.

(C8.1)  $v$  is adjacent to at most six 2-vertices. [CR12]

(C8.2) If  $t(v) = 1$ , then  $n_2(v) \leq 5$ . [CR12]

(C9) Let  $f$  be a 3-face  $[xyz]$  with  $d(x) \leq d(y) \leq d(z)$ .

(C9.1)  $d(x) \neq 2$ . [MRW07]

(C9.2)  $[xyz]$  does not satisfy  $d(x) = d(y) = 3$  and  $d(z) \leq 5$ . [MRW07]

(C9.3)  $[xyz]$  does not satisfy  $d(x) = 3$  and  $d(y) = d(z) = 4$ . [MRW07]

(C10) There does not exist a 5-face  $[x_1x_2x_3x_4x_5]$  such that  $d(x_1) = 2$ ,  $d(x_3) = 3$ , and  $d(x_2) = 5$ . [MRW07]

The following section is a proof of Lemma 1 and will demonstrate how the presence of these configurations would lead to a contradiction in the choice of  $G$  as a minimum counterexample.

### 3.1 Restrictions on a 1-vertex

*Proof.* (C1) Suppose that there exists a 1-vertex  $v \in V(G)$  adjacent to the vertices  $u$ , as in Figure 3.1.



Figure 3.1: A 1-vertex in  $G$ .

By minimality of  $G$ , the graph  $G - \{v\}$  admits acyclically list 5-colouring  $\pi$ . Extend this colouring of  $G - \{v\}$  to  $G$  by letting  $\pi(v)$  be a colour in  $L(v)$  different than  $\pi(u)$ . Now  $G$  is acyclically 5-choosable, which contradicts the choice of  $G$ . Hence  $G$  does not contain a 1-vertex.  $\square$

## 3.2 Restrictions on a 2-vertex

*Proof.* (C2) Suppose that there exists a 2-vertex  $v \in V(G)$  adjacent to the vertices  $u$  and  $w$  with  $d(w) \leq 4$  as in Figure 3.2.

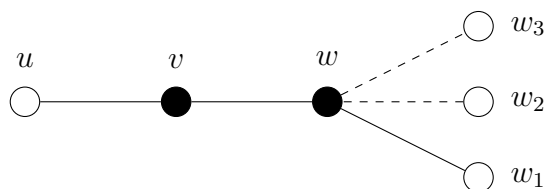


Figure 3.2: A 2-vertex in  $G$  with a neighbour having degree  $\leq 4$ .

Let  $w_1, \dots, w_k$  be the neighbours of  $w$  that are different from  $v$ , where  $k \leq 3$ . By minimality of  $G$ , the graph  $G - \{v\}$  admits an acyclic list 5-colouring  $\pi$ . Extend this colouring to  $G$  by considering the following:

If  $\pi(u) \neq \pi(w)$ , colour  $v$  with a colour in  $L(v)$  different from  $\pi(u)$  and  $\pi(w)$ . Otherwise,  $\pi(u) = \pi(w)$ . Colour  $v$  with a colour different from  $\pi(w), \pi(w_1), \dots, \pi(w_k)$ . Now  $G$  is acyclically 5-choosable, which contradicts the choice of  $G$ . Hence  $G$  does not contain a 2-vertex adjacent to a 4-vertex.  $\square$

Note that if a 2-vertex  $v$  were incident to a 3-face in the graph  $G$ , then its two neighbours  $u$  and  $w$  would always have the property that  $\pi(u) \neq \pi(w)$ , since  $u$  is adjacent to  $w$  in  $G - v$ . Hence, there is always a proper acyclic 5-colouring of  $G$  regardless of the degrees of the neighbours of  $v$ . This configuration will be examined in Condition (C9.1).

### 3.3 Restrictions on a 3-vertex

*Proof. (C3.1)* Suppose that  $v$  is adjacent to a 3-vertex  $v_1$ , a vertex  $v_2$  of degree at most 4, and a vertex  $v_3$  as in Figure 3.3.

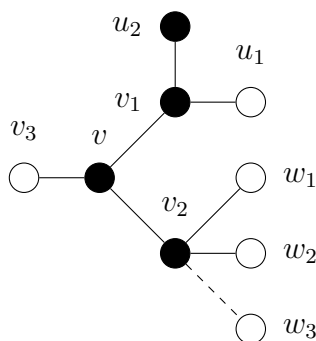


Figure 3.3: A 3-vertex  $v$  adjacent to a 3-vertex  $v_1$  and to a vertex  $v_2$  of degree at most 4.

Let  $u_1, u_2$  be the neighbours of  $v_1$  different from  $v$ , and  $w_1, w_2, \dots, w_k$ , for  $k \leq 3$ , be the neighbours of  $v_2$  different from  $v$ . Note that by Condition (C2),  $d(v_2) \neq 2$ . Let  $\pi$  be an acyclic list 5-colouring of  $G - \{v\}$ . If  $v_1, v_2, v_3$  have pairwise distinct colours, then colour  $v$  with a colour different from  $\pi(v_1), \pi(v_2), \pi(v_3)$ . If  $\pi(v_1) = \pi(v_2) \neq \pi(v_3)$ , colour  $v$  with a colour  $c \in L(v) \setminus \{\pi(v_1), \pi(v_3), \pi(u_1), \pi(u_2)\}$ . If  $\pi(v_1) = \pi(v_3) \neq \pi(v_2)$ , colour  $v$  with a colour different from  $\pi(v_1), \pi(v_2), \pi(u_1), \pi(u_2)$ . It remains to consider the following two cases.

- Assume that  $\pi(v_2) = \pi(v_3) \neq \pi(v_1)$ . If there exists a colour  $c \in L(v) \setminus \{\pi(v_1), \pi(v_2), \pi(w_1), \dots, \pi(w_k)\}$ , then colour  $v$  with  $c$ . Otherwise,



it follows that  $k = 3$  and  $w_1, w_2, w_3$  have pairwise distinct colours. Recolour  $v_2$  with a colour different from those of  $v_2, w_1, w_2, w_3$  and reduce to the previous case.

- Assume that  $\pi(v_1) = \pi(v_2) = \pi(v_3)$ . If either  $\pi(u_1) \neq \pi(u_2)$ , or  $\pi(w_1), \dots, \pi(w_k)$  are pairwise distinct, recolour  $v_1$  or  $v_2$  to reduce to the previous case. So suppose that  $\pi(u_1) = \pi(u_2)$  and, without loss of generality, assume  $\pi(w_1) = \pi(w_2)$ . In this case, colour  $v$  with a colour different from the colours assigned to  $v_1, u_1, w_1, \dots, w_k$ .

The resulting colouring is an acyclic list 5-colouring of  $G$ . This contradicts the choice of  $G$ . □

### 3.4 Restrictions on a pendant light 3-vertex

*Proof. (C3.2)(C4)* Let  $v$  be a pendant light 3-vertex of  $u$ , with  $f = [vv_1v_2]$  being a 3-face. Suppose, for contradiction, that  $d(u) \leq 4$ . Let  $u_1, \dots, u_k, k \leq 3$  be the neighbours of  $u$  different from  $v$  as in Figure 3.4. Note that by Condition (C2),  $d(u) \neq 2$ .

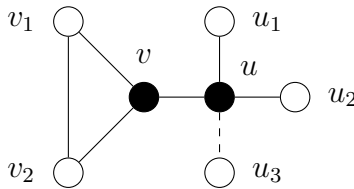


Figure 3.4: A 4-vertex  $v$  adjacent to a pendant light 3-vertex  $u$ .

Then by the minimality of  $G$ ,  $G - v$  admits an acyclic L-colouring  $\pi$ . If  $v_1, v_2$  and  $u$  are coloured with pairwise distinct colours, then colour  $v$  with a colour different from the colours assigned to its neighbours. Otherwise, by the symmetry, suppose  $\pi(v_1) = \pi(u)$ . Colour  $v$  with a colour  $c \in L(v) \setminus \{\pi(v_1), \pi(v_2), \pi(u_1), \dots, \pi(u_k)\}$ . If there is no such  $c$ , then  $k = 3$ , and  $L(v) = \{\pi(v_1), \pi(v_2), \pi(u_1), \pi(u_2), \pi(u_3)\}$ . If  $v$  cannot be acyclically coloured with  $\pi(u_1), \pi(u_2)$  or  $\pi(u_3)$ , then recolour  $u$  with a colour  $c_u$  different from  $\pi(u_1), \pi(u_2), \pi(u_3)$  and  $\pi(v_1)$ . If  $c_u \neq \pi(v_2)$  then colour  $v$  with a colour different from the colours assigned to its neighbours. Otherwise,  $c_u = \pi(v_2)$ . Then properly acyclically colour  $v$  with one of  $\pi(u_1), \pi(u_2)$  or  $\pi(u_3)$ , as  $G - v$  is planar and there does not exist a  $(\pi(u_i), \pi(v_2))$ -path from  $u_i$  to  $v_2$  for some  $1 \leq i \leq 3$ . The resulting colouring is an acyclic list 5-colouring of  $G$ . This contradicts the choice of  $G$ .  $\square$

### 3.5 Restrictions on a 5-vertex

*Proof. (C5.1)* Suppose that  $G$  contains a 5-vertex  $v$  adjacent to two 2-vertices  $v_1, v_2$  and other vertices  $v_3, v_4, v_5$  as in Figure 3.5

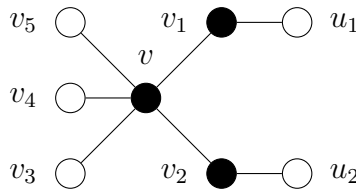


Figure 3.5: A 5-vertex  $v$  adjacent to two 2-vertices  $v_1, v_2$ .

For  $i = \{1, 2\}$ , let  $u_i$  be the neighbour of  $v_i$  different from  $v$ . By the minimality of  $G$ ,  $G - v_1$  has an acyclic list 5-colouring  $\pi$ . If  $\pi(v) \neq \pi(u_1)$ , colour  $v_1$  with a colour different from the colours assigned to  $v$  and  $u_1$ . Otherwise, if  $v_1$  can not be coloured acyclically, suppose that  $\pi(v) = \pi(u_1) = \pi(u_2)$ , and  $\pi(v_i)$  for  $i = 2, 3, 4, 5$  are pairwise distinct colours.  $L(v) = \{\pi(v), \pi(v_2), \dots, \pi(v_5)\}$  and there exists  $(\pi(v), \pi(v_i))$ -paths from  $u_1$  to  $v_i$  for  $i = 2, 3, 4, 5$ . If  $L(v) \neq L(v_1)$ , recolour  $v$  with a colour in  $L(v) \setminus L(v_1)$  and then give  $v_1$  a proper colouring. If  $L(v) = L(v_1)$ , recolour  $v$  with  $\pi(v_2)$ , then colour  $v_1$  with  $\pi(v_3)$  and  $v_2$  with a colour different from the colours assigned to  $v, u_2$ . The resulting colouring is an acyclic list 5-colouring of  $G$ . This contradicts the choice of  $G$ .  $\square$

*Proof. (C5.2)* Suppose that  $G$  contains a 5-vertex  $v$  adjacent to a 2-vertex  $v_1$  with neighbour of  $v$  and  $u_1$ , a pendant light 3-vertex  $v_2$  which is incident a 3-face  $[v_2xy]$ , and other vertices  $v_3, v_4, v_5$  as in Figure 3.6.

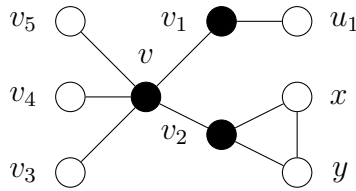


Figure 3.6: A 5-vertex  $v$  adjacent to a 2-vertex and a pendant light 3-vertex.

Let  $\pi$  be an acyclic list 5-colouring of  $G - \{v_1\}$ . If  $v_1$  cannot be acyclically coloured by any of its assigned acceptable colours, assume that  $|L(v)| = 5$ ,  $\pi(v) = \pi(x) = \pi(u_1)$  and  $\pi(v_i)$  for  $i = 2, 3, 4, 5$  are pairwise distinct colours.

$L(v) = \{\pi(v), \pi(v_2), \dots, \pi(v_5)\}$  and there exists  $(\pi(v), \pi(v_i))$ -paths from  $u_1$  to  $v_i$  for  $i = 2, 3, 4, 5$ . If  $L(v) \neq L(v_1)$ , recolour  $v$  with a colour in  $L(v) \setminus L(v_1)$  and then give  $v_1$  a proper colouring. If  $L(v) = L(v_1)$ , recolour  $v$  with  $\pi(v_2)$ , then colour  $v_1$  with  $\pi_3$  and  $v_2$  with a colour different from the colours assigned to  $v, x, y$ . The resulting colouring is an acyclic list 5-colouring of  $G$ . This contradicts the choice of  $G$ .  $\square$

*Proof. (C5.3)* Let  $v_1, v_2, \dots, v_5$  be the neighbours of a 5-vertex  $v$  with  $d(v_1) = 2$  and  $N(v_1) = v, u_1$ . Assume that  $v$  is incident to a 3-face  $f = [vv_2v_3]$  such that  $n_3(f) \geq 1$ . By Condition (C9.2),  $n_3(f) = 1$ , say  $d(v_2) = 3$ . Let  $x_2$  be the neighbour of  $v_2$  different from  $v$  and  $v_3$  as in Figure 3.7.

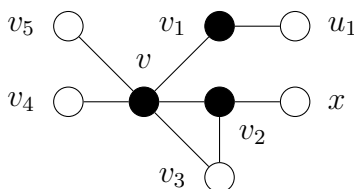


Figure 3.7: A 5-vertex  $v$  adjacent to a 2-vertex  $v_1$  and a 3-face  $[vv_2v_3]$  with  $d(v_2) = 3$ .

By the minimality of  $G$ ,  $G - \{v_1\}$  has an acyclic list 5-colouring  $\pi$ . If  $\pi(u_1) \neq \pi(v)$ , then let  $\pi(v_1) = c_1 \in L(v_1) \setminus \{\pi(u_1), \pi(v)\}$ . Otherwise,  $\pi(u_1) = \pi(v)$ . If there does not exist a colour  $c_1 \in L(v_1)$  which acyclically colours  $G$  properly, then it is the case that  $\pi(v) = \pi(u_1) = \pi(x_2)$ . If  $L(v) \neq L(v_1)$ , recolour  $v$  with a colour in  $L(v) \setminus L(v_1)$  and then  $v_1$  is properly coloured. If  $L(v) = L(v_1)$ , recolour  $v$  with  $\pi(v_2)$  and colour  $v_1$  with  $\pi(v_3)$ , then recolour

$v_2$  with a colour different from  $\pi(x_2), \pi(v)$  and  $\pi(v_3)$ . The resulting colouring is an acyclic list 5-colouring of  $G$ . This contradicts the choice of  $G$ .  $\square$

*Proof. (C5.4)* Assume to the contrary that  $p_3(v) \geq 4$  so that  $v_1, \dots, v_4$  are pendant light 3-vertices of  $v$  and  $v_5$  is the other neighbour of  $v$  with  $d(v_5) \geq 3$ . Let  $x_i, y_i$  be the neighbours of  $v_i$ , different from  $v$ , for  $i \leq 4$  as in Figure 3.8.

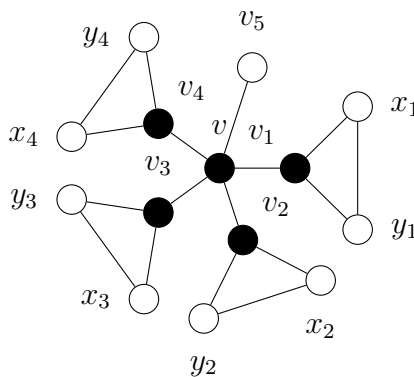


Figure 3.8: A 5-vertex  $v$  adjacent to at least four pendant light 3-vertices  $v_1, v_2, v_3, v_4$ .

By the minimality of  $G$ ,  $G - \{v, v_1, \dots, v_4\}$  admits an acyclic list 5-colouring  $\pi$ . Notice that  $\pi(x_i) \neq \pi(y_i)$  for all  $1 \leq i \leq 4$  since  $x_i$  and  $y_i$  are adjacent. Let  $S = \{x_1, y_1, \dots, x_4, y_4\}$ . Since  $|L(v) \setminus \{\pi(v_5)\}| \geq 4$  and  $|S| = 8$ , then there exists a colour  $c \in L(v) \setminus \{\pi(v_5)\}$  which appears at most twice on the set  $S$ , say  $\pi(x_1) = \pi(x_2) = c$ . Then, colour  $v$  with  $c, v_1$  with  $c_1 \in L(v_1) \setminus \{c, \pi(v_5), \pi(y_1)\}, v_2$  with  $c_2 \in L(v_2) \setminus \{c, c_1, \pi(v_5), \pi(y_2)\}$ , and  $v_i$  with a colour different from  $c, \pi(x_i), \pi(y_i)$  for  $i = 3, 4$ . The resulting colouring is an acyclic list 5-colouring of  $G$ . This contradicts the choice of  $G$ .  $\square$

*Proof. (C5.5)* Let  $v_1, v_2, v_3, v_4, v_5$  be neighbours of a 5-vertex  $v$  in clockwise order. Assume to the contrary that  $[vv_1v_2]$  is a 3-face and  $v_3, v_4, v_5$  are pendant light 3-vertices of  $v$ . Let  $x_3, y_3, x_4, y_4, x_5, y_5$  be, respectively, neighbours of  $v_3, v_4, v_5$  different from  $v$  as in Figure 3.9.

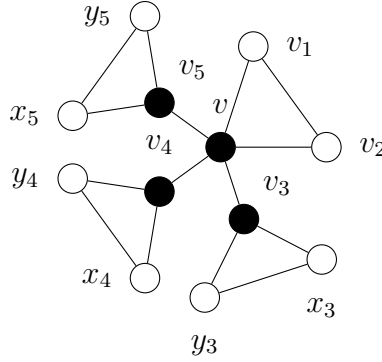


Figure 3.9: A 5-vertex  $v$  adjacent to at least three pendant light 3-vertices  $v_1, v_2, v_3, v_4$ .

Let  $G' = G - \{v, v_3, v_4, v_5\}$ . By the minimality of  $G$ ,  $G'$  admits an acyclic list 5-colouring  $\pi$ . Notice that,  $\pi(v_1) \neq \pi(v_2)$  and  $\pi(x_i) \neq \pi(y_i)$  for each  $i \in \{3, 4, 5\}$ . Let  $S = \{x_3, x_4, x_5, y_3, y_4, y_5\}$ . Notice also that  $|L(v) \setminus \{\pi(v_1), \pi(v_2)\}| \geq 3$  and  $|S| = 6$ . Then there exists a colour in  $L(v) \setminus \{\pi(v_1), \pi(v_2)\}$  appearing at most twice on the set  $S$ . Consider two cases:

1. If there exists a colour  $c \in L(v) \setminus \{\pi(v_1), \pi(v_2)\}$  which appears at most once on the set  $S$ , then assume, without loss of generality, that  $\pi(x_3) = c$ . Let  $\pi(v) = c, v_3$  with  $c_3 \in L(v_3) \setminus \{c, \pi(v_1), \pi(v_2), \pi(y_3)\}$ , and finally colour  $v_i$  with a colour different from  $c, \pi(x_i), \pi(y_i)$  for  $i = 4, 5$ .

2. Otherwise, each colour in  $L(v) \setminus \{\pi(v_1), \pi(v_2)\}$  appears exactly twice on the set  $S$ . Without loss of generality, assume that  $\pi(x_3) = \pi(x_5)$ ,  $\pi(y_4) = \pi(y_5)$  and  $\pi(y_3) = \pi(x_4)$ .

- If there is no alternating  $(\pi(y_3), \pi(v_1))$ -path in  $G'$  connecting  $y_3$  and  $v_1$ , then colour  $v$  with  $\pi(y_3)$ ,  $v_4$  with  $a \in L(v_4) \setminus \{\pi(v_1), \pi(v_2), \pi(y_4), \pi(y_3)\}$ ,  $v_3$  with  $b \in L(v_3) \setminus \{\pi(v_2), \pi(x_3), \pi(y_3), a\}$ , and finally colour  $v_5$  with a colour distinct from  $\pi(x_3), \pi(y_4), \pi(y_3)$ .
- If there is no alternating  $(\pi(y_5), \pi(v_2))$ -path in  $G'$  connecting  $y_5$  and  $v_2$ , then colour  $v$  with  $\pi(y_5)$ ,  $v_4$  with  $c \in L(v_4) \setminus \{\pi(v_1), \pi(v_2), \pi(y_4), \pi(y_3)\}$ ,  $v_5$  with  $d \in L(v_5) \setminus \{\pi(v_1), \pi(x_3), \pi(y_4)c\}$ , and finally colour  $v_3$  with a colour distinct from  $\pi(x_3), \pi(y_4), \pi(y_3)$ .

Since there cannot be both an alternating  $(\pi(y_3), \pi(v_1))$ -path connecting  $y_3$  and  $v_1$ , and an alternating  $(\pi(y_5), \pi(v_2))$ -path connecting  $y_5$  and  $v_2$  due to the planarity of  $G$ , then the colouring of  $G'$  can be extended to  $G$ . The resulting colouring is an acyclic list 5-colouring of  $G$ . This contradicts the choice of  $G$ . □

*Proof. (C5.6)* Let  $v_1, v_2, v_3, v_4, v_5$  be neighbours of a 5-vertex  $v$  in clockwise order. Assume that  $[vv_1v_2]$  is an incident  $(5, 3, 4)$ -face of  $v$ , with  $d(v_1) = 3$ ,  $d(v_2) = 4$  and  $v_3, v_4$  are pendant light 3-vertices of  $v$ . Let  $u$  be the neighbour of  $v_1$  different from  $v$  and  $v_2$ , and let  $x_2$  and  $y_2$  be the neighbour of  $v_2$  different from  $v$  and  $v_1$  and let  $x_3, y_3, x_4, y_4$  be, respectively, neighbours of  $v_3, v_4$  different from  $v$  as in Figure 3.10.

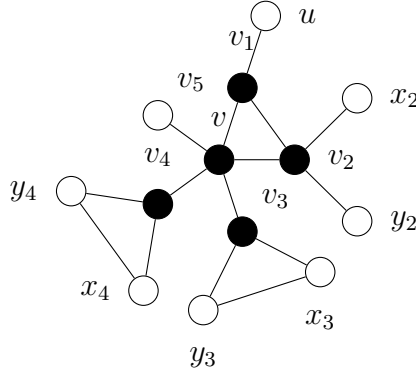


Figure 3.10: A 5-vertex  $v$  adjacent to a  $(3, 4, 5)$ -face and two pendant light 3-vertices.

Let  $G' = G - \{v, v_1, v_3, v_4\}$ . By the minimality of  $G$ ,  $G'$  admits an acyclic list 5-colouring  $\pi$ . Let  $S = \{u, x_3, y_3, x_4, y_4\}$ , and consider the following two cases:

1. If  $\pi(v_2) \neq \pi(v_5)$ , then since  $|L(v) \setminus \{\pi(v_2), \pi(v_5)\}| \geq 3$  and  $|S| = 5$ , there exists a colour  $c \in L(v) \setminus \{\pi(v_2), \pi(v_5)\}$  which appears at most once on the set  $S$ . Let  $\pi(v) = c$ . If  $\pi(u) = c$ , then let  $\pi(v_1) = c_1 \in L(v_1) \setminus \{\pi(v_2), \pi(v_5), c\}$ , and then colour  $v_i$  with a colour different from  $c, \pi(x_i), \pi(y_i)$  for  $i = \{3, 4\}$ . Otherwise assume, without loss of generality, that  $\pi(x_3) = c$ . Let  $\pi(v_3) = c_3 \in L(v_3) \setminus \{c, \pi(v_2), \pi(v_5), \pi(y_3)\}$ , colour  $v_4$  with a colour different from  $c, \pi(x_4), \pi(y_4)$ , then colour  $v_1$  as follows:

- If  $\pi(u) \neq \pi(v_2)$ , colour  $v_1$  different from  $c, \pi(u)$  and  $\pi(v_2)$ .
- If  $\pi(u) = \pi(v_2)$ , colour  $v_1$  different from  $c, \pi(v_2), \pi(x_2)$  and  $\pi(y_2)$ .



2. If  $\pi(v_2) = \pi(v_5)$ , consider the following:

- If  $\pi(x_2) = \pi(y_2)$ , then there exists a colour in  $L(v) \setminus \{\pi(v_2), \pi(x_2)\}$  which appears at most once on the set  $S$ . Then the proof can also be given with a similar argument to the previous case.
- Otherwise,  $x_2 \neq y_2$ . Recolour  $v_2$  with a colour different from  $\pi(v_2), \pi(x_2), \pi(y_2)$  and then reduce the proof to the former case.

The resulting colouring is an acyclic list 5-colouring of  $G$ . This contradicts the choice of  $G$ .  $\square$

*Proof. (C5.7)* Let  $v_1, v_2, v_3, v_4, v_5$  be neighbours of a 5-vertex  $v$  in clockwise order. Assume to the contrary that  $[vv_1v_2]$  is a  $(5, 3, 5^+)$ -face with  $d(v_1) = 3$  and  $d(v_2) \geq 5$ . Let  $v_3, v_4$  be pendant light 3-vertices of  $v$  and let  $v_5$  be a 3-vertex. Let  $u$  be the neighbour of  $v_1$  different from  $v$  and  $v_2$  and let  $N(v_i) = \{v, x_i, y_i\}$  for  $i = 3, 4, 5$  as in Figure 3.11.

Consider the graph  $G - \{v, v_3, v_4\}$ . By the minimality of  $G$ ,  $G - \{v, v_3, v_4\}$  admits an acyclic  $L$ -colouring  $\pi$ . Let  $S = \{x_3, y_3, x_4, y_4\}$ ,  $\alpha = |\{\pi(x_3), \pi(y_3), \pi(x_4), \pi(y_4)\}|$ , and consider the following three cases.

- If  $\pi(v_1), \pi(v_2), \pi(v_5)$  are pairwise distinct colours, then consider the following:
  - ◊ If there exists a colour  $c \in L(v) \setminus \{\pi(v_1), \pi(v_2), \pi(v_5)\}$  which appears at most once on the set  $S$ , say  $\pi(x_3) = c$ , then let  $\pi(v) = c$  and

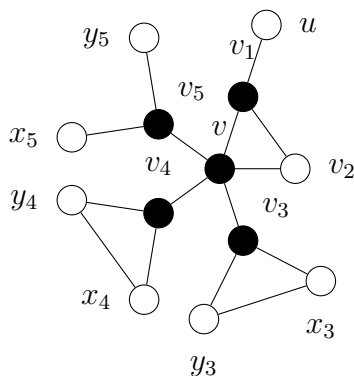


Figure 3.11: A 5-vertex  $v$  adjacent to a (3, 4, 5)-face and two pendant light 3-vertices.

$\pi(v_4) = c_4 \in L(v_4) \setminus \{c, \pi(x_4), \pi(y_4)\}$ . Then let  $\pi(v_3) = c_3 \in L(v_3) \setminus \{c, \pi(y_3), \pi(v_1), \pi(v_2), \pi(v_5)\}$ .

If there is no such colour  $c_3$ , then  $L(v_3) = \{c, \pi(y_3), \pi(v_1), \pi(v_2), \pi(v_5)\}$ . If  $v_3$  cannot be acyclically coloured, then  $\pi(u) = c$ . So, let  $\pi(v_3) = \pi(v_1)$  and recolour  $v_1$  with  $c_1 \in L(v_1) \setminus \{c, \pi(v_2), \pi(v_3), \pi(v_5)\}$ .

- ◇ Now assume, that all colours in  $L(v) \setminus \{\pi(v_1), \pi(v_2), \pi(v_5)\}$  each appear twice in  $S$ , so assume without loss of generality that  $\pi(x_3) = \pi(x_4)$  and  $\pi(y_3) = \pi(y_4)$ . If  $\pi(u) \neq \pi(v_2)$ , then colour  $v$  with  $\pi(v_1)$ , recolour  $a \in L(v_1) \setminus \{\pi(v_1), \pi(v_2), \pi(v_5), \pi(u)\}$ , and colour  $v_i$  with a colour distinct from  $\pi(v), \pi(x_3), \pi(y_3)$  for  $i = \{3, 4\}$ .

Otherwise, suppose that  $\pi(u) = \pi(v_2)$ . If  $\pi(x_5) = \pi(y_5)$ , then let  $\pi(v) = b \in L(v) \setminus \{\pi(x_3), \pi(y_3), \pi(x_5)\}$ ,  $\pi(v_3) = c \in L(v_3) \setminus \{\pi(v_2), \pi(x_3), \pi(y_3)\}$  and  $\pi(v_4) = d \in L(v_4) \setminus \{\pi(v_2), \pi(x_3), \pi(y_3)\}$ .

If  $\pi(x_5) \neq \pi(y_5)$ , colour  $v$  with  $\pi(v_5)$ , then recolour  $v_5$  with a colour different from  $\pi(v_2), \pi(v_5), \pi(x_5), \pi(y_5)$ , and give a proper colouring for  $v_3$  and  $v_4$ .

- Assume that  $\pi(v_5) = \pi(v_1) \neq \pi(v_2)$ . If  $\pi(u) \neq \pi(v_2)$ , recolour  $v_1$  with a colour different from  $\pi(v_1), \pi(v_2), \pi(u)$  and then colour the other vertices as in the previous case where  $\pi(v_1), \pi(v_2), \pi(v_5)$  are pairwise distinct colours. Now suppose that  $\pi(u) = \pi(v_2)$ . There exists a colour  $c \in L(v) \setminus \{\pi(v_1), \pi(v_2)\}$  which appears at most once on the set  $S$ , say  $\pi(x_3) = c$ . Colour  $v$  with  $c$ ,  $v_3$  with a colour in  $L(v_3) \setminus \{\pi(v_1), \pi(v_2), c, \pi(y_3)\}$ , and  $v_4$  with a colour different from  $c, \pi(x_4), \pi(y_4)$ .
- Assume that  $\pi(v_5) = \pi(v_2) \neq \pi(v_1)$ . If  $\pi(x_5) \neq \pi(y_5)$ , then recolour  $v_5$  with a colour different from  $\pi(v_5), \pi(v_2), \pi(x_5), \pi(y_5)$  and thus reduce the proof to the previous case where  $\pi(v_1), \pi(v_2), \pi(v_5)$  are pairwise distinct colours. Suppose now that  $\pi(x_5) = \pi(y_5)$ . If there exists a colour  $c \in L(v) \setminus \{\pi(v_5), \pi(v_1), \pi(x_5)\}$  appearing at most once on the set  $S$ , say  $\pi(x_3) = c$ , then let  $\pi(v) = c$ ,  $v_3$  with a colour distinct from  $\pi(v_1), \pi(v_5), c, \pi(y_3)$ , and colour  $v_4$  with a colour different from  $c, \pi(x_4), \pi(y_4)$ . Otherwise, assume, without loss of generality that  $L(v) = \{1, 2, \pi(x_5), 4, 5\}$  and that  $\pi(v_5) = \pi(v_2) = 1, \pi(v_1) = 2, \pi(x_3) = \pi(x_4) = 4$  and  $\pi(y_3) = \pi(y_4) = 5$ . If  $\pi(u) \neq 1$ , recolour  $v_1$  with  $a \in L(v_1) \setminus \{1, 2, \pi(u)\}$  and then reduce the proof to the previous case. Otherwise,  $\pi(u) = 1$ . Colour  $v$  with 4,  $v_3$  with a colour

$b \in L(v_3) \setminus \{1, 4, 5\}$  and  $v_4$  with a colour in  $L(v_4) \setminus \{1, 4, 5, b\}$ .

The resulting colouring is an acyclic list 5-colouring of  $G$ . This contradicts the choice of  $G$ .  $\square$

### 3.6 Restrictions on a 6-vertex

*Proof. (C6.1)* Suppose that  $G$  contains a 6-vertex  $v$  adjacent to five 2-vertices  $v_1, \dots, v_5$  and a vertex  $v_6$  as in Figure 3.12.

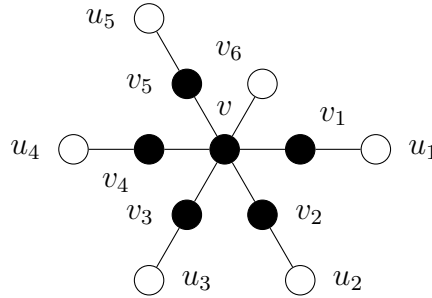


Figure 3.12: A 6-vertex adjacent to five 2-vertices.

Let  $u_i$  be the neighbour of  $v_i$  different from  $v$  for  $i = 1, 2, \dots, 5$ . Let  $\pi$  be an acyclic list 5-colouring of  $G - \{v, v_1, \dots, v_5\}$ . Then there exists a colour  $c \in L(v) \setminus \{\pi(v_6)\}$  which appears at most once on the vertices  $u_1, u_2, \dots, u_5$ . Without loss of generality, suppose that  $c$  appears on  $u_1$ . Colour  $v$  with  $c$ ,  $v_1$  with a colour different from  $u_1, v, v_6$ , and for each  $i = 2, 3, 4, 5$  and colour  $v_i$  with a colour different from  $v$  and  $u_i$ . The resulting colouring is an acyclic list 5-colouring of  $G$ . This contradicts the choice of  $G$ .

□

*Proof. (C6.2)* Suppose that  $G$  contains a 6-vertex  $v$  adjacent to four 2-vertices  $v_1, v_2, v_3, v_4$ , a 3-vertex  $v_5$ , and a vertex  $v_6$  as in Figure 3.13. Let

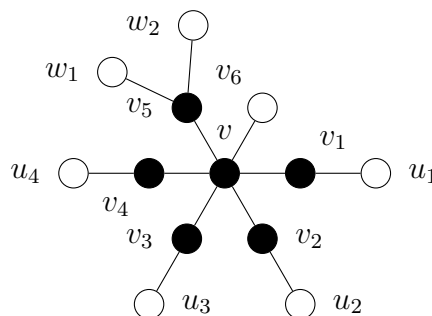


Figure 3.13: A 6-vertex adjacent to four 2-vertices and a 3-vertex.

$u_i$  be the neighbour of  $v_i$  different from  $v$  for  $i = 1, 2, 3, 4$ , and let  $w_1, w_2$  be the neighbours of  $v_5$  different from  $v$ . Suppose that  $\pi$  is an acyclic list 5-colouring of  $G - \{v, v_1, v_2, v_3, v_4\}$ . If  $\pi(v_5) \neq \pi(v_6)$ , then there exists a colour  $c \in L(v) \setminus \{\pi(v_5), \pi(v_6)\}$  which appears at most once on the vertices  $u_1, u_2, u_3, u_4$ . Suppose that  $c$  appears on  $u_1$  possibly. Colour  $v$  with  $c, v_1$  with a colour different from the colours assigned to  $c, v_5, v_6$ , and  $v_i$  with a colour different from  $c, \pi(u_i)$  for  $i = 2, 3, 4$ . If  $\pi(v_5) = \pi(v_6)$ , then if  $\pi(w_1) \neq \pi(w_2)$ , recolour  $v_5$  with a colour different from the colours assigned to  $w_1, w_2, v_6$ , and then reduce the proof to the previous case. Suppose then that  $\pi(w_1) = \pi(w_2)$ . Again, there exists a colour  $c \in L(v) \setminus \{\pi(v_5), \pi(w_1)\}$  which appears at most once on the vertices  $u_1, u_2, u_3, u_4$ . Without loss of generality, suppose  $\pi(u_1) = c$ . Colour  $v$  with  $c, v_1$  with a colour different from

the colours assigned to  $v, u_1, v_5, w_1$ , and  $v_i$  with a colour different from the colours assigned to  $v, u_i$  for  $i = 2, 3, 4$ . The resulting colouring is an acyclic list 5-colouring of  $G$ . This contradicts the choice of  $G$ .  $\square$

*Proof. (C6.3)* Assume to the contrary that  $v_1, v_2, v_3, v_4$  are 2-vertices and  $[v v_5 v_6]$  is a 3-face as in Figure 3.13.

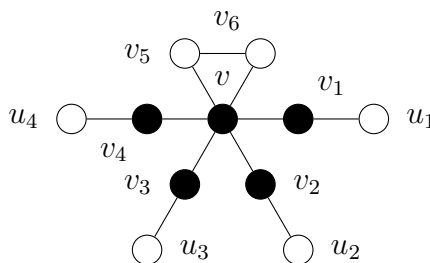


Figure 3.14: A 6-vertex adjacent to four 2-vertices and incident a 3-face.

Let  $\pi$  be an acyclic list 5-colouring of  $G - \{v, v_1, v_2, v_3, v_4\}$ . Obviously,  $\pi(v_5) \neq \pi(v_6)$ . There exists a colour  $c \in L(v) \setminus \{\pi(v_5), \pi(v_6)\}$  appearing at most once on  $u_1, u_2, u_3, u_4$ . Suppose that  $c$  appears on  $u_1$  possibly. Colour  $v$  with  $c$ ,  $v_1$  with a colour different from the colours assigned to  $v, u_1, v_5, v_6$ , and  $v_i$  with a colour different from the colours assigned to  $v, u_i$  for  $i = 2, 3, 4$ . The resulting colouring is an acyclic list 5-colouring of  $G$ . This contradicts the choice of  $G$ .  $\square$

*Proof. (C6.5)* Assume to the contrary that  $[v v_1 v_2]$  is an incident 3-face,  $v_3, v_4, v_5$  are 2-vertices and  $v_6$  is an pendant light 3-vertex of  $v$  as in Figure 3.15. Let  $u_1, u_2, u_3$  be neighbours of  $v_1, v_2, v_3$  different from  $v$ , let  $[v_4 v v_5]$  be a 3-face

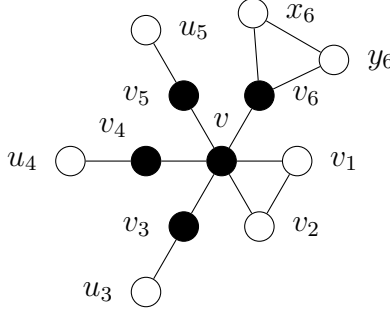


Figure 3.15: A 6-vertex with  $n_2(v) = 3$ ,  $t(v) = 1$  and  $p_3(v) = 1$

and let  $x_6, y_6$  be neighbours of  $v_6$  different from  $v$ .

By minimality,  $G - \{v, v_3, v_4, v_5, v_6\}$  has an acyclic 5-colouring  $\pi$ . Let  $S = \{u_3, u_4, u_5, x_6, y_6\}$ . Since  $|L(v) \setminus \{\pi(v_1), \pi(v_2)\}| \geq 3$  and  $|S| = 5$ , there exists a colour  $c \in L(v) \setminus \{\pi(v_1), \pi(v_2)\}$  appearing at most once on the set  $S$ . First colour  $v$  with  $c$  and in order to colour the remaining uncoloured vertices, without loss of generality, consider the following two cases.

- If  $\pi(u_3) = c$  then colour  $v_i$  with a colour different from  $c, \pi(v_1), \pi(v_2), \pi(u_i)$  for  $i = 3, 4, 5$ , and  $v_6$  with a colour different from  $c, \pi(x_6), \pi(y_6)$ .
- If  $\pi(x_6) = c$ , then colour  $v_i$  with a colour different from  $c, \pi(u_i)$  for  $i = 3, 4, 5$ , and  $v_6$  with a colour different from  $c, \pi(v_1), \pi(v_2), \pi(y_6)$ .

□

*Proof. (C6.5)* Suppose to the contrary that  $v_1, v_2$  are 2-vertices and  $v_3, v_4, v_5$  are pendant light 3-vertices of  $v$  as in Figure 3.16.

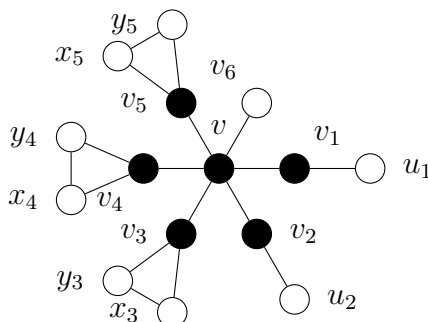


Figure 3.16: A 6-vertex adjacent to two 2-vertices and three pendant light 3-vertices.

By the minimality of  $G$ ,  $G - \{v, v_1, v_2, \dots, v_5\}$  has an acyclic list 5-colouring  $\pi$ . It is obvious that  $\pi(x_i) \neq \pi(y_i)$  for all  $i = 3, 4, 5$ . Let  $S = \{u_1, u_2, x_3, y_3, x_4, y_4, x_5, y_5\}$ . Since  $|L(v) \setminus \{\pi(v_6)\}| \geq 4$  and  $|S| = 8$ , there exists a colour belonging to  $L(v) \setminus \{\pi(v_6)\}$  appearing at most twice on the set  $S$ . First assume that there exists a colour  $c \in L(v) \setminus \{\pi(v_6)\}$  which appears at most once on the set  $S$ . Colour  $v$  with  $c$ ,  $v_i$  with a colour different from  $c, \pi(v_6), \pi(u_i)$  for  $i = 1, 2$ , and  $v_j$  with a colour different from  $c, \pi(v_6), \pi(x_j), \pi(y_j)$  for  $j = 3, 4, 5$ . Now assume, without loss of generality, that  $L(v) = \{1, 2, 3, 4, 5\}$ ,  $\pi(v_6) = 1$ , and each colour belonging to  $\{2, 3, 4, 5\}$  appears exactly twice on the set  $S$ . One can easily observe that there exist two vertices  $x$  and  $y$ , where  $x, y \in S \setminus \{u_1, u_2\}$ , such that  $\pi(x) = \pi(y)$ . Without loss of generality, assume that  $\pi(x_3) = \pi(x_4) = 2$ . Colour  $v$  with 2,  $v_3$  with a colour  $a \in L(v_3) \setminus \{1, 2, \pi(y_3)\}$ ,  $v_4$  with a colour  $b \in L(v_4) \setminus \{1, 2, a, \pi(y_4)\}$ ,  $v_i$  with a colour different from 2,  $\pi(u_i)$  for  $i = 1, 2$ , and finally colour  $v_6$  with a colour different from 2,  $\pi(x_6), \pi(y_6)$ .  $\square$



*Proof. (C6.6)* Assume to the contrary that  $[vv_1v_2]$  is a incident 3-face,  $v_3, v_4$  are 2-vertices and  $v_5, v_6$  are pendant light 3-vertices of  $v$  as in Figure 3.17.

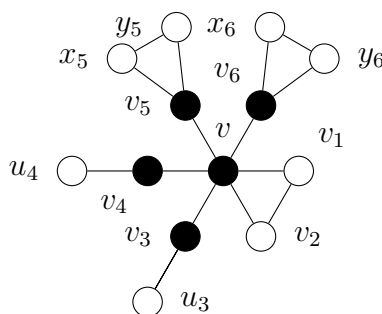


Figure 3.17: A 6-vertex adjacent to two 2-vertices, two pendant light 3-vertices and incident a 3-face.

By the minimality of  $G, G - \{v, v_3, v_4, v_5, v_6\}$  admits an acyclic list 5-colouring  $\pi$ . Let  $S = \{u_3, u_4, x_5, y_5, x_6, y_6\}$ . It is easy to observe that  $|L(v) \setminus \{\pi(v_1), \pi(v_2)\}| \geq 3$  and  $|S| = 6$ . Based on this fact, there exists a colour belonging to  $L(v) \setminus \{\pi(v_1), \pi(v_2)\}$  appearing at most twice on the set  $S$ . First assume that there exists a colour  $c \in L(v) \setminus \{\pi(v_1), \pi(v_2)\}$  appearing at most once on the set  $S$ . By symmetry, colour  $v$  with  $c$ . Then colour the remaining uncoloured vertices in the following way: If  $\pi(u_3) = c$ , colour  $v_3$  with a colour different from  $c, \pi(v_1), \pi(v_2)$ , and then assign  $v_i$  with a colour different from the colours assigned to that of its neighbours for  $i = 4, 5, 6$ . If  $\pi(x_5) = c$ , colour  $v_5$  with a colour different from  $c, \pi(v_1), \pi(v_2), \pi(y_5)$ , and then assign  $v_j$  with a colour different from the colours assigned to that of its neighbours for  $j = 3, 4, 6$ . Now, assume that  $L(v) = \{1, 2, 3, 4, 5\}, \pi(v_1) = 1, \pi(v_2) = 2$  and each colour in  $\{3, 4, 5\}$  appears exactly twice on the set  $S$ . If

$\pi(u_3) = \pi(u_4)$ , say  $\pi(u_3) = \pi(u_4) = 3$ , then colour  $v$  with 3,  $v_3$  with a colour  $a \in L(v_3) \setminus \{1, 2, 3\}$ ,  $v_4$  with a colour  $b \in L(v_4) \setminus \{1, 2, 3, a\}$ , and finally colour  $v_i$  with a colour distinct from 3,  $\pi(x_i)$ ,  $\pi(y_i)$  for  $i = 5, 6$ . Otherwise, without loss of generality, suppose that  $\pi(u_3) = \pi(x_5) = 3$ . Then colour  $v$  with 3,  $v_5$  with  $c \in L(v_5) \setminus \{1, 2, 3, \pi(y_5)\}$ ,  $v_3$  with  $d \in L(v_3) \setminus \{1, 2, 3, c\}$ , and finally assign a proper colouring for  $v_4$  and  $v_6$  easily.  $\square$

*Proof. (C6.7)* Assume to the contrary that  $[vv_1v_2]$  is a  $(6, 3, 3)$ -face, i.e.  $d(v_1) = d(v_2) = 3$ , and  $v_3, v_4, v_5$  are pendant light 3-vertices of  $v$  as in Figure 3.18. Let  $N(v_1) = \{u_1, v_2, v\}$  and  $N(v_2) = \{u_2, v_1, v\}$ . By the minimality

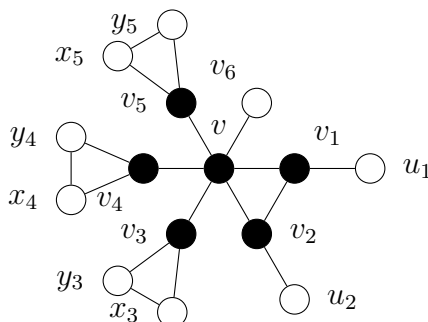


Figure 3.18: A 6-vertex adjacent to three pendant light 3-vertices and incident a  $(3, 3, 6)$ -face.

of  $G, G - \{v, v_1, v_2, \dots, v_5\}$  has an acyclic list 5-colouring  $\pi$ . Notice that  $\pi(x_i) \neq \pi(y_i)$  for each  $i \in \{3, 4, 5\}$ . Let  $S = \{u_1, u_2, x_3, y_3, x_4, y_4, x_5, y_5\}$ . Since  $|L(v) \setminus \{\pi(v_6)\}| \geq 4$  and  $|S| = 8$ , there exists a colour belonging to  $L(v) \setminus \{\pi(v_6)\}$  appearing at most twice on the set  $S$ . If there exists a colour in  $L(v) \setminus \{\pi(v_6)\}$  appearing at most once on  $S$ , the proof can also be given with

a similar argument to the previous lemma ?? . Now assume, without loss of generality, that  $L(v) = \{1, 2, 3, 4, 5\}$ ,  $\pi(v_6) = 1$ , and each colour in  $\{2, 3, 4, 5\}$  appears exactly twice on the set  $S$ . It is easy to see that there exist two vertices  $x, y \in \{x_3, y_3, x_4, y_4, x_5, y_5\}$  having the same colour, set  $\pi(x_3) = \pi(x_4) = 2$ . Colour  $v$  with 2,  $v_1$  with a colour  $a$  different from 2,  $\pi(u_1)$ ,  $\pi(u_2)$ ,  $v_2$  with a colour different from  $a, 2, \pi(u_2)$ ,  $v_3$  with a colour  $b \in L(v_3) \setminus \{1, 2, \pi(y_3)\}$ ,  $v_4$  with a colour  $c \in L(v_4) \setminus \{1, 2, b, \pi(y_4)\}$ , and finally assign a proper colouring for  $v_5$ .  $\square$

*Proof. (C6.8)* Suppose that  $[vv_1v_2]$  is  $(6, 3, 4)$ -face such that  $d(v_1) = 3$  and  $d(v_2) = 4$ . Let  $N(v_1) = \{u_1, v_2, v\}$  and  $N(v_2) = \{x_2, y_2, v_1, v\}$ . Assume to the contrary that  $v_3$  is a 2-vertex with a neighbour  $u_3$  different from  $v$  and  $v_4, v_5, v_6$  are pendant light 3-vertices of  $v$  as in Figure 3.19. By the

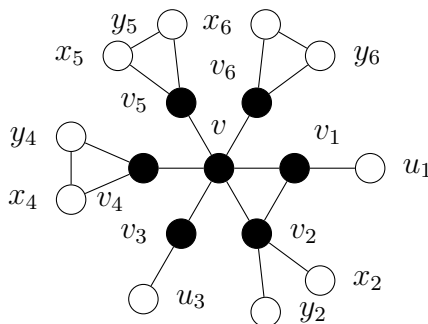


Figure 3.19: A 6-vertex adjacent to two 2-vertices and incident a  $(3, 4, 6)$ -face.

minimality of  $G$ ,  $G - \{v, v_1, v_3, v_4, v_5, v_6\}$  admits an acyclic list 5-colouring  $\pi$ . Let  $S = \{u_1, u_3, x_4, y_4, x_5, y_5, x_6, y_6\}$ . It is easy to see that there exists a colour belonging to  $L(v) \setminus \{\pi(v_2)\}$  appearing at most twice on the set  $S$ , since

$|L(v) \setminus \{\pi(v_2)\}| \geq 4$  and  $|S| = 8$ . Consider the following two cases. First assume that there exists a colour  $c \in L(v) \setminus \{\pi(v_2)\}$  which appears at most once on the set  $S$ . Colour  $v$  with  $c$  first, then colour  $v_3$  with a colour different from  $c, \pi(u_3), \pi(v_2)$ , and  $v_i$  with a colour different from  $c, \pi(v_2), \pi(x_i), \pi(y_i)$  for  $i = 4, 5, 6$ . Then colour  $v_1$  in the following way: If  $\pi(u_1) = \pi(v_2)$ , then assign  $v_1$  a colour in  $L(v_1) \setminus \{c, \pi(v_2), \pi(x_2), \pi(y_2)\}$ . Otherwise, assign a colour in  $L(v_1) \setminus \{c, \pi(v_2), \pi(u_1)\}$  to  $v_1$ .

Now assume, without loss of generality., that  $L(v) = \{1, 2, 3, 4, 5\}, \pi(v_2) = 1$ , and each colour in  $\{2, 3, 4, 5\}$  appears exactly twice on the set  $S$ . Then there exist two vertices  $x$  and  $y$  belonging to  $\{x_4, y_4, x_5, y_5, x_6, y_6\}$  having the same colour. Without loss of generality., assume that  $\pi(x_4) = \pi(x_5) = 2$ . First colour  $v$  with 2,  $v_3$  with a colour different from 2,  $\pi(u_3)$ ,  $v_4$  with a colour  $a \in L(v_4) \setminus \{1, 2, \pi(y_4)\}$ ,  $v_5$  with a colour  $b \in L(v_5) \setminus \{1, 2, a, \pi(y_5)\}$ ,  $v_6$  with a colour different from 2,  $\pi(x_6), \pi(y_6)$ , and finally colour  $v_1$  in the following way: If  $\pi(u_1) = \pi(v_2) = 1$ , then assign  $v_1$  with a colour in  $L(v_1) \setminus \{1, 2, \pi(x_2), \pi(y_2)\}$ . Otherwise, assign a colour in  $L(v_1) \setminus \{1, 2, \pi(u_1)\}$  to  $v_1$ .  $\square$

### 3.7 Restrictions on a 7-vertex

*Proof. (C7.1)* Suppose that  $G$  contains a 7-vertex  $v$  adjacent to six 2-vertices  $v_1, \dots, v_6$  and one other vertex  $v_7$  as in Figure 3.20

Let  $G' = G - \{v, v_i\}$  for  $1 \leq i \leq 6$ . Then  $G'$  admits an acyclic list 5-colouring  $\pi$  by the minimality of  $G$ . There exists a colour  $c \in L(v) \setminus \{\pi(v_7)\}$

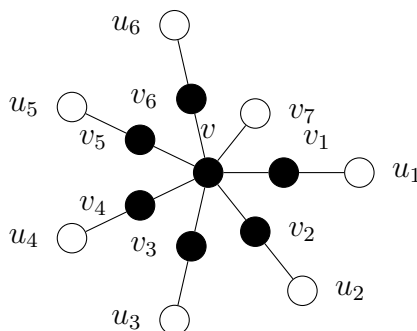


Figure 3.20: A 7-vertex adjacent to six 2-vertices

which appears at most once in  $u_i$  for  $1 \leq i \leq 6$ , so let  $\pi(v) = c$ . Assuming  $\pi(u_1) = c$ , let  $\pi(v_i) = c_i \in L(v_i) \setminus \{c, \pi(u_i)\}$  for  $2 \leq i \leq 6$  and let  $\pi(v_1) = c_1 \in L(v_1) \setminus \{c, \pi(v_7)\}$ .  $\square$

*Proof. (C7.2)* Assume to the contrary that the neighbours of  $v$  have degrees as follows:  $v_1, v_2, v_3, v_4$  are 2-vertices and  $v_5, v_6, v_7$  are 3-vertices. Let  $u_1, u_2, u_3, u_4$  be, respectively, the neighbours of  $v_1, v_2, v_3, v_4$  that are different from  $v$ . Let  $x_5, y_5, x_6, y_6, x_7, y_7$  be, respectively, the neighbours of  $v_5, v_6, v_7$  that are different from  $v$  as in Figure 3.21.

By the minimality of  $G$ ,  $G' = G - \{v, v_1, v_2, v_3, v_4\}$  has an acyclic list 5-colouring  $\pi$ . Let  $\alpha = |\{\pi(v_5), \pi(v_6), \pi(v_7)\}|$  and let  $S = \{u_1, u_2, u_3, u_4\}$ . Consider the following possibilities:

1.  $\alpha = 3$ . If there exists a colour  $c \in L(v) \setminus \{\pi(v_5), \pi(v_6), \pi(v_7)\}$  appearing at most once on  $S$ , say  $\pi(u_1) = c$ , then colour  $v$  with  $c$ , let  $\pi(v_1) = c_1 \in L(v_1) \setminus \{c, \pi(v_5), \pi(v_6), \pi(v_7)\}$  and colour  $v_i$  with a colour

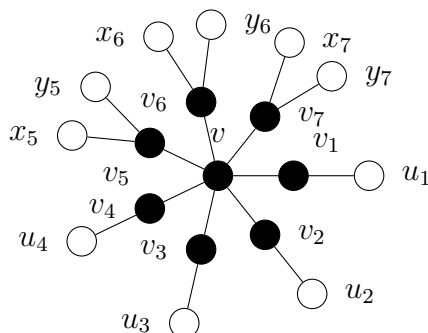


Figure 3.21: A 7-vertex adjacent to four 2-vertices and three 3-vertices.

different from  $c$  and  $\pi(u_i)$  for  $i = 2, 3, 4$ .

Otherwise, colours  $c$  and  $c'$  each appear twice in the colouring of  $S$ , so assume that  $\pi(v_1) = \pi(v_2) = c$ . If  $L(v_1) \neq L(v_2)$ , then let  $\pi(v) = c$ , let  $\pi(v_1) = c_1 \in L(v_1) \setminus \{c, \pi(v_5), \pi(v_6), \pi(v_7)\}$  and let  $\pi(v_2) = c_2 \in L(v_2) \setminus \{c, c_1, \pi(v_5), \pi(v_6), \pi(v_7)\}$ . Finally colour  $v_3$  and  $v_4$  differently from their respective neighbours. If  $L(v_1) = L(v_2)$  and a proper acyclic list 5-colouring  $G'$  cannot be extended to  $G$  when  $\pi(v) = c$ , then  $|L(v_1) - \{c, \pi(v_5), \pi(v_6), \pi(v_7)\}| = 1$  and there exists a  $(c, \pi(v_j))$ -path from  $u_i$  to  $v$  for  $i = 1, 2$  and  $j = 5, 6, 7$ . Then colour  $v$  with  $c'$ ,  $v_1$  and  $v_2$  with a colour different from  $c$  and  $c'$ , then let  $\pi(v_3) = c_3 \in L(v_3) \setminus \{c', \pi(v_5), \pi(v_6), \pi(v_7)\}$  and  $\pi(v_4) = c_4 \in L(v_4) \setminus \{c', c_3, \pi(v_5), \pi(v_6), \pi(v_7)\}$ . If there is no such  $c_4$ , then colour  $v_4$  with one of  $\pi(v_5), \pi(v_6), \pi(v_7)$  as there does not exist a  $(c', \pi(v_j))$ -path from  $u_4$  to  $v$  for some  $j = 5, 6, 7$  because  $G$  is planar.

2.  $\alpha = 2$ . Without loss of generality, assume that  $\pi(v_5) = \pi(v_6)$ . If

$\pi(x_5) \neq \pi(y_5)$  or  $\pi(x_6) \neq \pi(y_6)$ , recolour  $v_5$  or  $v_6$  to reduce to the previous case (1). Thus, suppose  $\pi(x_5) = \pi(y_5)$  and  $\pi(x_6) = \pi(y_6)$ . There exists a colour  $c \in L(v) \setminus \{\pi(v_5), \pi(v_7), \pi(x_5)\}$  appearing at most twice on  $S$ , say  $\pi(u_1) = \pi(u_2) = c$ . Colour  $v$  with  $c$ ,  $v_1$  with a colour  $c_1$  different from  $\{c, \pi(v_5), \pi(v_7)\}$ ,  $v_2$  with a colour different from  $\{c, c_1, \pi(v_5), \pi(v_7)\}$ , and give a proper colouring for  $v_3$  and  $v_4$ .

3.  $\alpha = 1$ . If there exists  $j \in \{5, 6, 7\}$  such that  $\pi(x_j) \neq \pi(y_j)$ , then recolour  $v_j$  to reduce to the former case (2).

Otherwise,  $\pi(x_j) = \pi(y_j)$  for all  $j \in \{5, 6, 7\}$ . There exists a colour  $c \in L(v) \setminus \{\pi(v_5), \pi(x_5), \pi(x_6)\}$  appearing at most twice on  $S$ , say  $\pi(u_1) = \pi(u_2) = c$ . Colour  $v$  with  $c$ ,  $v_1$  with a colour in  $L(v_1) \setminus \{\pi(v_5), c\}$ ,  $v_2$  with a colour different from  $\{\pi(v_5), c, \pi(v_1)\}$ , then properly colour  $v_3$  and  $v_4$ .

The resulting colouring is an acyclic  $L$ -colouring of  $G$ . This contradicts the choice of  $G$ .  $\square$

*Proof. (C7.3)* Let  $v$  be a 7-vertex with neighbours  $v_1, \dots, v_7$  such that  $v_1, \dots, v_5$  are 2-vertices having neighbours  $u_1, \dots, u_5$  different from  $v$  as in Figure 3.22. By the minimality of  $G$ ,  $G' = G - \{v, v_1, v_2, v_3, v_4, v_5\}$  has an acyclic list 5-colouring  $\pi$ . Consider the following two cases:

1. Assume for contradiction that  $f = [vv_6v_7]$  is a 3-face in  $G$ . Then there exists a colour  $c \in L(v) \setminus \{\pi(v_6), \pi(v_7)\}$  which appears at most once in the colouring of  $u_1, \dots, u_5$ , say  $\pi(u_1) = c$ . Let  $\pi(v) = c$ , let

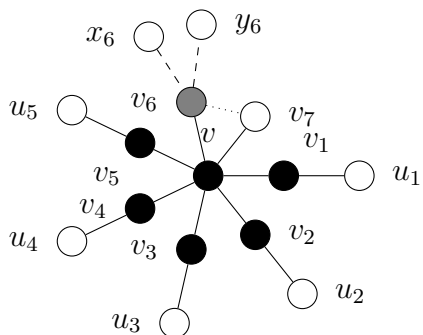


Figure 3.22: A 7-vertex adjacent to five 2-vertices and a 3-vertex or incident a 3-face ( $v_6$  is gray as it is either a 3-vertex or incident to a 3-face).

$\pi(v_1) = c_1 \in L(v_1) \setminus \{c, \pi(v_6), \pi(v_7)\}$  and give a proper colouring to  $v_i$  which is different from  $c$  and  $\pi(u_i)$  for  $i = 2, 3, 4, 5$ .

2. Assume for contradiction that  $d(v_6) = 3$ . If  $\pi(v_6) \neq \pi(v_7)$ , then the colouring of  $v, v_1, v_2, v_3, v_4, v_5$  is as in case (1), so assume  $\pi(v_6) = \pi(v_7)$ . If the neighbours,  $x_6, y_6$  of  $\pi(v_6)$  different from  $v$  are not assigned the same colour; i.e.,  $\pi(x_6) \neq \pi(y_6)$ , then recolour  $\pi(v_6)$  with a colour in  $L(v_6)$  which is different from  $\pi(v_6), \pi(x_6), \pi(y_6)$  and colour  $v, v_1, v_2, v_3, v_4, v_5$  as in case (1). Assume then that  $\pi(x_6) = \pi(y_6)$ . There exists a colour  $c \in L(v) \setminus \{\pi(v_6), \pi(x_6)\}$  which appears at most once in the colouring of  $u_1, \dots, u_5$ , say  $\pi(u_1) = c$ . Let  $\pi(v) = c$ , let  $\pi(v_1) = c_1 \in L(v_1) \setminus \{c, \pi(v_6)\}$  and give a proper colouring to  $v_i$  which is different from  $c$  and  $\pi(u_i)$  for  $i = 2, 3, 4, 5$ .

The resulting colouring is an acyclic  $L$ -colouring of  $G$ . This contradicts the choice of  $G$ . □



*Proof. (C7.4)* Suppose to the contrary that  $v_1, \dots, v_4$  are 2-vertices,  $[vv_5v_6]$  is a 3-face, and  $v_7$  is a pendant light 3-vertex of  $v$  as in Figure 3.23. By the minimality of  $G$ ,  $G' = G - \{v, v_1, v_2, v_3, v_4, v_7\}$  admits an acyclic list 5-colouring  $\pi$ . Let  $S = \{u_1, u_2, u_3, u_4, x_7, y_7\}$ , the neighbours of  $v_1, v_2, v_3, v_4, v_7$  respectively.

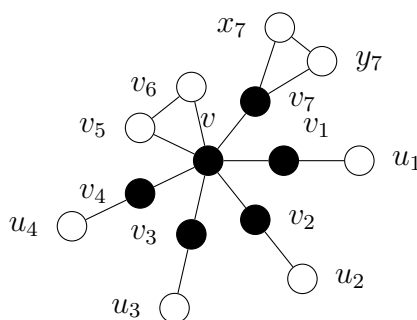


Figure 3.23: A 7-vertex adjacent to four 2-vertices a pendant light 3-vertex and a 3-face.

Obviously,  $|L(v) \setminus \{\pi(v_5), \pi(v_6)\}| \geq 3$  and  $|S| = 6$ . This fact implies that there exists a colour  $c$  belonging to  $L(v) \setminus \{\pi(v_5), \pi(v_6)\}$  appearing at most twice on the set  $S$ . If  $c$  appears at most once on the set  $S$ , then let  $\pi(v) = c$ , colour  $v_i$  different from  $c, \pi(v_5), \pi(v_6)$  (and, without loss of generality,  $\pi(y_7)$  if  $\pi(x_7) = c$ ), and colour the remaining vertices the set  $\{v_1, v_2, v_3, v_4, v_7\}$  which have not yet been coloured. If  $c$  appears exactly twice in  $S$ , say  $\pi(v_i) = \pi(v_j) = c$  for  $(i \neq j) \in \{1, 2, 3, 4, 7\}$ , let  $\pi(v) = c$ , let  $\pi(v_i) = c_i \in L(v_i) \setminus \{c, \pi(v_5), \pi(v_6)\}$  (and, without loss of generality,  $\pi(y_7)$  if  $\pi(x_7) = c$ ), let  $\pi(v_j) = c_j \in L(v_j) \setminus \{c, c_i, \pi(v_5), \pi(v_6)\}$ , and colour the remaining vertices the set  $\{v_1, v_2, v_3, v_4, v_7\}$  which have not yet been coloured.

The resulting colouring is an acyclic list 5-colouring of  $G$ . This contradicts the choice of  $G$ .  $\square$

*Proof. (C7.5)* Suppose to the contrary that  $[vv_1v_2]$  is a  $(7, 3, 3)$ -face such that  $d(v_1) = d(v_2) = 3$ ,  $v_3, v_4, v_5$  are 2-vertices and  $v_6, v_7$  are pendant light 3-vertices of  $v$ . By the minimality of  $G$ ,  $G' = G - \{v, v_1, \dots, v_7\}$  admits an acyclic list 5-colouring  $\pi$ . Let  $N(v_1) = (u_1, v, v_2)$  and  $N(v_2) = (u_2, v, v_1)$ . Let  $u_3, u_4, u_5$  be, respectively, the neighbours of  $v_3, v_4, v_5$ . Let  $S = \{u_1, u_2, u_3, u_4, u_5, x_6, y_6, x_7, y_7\}$  where  $x_6, y_6, x_7, y_7$  are the neighbours of  $v_6$  and  $v_7$  different from  $v$  as in Figure 3.24.

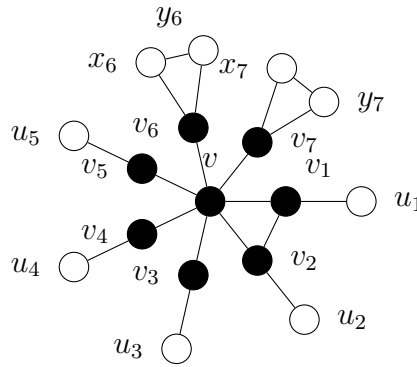


Figure 3.24: A 7-vertex adjacent to three 2-vertices, two pendant light 3-vertex and a  $(7, 3, 3)$ -face.

Since  $|S| = 9$  and  $|L(v)| = 5$ , then there exists a colour  $c \in L(v)$  which appears at most once on the set  $S$ . The colouring of  $G'$  can be extended in the following way: colour  $v$  with  $c$ ,  $v_1$  with a colour  $c_1 \in L(v_1)$  different from  $c, \pi(u_1)$  and  $\pi(u_2)$ ,  $v_2$  with a colour  $c_2 \in L(v_2)$  different from  $c, c_1, \pi(u_1)$

and  $\pi(u_2)$ ,  $v_i$  with a colour different from  $c$  and  $\pi(u_i)$  for  $i = 3, 4, 5$ , and  $v_j$  with a colour different from the colours assigned to  $c, x_j$  and  $y_j$  for  $j = 6, 7$ . The resulting colouring is an acyclic  $L$ -colouring of  $G$ . This contradicts the choice of  $G$ .  $\square$

### 3.8 Restrictions on a 8-vertex

*Proof. (C8.1)* Suppose that  $G$  contains an 8-vertex  $v$  adjacent to seven 2-vertices

$v_1, v_2, v_3, v_4, v_5, v_6, v_7$  and one other vertex  $v_8$  as in Figure 3.25.

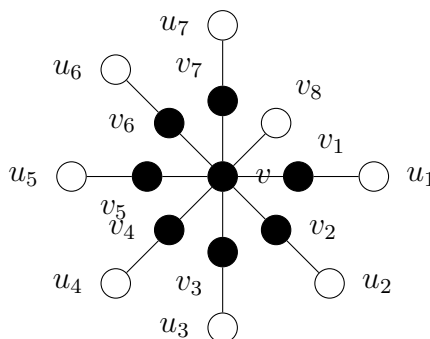


Figure 3.25: An 8-vertex adjacent to seven 2-vertices

Let  $G' = G - \{v, v_i\}$  for  $1 \leq i \leq 7$ . Then  $G'$  admits an acyclic list 5-colouring  $\pi$  by the minimality of  $G$ . There exists a colour  $c \in L(v) \setminus \{\pi(v_8)\}$  which appears at most once in  $u_i$  for  $1 \leq i \leq 7$ , so let  $\pi(v) = c$ . Assuming  $\pi(u_1) = c$ , let  $\pi(v_i) = c_i \in L(v_i) \setminus \{c, \pi(u_i)\}$  for  $2 \leq i \leq 7$  and let  $\pi(v_1) = c_1 \in L(v_1) \setminus \{c, \pi(v_8)\}$ . The resulting colouring is an acyclic  $L$ -colouring of

$G$ . This contradicts the choice of  $G$ .  $\square$

*Proof. (C8.2)* Assume to the contrary that  $v_1, v_2, v_3, v_4, v_5, v_6$  are 2-vertices with neighbours  $u_1, u_2, u_3, u_4, u_5, u_6$  different from  $v$  and let  $[vv_7v_8]$  be a 3-face as in Figure 3.26. Note that by (C9.1), no 2-vertex is contained in a 3-face.

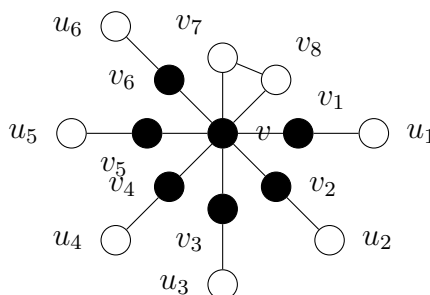


Figure 3.26: An 8-vertex adjacent to six 2-vertices and incident to a 3-face

Let  $\pi$  be an acyclic list 5-colouring of  $G - \{v, v_1, \dots, v_6\}$ . Obviously,  $\pi(v_7) \neq \pi(v_8)$ . Let  $S = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ . Then there exists a colour  $c \in L(v) \setminus \{\pi(v_7), \pi(v_8)\}$  appearing at most twice on the set  $S$ , say  $\pi(u_1) = \pi(u_2) = c$ . Then colour  $v$  with  $c$ , colour  $v_1$  with a colour  $c_1$  different from  $\{c, \pi(v_7), \pi(v_8)\}$  and colour  $v_2$  with a colour  $c_2$  different from  $\{c, c_1, \pi(v_7), \pi(v_8)\}$ . Finally colour  $v_i$  with a colour different from  $\{c, \pi(u_i)\}$  for  $i = 3, 4, 5, 6$ . The resulting colouring is an acyclic  $L$ -colouring of  $G$ . This contradicts the choice of  $G$ .  $\square$

### 3.9 Restrictions on a 3-face

*Proof. (C9.1)* Let  $f = [xyz]$  be a 3-face and assume for contradiction that  $d(x) = 2$  as in Figure 3.27. By the minimality of  $G$ , the graph  $G - x$  admits

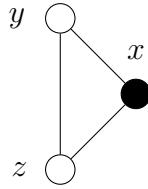


Figure 3.27: A 3-face  $[xyz]$  with  $d(x) = 2$ .

an acyclic  $L$ -colouring  $\pi$ . Since  $\pi(y) \neq \pi(z)$  in  $G - x$ , extend this acyclic list 5-colouring letting  $\pi(x) = c \in L(x) \setminus \pi(y), \pi(z)$ . The resulting colouring is an acyclic  $L$ -colouring of  $G$ . This contradicts the choice of  $G$ .  $\square$

*Proof. (C9.2)* Suppose that  $G$  contains a 3-face  $[xyz]$  with  $d(x) = d(y) = 3$  and  $d(z) = 5$  as in Figure 3.28. Notice that by Lemma ??,  $d(z) \geq 5$

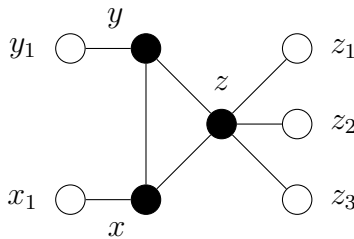


Figure 3.28: A 3-face  $[xyz]$  with  $d(x) = d(y) = 3$  and  $d(z) = 5$ .

Let  $x_1, y_1$ , and  $z_1, z_2, z_3$ , be, respectively, the neighbours of  $x, y$ , and  $z$  that are not on the face  $[xyz]$ . Let  $\pi$  be an acyclic list 5-colouring of

$G - \{x, y\}$  and consider the extension of this colouring in the following four cases. If the colours of  $x_1, y_1, z$  are pairwise distinct, colour  $x$  with a colour  $c_x \in L(x) \setminus \{\pi(x_1), \pi(z)\}$  and  $y$  with a colour  $c_y \in L(y) \setminus \{c_x, \pi(y_1), \pi(z)\}$ . By symmetry, consider if  $\pi(x_1) = \pi(z)$  and  $\pi(y_1) \neq \pi(z)$ , colour  $x$  with  $c_x \in L(x) \setminus \{\pi(z), \pi(z_1), \pi(z_2), \pi(z_3)\}$  and  $y$  with a colour  $c_y \in L(y) \setminus \{c_x, \pi(y_1), \pi(z)\}$ . If  $\pi(x_1) = \pi(y_1)$  and  $\pi(z) \neq \pi(x_1)$ , colour  $x$  with  $c_x \in L(x) \setminus \{\pi(x_1), \pi(z)\}$  and  $y$  with a colour  $c_y \in L(y) \setminus \{c_x, \pi(y_1), \pi(z)\}$ . Finally suppose that  $\pi(x_1) = \pi(y_1) = \pi(z)$ . If  $z_1, z_2, z_3$  have pairwise distinct colours, recolour  $z$  with a colour different from the colours assigned to  $z, z_1, z_2, z_3$  and then reduce to the previous case. If at least two of  $z_1, z_2, z_3$  have the same colour, say  $\pi(z_1) = \pi(z_2)$ , colour  $x$  with a colour different from the colours assigned to  $z, z_2, z_3$ , and  $y$  with a colour different from the colours assigned to  $x, z, z_2, z_3$ . The resulting colouring is an acyclic  $L$ -colouring of  $G$ . This contradicts the choice of  $G$ .  $\square$

*Proof. (C9.3)* Suppose that  $G$  contains a 3-face  $[xyz]$  with  $d(x) = 3$  and  $d(y) = d(z) = 4$  as in Figure 3.29. Let  $x_1, y_1, y_2$ , and  $z_1, z_2$  be, respectively, the neighbours of  $x, y$ , and  $z$  that are not on the face  $[xyz]$ . Let  $\pi$  be an acyclic list 5-colouring of  $G - x$ .

If  $x_1, y, z$  have pairwise distinct colours, colour  $x$  properly. Otherwise, suppose that  $\pi(y) = \pi(x_1) \neq \pi(z)$ . In this case, colour  $x$  with a colour different from the colours assigned to  $y, z, y_1, y_2$ . The resulting colouring is an acyclic  $L$ -colouring of  $G$ . This contradicts the choice of  $G$ .  $\square$

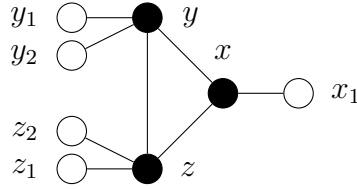


Figure 3.29: A 3-face  $[xyz]$  with  $d(x) = 3$  and  $d(y) = d(z) = 4$ .

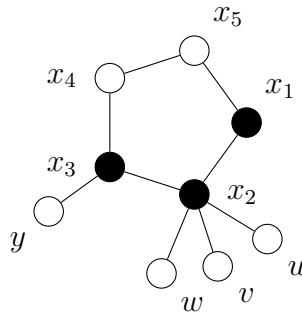


Figure 3.30: A 5-face  $[x_1, \dots, x_5]$  with  $d(x_1) = 2$ ,  $d(x_2) = 5$  and  $d(x_3) = 3$ .

### 3.10 Restrictions on a 5-face

Suppose to the contrary that  $G$  contains a 5-face  $f$  with boundary vertices  $x_1, \dots, x_5$  such that  $d(x_1) = 2$ ,  $d(x_2) = 5$  and  $d(x_3) = 3$ . Let  $u, v, w$  be the neighbours of  $x_2$  different from  $x_1$  and  $x_3$ . Let  $y$  be the neighbour of  $x_3$  different from  $x_2$  and  $x_4$  as in Figure 3.30.

By the minimality of  $G$ , the graph  $G - x_1$  admits an acyclic L-colouring  $\pi$ . To extend the colouring of  $G - x_1$  to  $G$ , consider the following. If  $\pi(x_2) \neq \pi(x_5)$ , let  $x_1$  be coloured with  $c \in L(x_1) \setminus \{\pi(x_2), \pi(x_5)\}$ . If  $\pi(x_2) = \pi(x_5)$ , then let  $x_1$  be coloured with  $c \in L(x_1) \setminus \{\pi(x_2), \pi(x_3), \pi(u), \pi(v), \pi(w)\}$ . If there

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is no such colour  $c$  and if  $x_1$  cannot be acyclically coloured with  $\pi(x_3)$ ,  $L(x_1) = \{\pi(x_2), \pi(x_3), \pi(u), \pi(v), \pi(w)\}$  and  $\pi(y) = \pi(x_2)$ . If  $L(x_2) \neq L(x_1)$ , recolour  $x_2$  with a colour in  $L(x_2) \setminus L(x_1)$  and give a proper colouring to  $x_1$ . If  $L(x_2) = L(x_1)$  consider that  $\pi(x_4) \neq \pi(x_2)$  and  $\pi(x_3)$ . Recolour  $x_2$  with  $\pi(x_3)$ , let  $x_3$  be recoloured with a colour different than those assigned to  $y, x_4, x_2$ , and give a proper colouring to  $x_1$ . The resulting colouring is an acyclic  $L$ -colouring of  $G$ . This contradicts the choice of  $G$ .



# Chapter 4

## Proof of Theorem

### 4.1 New Reducible Configurations

The list of reducible figures provided in previous work is not sufficient to ensure that a proof using the discharging method would lead to a contradiction. A closer look at the possible structure of a minimum counterexample  $G$  need exploration. Many shortcomings during the discharging portion of this proof came about when

- a 6-vertex had many neighbours of degree 2;
- a vertex had degree 9, 10 and 11.

The following proofs reveal more about the structure  $G$  in the same manor as in the previous chapter.

### 4.1.1 Weak 6-vertex

**Definition 1.** We define a weak 6-vertex as a vertex of degree 6 with the following properties:

1.  $v$  is adjacent to exactly four 2-vertices
2.  $v$  is incident to either
  - three 4-faces or
  - two 4-faces containing all four 2-vertices in  $N(v)$ , and one 5-face.

Since, by assumption, a 4-face is not adjacent to a 4- or 5-face, and by condition (C6.3),  $v$  is not incident to a 3-face, then the number 4- and 5-faces incident to  $v$  is less than or equal to 3. Figures 4.1 and 4.2 show the three possible configurations of a weak 6-vertex.

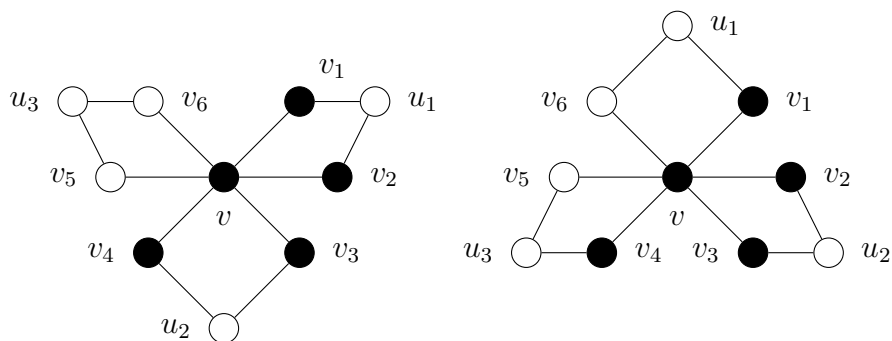


Figure 4.1: A weak 6-vertex incident to three 4-faces.

As a consequence of this definition, a weak 4- or 5-face is a 4- or 5-face which is incident to a weak 6-vertex. Let  $w_6(f)$  denotes the number of weak

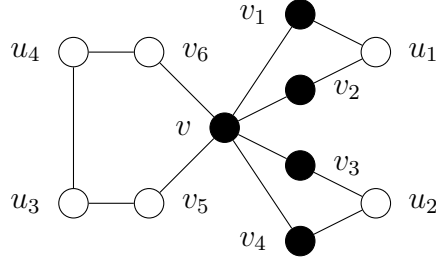


Figure 4.2: A weak 6-vertex incident to a 5-face.

6-vertices incident to  $f$ .

The following lemma will prove the non-adjacency of two weak 6-vertices.

**Lemma 2.** *Every weak 4-face  $[wxyz]$ ; where  $w$  is a weak 6-vertex, satisfies one of the following:*

(A1.1)  $x$  is not a weak 6-vertex.

(A1.2) If  $d(x) = 2$  and  $d(z) \neq 2$ , then  $y$  is not a weak 6-vertex.

*Proof(A1.1)* Assume otherwise and let  $x$  be a weak 6-vertex as in Figure 4.3.

Let  $w_1 \dots w_4$  be the neighbours of  $w$  different from  $x$  and  $z$ . Let  $d(w_1) = d(w_2) = d(w_3) = 2$ . Since  $w$  is weak, then either  $d(w_4) = 2$  or  $d(z) = 2$ . With a similar labelling, let  $x_1 \dots x_4$  be the neighbours of  $x$  different from  $w$  and  $y$ , and let  $d(x_1) = d(x_2) = d(x_3) = 2$ . Again, since  $x$  is weak, then either  $d(x_4) = 2$  or  $d(y) = 2$ . By (C2), it is not the case that  $d(y) = d(z) = 2$ . Let  $G' = G - \{w, w_1, w_2, w_3, x, x_1, x_2, x_3\}$ . Then  $G'$  admits an  $L$ -colouring  $\pi$  by the minimality of  $G$ . Let  $S$  denote the

set of colours which  $\pi$  has assigned to  $\{y, z, w_4, u_1, u_2\}$ . If there exists a colour  $c \in L(w) \setminus S$  then colour  $w$  with  $c$ . Otherwise, consider the following two cases.

Assume first that  $d(w_4) = 2$ . Colour  $w$  with  $\pi(w_4)$ , recolour  $w_4$  different from  $w$  and  $u_2$ , and colour each of  $w_1, w_2, w_3$ , differently than its two neighbours. Now, if  $d(z) = 2$ , then let  $\pi(w) = \pi(z)$ , let  $z$  be recoloured with a colour different from  $\pi(w)$  and  $\pi(y)$  and colour each of  $w_1, w_2, w_3$  differently than its two neighbours. All that remains is to colour  $x, x_1, x_2, x_3$ .

Let  $T$  denote the set of colours which have been assigned to  $\{x_4, y, v_1, v_2, \pi(w)\}$ . If there exists a colour  $c' \in L(x) \setminus T$ , then colour  $x$  with  $c'$  and colour each of  $x_1, x_2, x_3$  differently from their neighbours. Otherwise, consider the following two cases. First assuming that  $d(x_4) = 2$ . Colour  $x$  with  $\pi(x_4)$  and give a proper colouring to  $x_1, x_2, x_3$ .

Assuming now that  $d(y) = 2$ , let  $\pi(x) = \pi(y)$ , let  $y$  be recoloured with a colour different from  $\pi(x)$  and  $\pi(z)$  and colour each of  $x_1, x_2, x_3$  differently than its two neighbours. Since the colouring of  $G'$  has been extended to  $G$ , then this contradicts our choice of  $G$ .

(A1.2) Let  $d(x) = 2$ ,  $d(z) > 2$  and assume that both  $w$  and  $y$  are weak 6-vertices, as in Figure 4.4. Let  $w_1, \dots, w_4$  be the neighbours of  $w$  different from  $x$  and  $z$ . Let  $d(w_1) = d(w_2) = d(w_3) = 2$ . With a similar labelling,

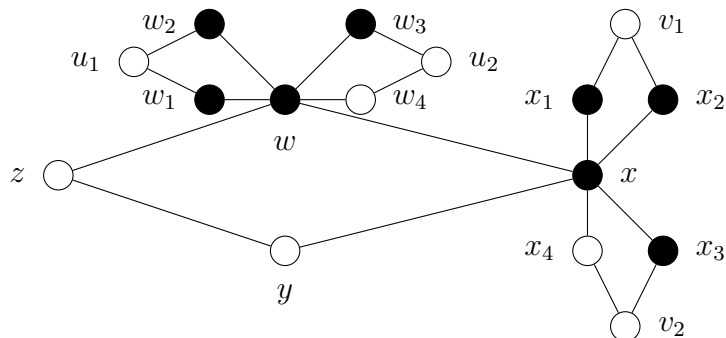


Figure 4.3: A weak 4-face incident with adjacent weak 6-vertices.

let  $y_1, \dots, y_4$  be the neighbours of  $y$  different from  $x$  and  $z$ , and let  $d(y_1) = d(y_2) = d(y_3) = 2$ .

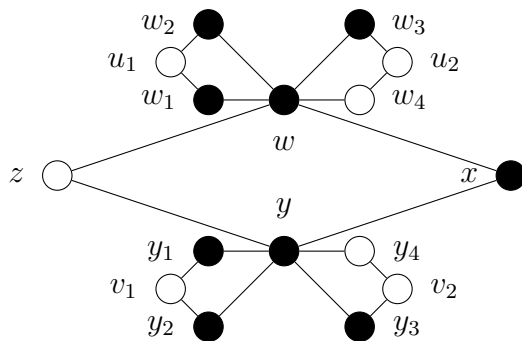


Figure 4.4: A weak 4-face incident with non adjacent weak 6-vertices.

Let  $G' = G - \{w, w_1, w_2, w_3, x, y, y_1, y_2, y_3\}$ . Then  $G'$  admits an  $L$ -colouring  $\pi$  by the minimality of  $G$ . Let  $S$  denote the set of colours which  $\pi$  has assigned to  $\{z, w_4, u_1, u_2\}$ . Colour  $w$  with  $c \in L(w) \setminus S$ . Let  $T$  denote the set of colours which  $\pi$  has assigned to  $\{z, y_4, v_1, v_2\}$ .

Colour  $y$  with  $c' \in L(y) \setminus T$ . Let  $\pi(x) = a \in L(x) \setminus \{\pi(z), c, c'\}$ . Now let  $w_1, w_2, w_3, y_1, y_2, y_3$  each be coloured with a colour different from its neighbours. Since the colouring of  $G'$  has been extended to  $G$ , then this contradicts our choice of  $G$ .

□

**Lemma 3.** *Let  $f$  be a weak 5-face. Then  $f$  is not incident to two adjacent weak 6-vertices.*

*Proof.* Assume the contrary, that a 5-face  $[vwxyz]$  has two adjacent weak 6-vertices  $v$  and  $w$  as in Figure 4.5. Let  $v_1, \dots, v_4$  be the neighbours of  $v$  different from  $w$  and  $z$ . Let  $d(v_1) = d(v_2) = d(v_3) = d(v_4) = 2$ . With a similar labelling, let  $w_1, \dots, w_4$  be the neighbours of  $w$  different from  $v$  and  $x$ , and let  $d(w_1) = d(w_2) = d(w_3) = d(w_4) = 2$ .

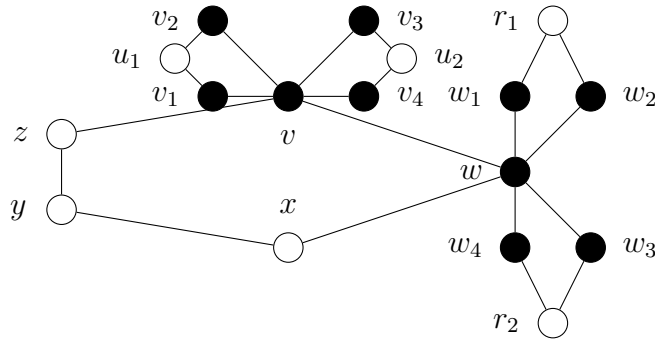


Figure 4.5: A weak 5-face incident with adjacent weak 6-vertices.

Let  $G' = G - \{v, v_1, v_2, v_3, v_4, w, w_1, w_2, w_3, w_4\}$ . Then  $G'$  admits an acyclic  $L$ -colouring  $\pi$  by the minimality of  $G$ . Let  $S$  denote the set of colours which

$\pi$  has assigned to  $\{z, x, u_1, u_2\}$ . Colour  $v$  with  $c \in L(v) \setminus S$ . Let  $T$  denote the set of colours which  $\pi$  has assigned to  $\{x, r_1, r_2\}$  and  $c$ . Colour  $w$  with  $c' \in L(w) \setminus T$ . There is a proper colouring of  $\{v_1, v_2, v_3, v_4, w_1, w_2, w_3, w_4\}$  by colouring each differently from its two neighbours. Since the colouring of  $G'$  has been extended to  $G$ , then this contradicts our choice of  $G$ .  $\square$

### 4.1.2 Restrictions On A 6-vertex

The following lemmas are new and explore some necessary conditions in which a 6-vertex may appear in  $G$ .

**Lemma 4.** *Let  $v$  be a vertex of degree 6.*

(B1.1) *If  $n_2(v) = 3$  and incident to a  $(3, 5^+, 6)$ -face, and two 4-faces, then  $n_3(v) = 1$ .*

(B1.2) *If  $n_2(v) = 3$  and incident to a  $(3, 5^+, 6)$ -face, and two 4-faces, then  $w_6(v) = 0$ .*

(B2) *If  $n_2(v) = 2$  and  $v$  is incident to a  $(6, 3, 4^-)$ -face, then  $t(v) = 1$ .*

*Proof*(B1.1) Suppose that  $v$  is a 6-vertex  $v$  with neighbours  $v_1, \dots, v_6$  where  $d(v_1) = d(v_2) = d(v_3) = 2$  and  $d(v_5) = 3$ . Suppose to the contrary that  $d(v_4) = 3$ . Let  $[vv_1wv_2]$  and  $[vv_3xv_4]$  be two 4-faces and, by (C9.1), let  $[v_5v_6v]$  be a  $(3, 5^+, 6)$ -face as in Figure 4.6. Let  $y$  be the neighbour of  $v_4$  different from  $v$  and  $x$ . Let  $z$  be the neighbour of  $v_5$  different from  $v$  and  $v_6$ .

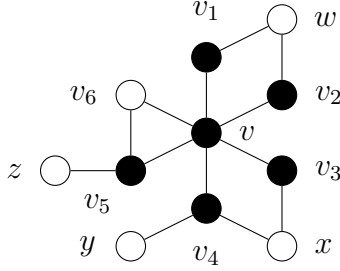


Figure 4.6: A degree 6-vertex  $v$  with  $n_2(v) = 3$ ,  $n_3(v) = 2$  and incident to a  $(3, 5^+, 6)$ -face and two 4-faces.

Consider the graph  $G' = G - \{v_1, v_2, v_3\}$ . By the minimality of  $G$ ,  $G'$  admits an acyclic  $L$ -colouring  $\pi$ . Let  $\alpha = |\{\pi(v_4), \pi(v_5), \pi(v_6)\}|$  and consider the following two cases:

$\alpha = 3$  If there exists a colour  $c \in L(v) \setminus \{\pi(v_4), \pi(v_5), \pi(v_6)\}$  which does not appear in  $\pi(w)$  and  $\pi(x)$ , then recolour  $v$  with  $c$  and properly colour  $v_1, v_2$  and  $v_3$  with an admissible colour different from their two neighbours (note that if  $\pi(v)$  does not appear in  $\pi(w)$  and  $\pi(x)$ , then no recolouring of  $v$  occurs). Otherwise each colour in  $L(v) \setminus \{\pi(v_4), \pi(v_5), \pi(v_6)\}$  appears exactly once in  $\pi(w)$  and  $\pi(x)$ . Recolour  $v$  with  $\pi(x)$  (recolouring will not occur if  $\pi(v) = \pi(x)$ ), properly colour  $v_1$  and  $v_2$  with an admissible colour different from  $\pi(w)$  and  $\pi(v)$  and colour  $v_3$  with an admissible colour different from  $\pi(v), \pi(v_4), \pi(v_5), \pi(v_6)$ .

$\alpha = 2$  If  $\pi(v)$  appears at most once in  $\pi(w)$  and  $\pi(x)$ , then if  $\pi(x) = \pi(v)$



colour  $v_1, v_2$  different from  $\pi(v), \pi(w)$  and colour  $v_3$  different from  $\pi(v), \pi(v_4), \pi(v_5), \pi(v_6)$ . Otherwise,  $\pi(w) = \pi(v)$ . Colour  $v_3$  different from  $\pi(v), \pi(x)$ , colour  $v_1$  different from  $\pi(v), \pi(v_4), \pi(v_5), \pi(v_6)$  and  $v_2$  different from  $\pi(v), \pi(v_4), \pi(v_5), \pi(v_6), \pi(v_1)$ .

Since  $\pi(v_4)$  is coloured the same as one of  $\pi(v_5)$  or  $\pi(v_6)$ , then if  $\pi(v) = \pi(w) = \pi(x)$  recolour  $v$  with a colour  $c \in L(v) \setminus \{\pi(v), \pi(v_4), \pi(v_5), \pi(v_6), \pi(y)\}$  and give a proper colouring to  $v_1, v_2$  and  $v_3$  with colours different from their two neighbours.

Since the colouring of  $G'$  has been extended to  $G$ , then this contradicts our choice of  $G$ .

(B1.2) Suppose that  $v$  is a 6-vertex  $v$  with neighbours  $v_1, \dots, v_6$  where  $d(v_1) = d(v_2) = d(v_3) = 2$  and  $d(v_5) = 3$ . Suppose to the contrary that  $v_4$  is a weak 6-vertex. Let  $[vv_1wv_2]$  and  $[vv_3xv_4]$  be two 4-faces. Since a weak 6-vertex is not incident to a 3-face and by (C9.1), let  $[v_5v_6v]$  be a  $(3, 5^+, 6)$ -face as in Figure 4.7. Let  $u_1, \dots, u_4$  be the neighbours of  $v_4$  different from  $v$  and  $x$  such that all are 2-vertices. Let  $z$  be the neighbour of  $v_5$  different from  $v$  and  $v_6$ .

Consider the graph  $G' = G - \{v, v_1, v_2, v_3, v_4, u_1, u_2, u_3, u_4\}$ . By the minimality of  $G$ ,  $G'$  admits an acyclic  $L$ -colouring  $\pi$ . Let  $\pi(v) = c \in L(v) \setminus \{\pi(v_5), \pi(v_6), \pi(w), \pi(x)\}$ , let  $v_4 = c_4 \in L(v_4) \setminus \{c, \pi(x), \pi(y_1), \pi(y_2)\}$ . Since both  $\pi(w)$  and  $\pi(x) \neq \pi(v)$  and both  $\pi(y_1)$  and  $\pi(y_2) \neq \pi(v_4)$ , colour each of  $v_1, \dots, v_3, u_1, \dots, u_4$  with an admissible colour which is

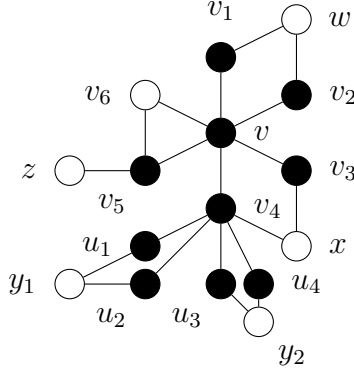


Figure 4.7: A degree 6-vertex  $v$  with  $n_2(v) = 3$ ,  $w_6(v) = 1$  and incident to a  $(3, 5^+, 6)$ -face and two 4-faces.

different from that of its two neighbours. Since the colouring of  $G'$  has been extended to  $G$ , then this contradicts our choice of  $G$ .

(B2) Suppose to the contrary that  $v$  is adjacent to two vertices with degree 2, say  $v_1$  and  $v_2$ , a  $(6, 3, 4)$ -face  $[vv_3v_4]$  and, by (C9.1), a second 3-face  $[vv_5v_6]$ . Let  $u_i$  be the neighbour of  $v_i$  different from  $v$  for  $i = \{1, 2\}$  and  $u_3$  be the neighbour of  $v_3$  different from  $v$  and  $v_4$ . Let  $x$  (and  $y$ ) be the neighbours of  $v_4$  different from  $v$  and  $v_3$  as shown in Figure 4.8.

Consider the graph  $G' = G - \{v, v_1, v_2, v_3\}$ . By the minimality of  $G$ ,  $G'$  admits an acyclic  $L$ -colouring  $\pi$ . Let  $\alpha = |\{\pi(v_4), \pi(v_5), \pi(v_6)\}|$  and consider the following two cases:

- If  $\alpha = 3$  then there exists a colour  $c \in L(v) \setminus \{\pi(v_4), \pi(v_5), \pi(v_6)\}$  which appears at most once in the colouring of  $u_1, u_2$  and  $u_3$ .

◊ If  $\pi(u_1) = c$ , then let  $\pi(v) = c$ , let  $\pi(v_1)$  be coloured with  $c_1 \in$

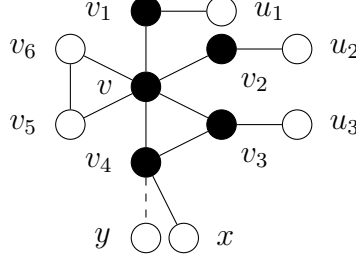


Figure 4.8: A degree 6-vertex  $v$  with  $n_2(v) = 2$  and  $t(v) = 2$ .

$L(v_1) \setminus \{c, \pi(v_4), \pi(v_5), \pi(v_6)\}$ , let  $\pi(v_2) = c_2 \in L(v_2) \setminus \{c, \pi(u_2)\}$  and let  $\pi(v_3) = c_3 \in L(v_3) \setminus \{c, \pi(v_4), \pi(u_3)\}$ .

- ◇ If  $\pi(u_2) = c$ , then with a similar argument to  $\pi(u_1) = c$ , let  $\pi(v) = c$ , let  $\pi(v_2)$  be coloured with  $c_2 \in L(v_2) \setminus \{c, \pi(v_4), \pi(v_5), \pi(v_6)\}$ , let  $\pi(v_1) = c_1 \in L(v_1) \setminus \{c, \pi(u_1)\}$  and let  $\pi(v_3) = c_3 \in L(v_3) \setminus \{c, \pi(v_4), \pi(u_3)\}$ .
- ◇ Otherwise  $\pi(u_3) = c$ . Let  $\pi(v) = c$ , let  $\pi(v_i)$  be a colour different from  $c$  and  $\pi(u_i)$  for  $i = \{1, 2\}$ , and let  $\pi(v_3)$  be a colour different from  $c, \pi(v_4), \pi(v_5), \pi(v_6)$ .
- If  $\alpha = 2$ , then consider the following.
  - ◇ If  $\pi(x) \neq \pi(y)$  then recolour  $v_4$  with a colour different from  $\pi(x)$ ,  $\pi(y)$ ,  $\pi(v_5)$ ,  $\pi(v_6)$ , then colour  $v$ ,  $v_1$ ,  $v_2$  and  $v_3$  as above, since  $\pi(v_4)$ ,  $\pi(v_5)$  and  $\pi(v_6)$  are now pairwise distinct.
  - ◇ Otherwise  $\pi(x) = \pi(y)$ . Since  $|\{\pi(v_4), \pi(v_5), \pi(v_6), \pi(x), \pi(y)\}| \leq 3$ , then there exists a colour  $c \in L(v) \setminus \{\pi(v_4), \pi(v_5), \pi(v_6), \pi(x), \pi(y)\}$

which appears at most once in the colouring of  $u_1, u_2$  and  $u_3$ . With a similar argument to the case when  $\alpha = 3$ , let  $\pi(v) = c$ , colour two of  $v_1, v_2$  and  $v_3$  with a colour different from its neighbours if  $v$  is the only neighbour coloured with  $c$ , and colour one of  $v_1, v_2$  and  $v_3$  with a colour different from  $c, \pi(v_4), \pi(v_5), \pi(v_6)$  if two of its neighbours are coloured with  $c$ . This contradicts our choice of  $G$ , as we can obtain an acyclic colouring of  $G$ .

□

### 4.1.3 Restrictions On A 9-, 10-, 11-vertex

**Lemma 5.** *Let  $v$  be a 9-, 10- or 11-vertex. Then  $n_2(v) \leq d(v) - 2$ .*

*Proof.* Let  $v$  be a 9- 10- or 11-vertex and let  $N(v) = \{v_1, \dots, v_k\}$  for  $9 \leq k \leq 11$  be the neighbours of  $v$ . Assume for contradiction that  $n_2(v) = d(v) - 1$ , with  $d(v_2) = \dots = d(v_k) = 2$ . Let  $u_1, \dots, u_k$  be the neighbour of  $v_1, \dots, v_k$  different than  $v$ , as in Figure 4.9.

Let  $G' = G - \{v, v_2, \dots, v_k\}$ . Then  $G'$  admits an acyclic  $L$ -colouring  $\pi$  by the minimality of  $G$ . There exists a colour  $c \in L(v) \setminus \{\pi(v_1)\}$  which appears at most twice in  $u_2, \dots, u_k$ . Let  $\pi(v) = c$  and assume without loss of generality that  $\pi(u_2) = \pi(u_3) = c$ . Let  $\pi(v) = c$ , let  $\pi(v_2) = c_2 \in L(v_2) \setminus \{c, \pi(v_1)\}$ ,  $\pi(v_3) = c_3 \in L(v_3) \setminus \{c, c_2, \pi(v_1)\}$  and colour each of  $v_4, \dots, v_k$  with a colour different  $c, \pi(u_i)$  for  $i = 4, \dots, k$ . Since the colouring of  $G'$  has been extended to  $G$ , then this contradicts our choice of  $G$ .

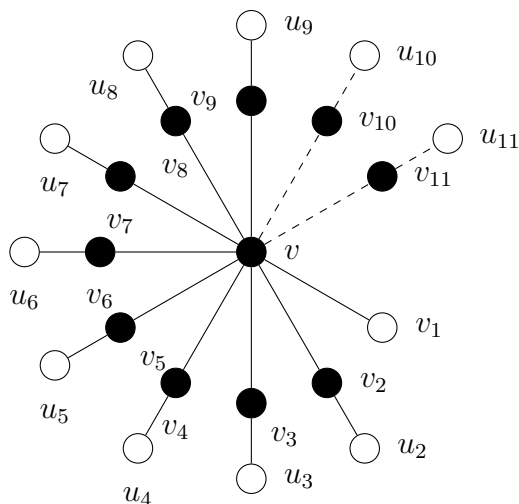


Figure 4.9: A 9-, 10-, 11-vertex with  $n_2(v) = d(v) - 1$ .

□

## 4.2 Initial Charging

In order to complete the proof, we suppose that  $G$  is a counterexample to Theorem 1 with the least number of vertices. Let  $L$  be a list assignment such that  $|L(v)| \geq 5$  for all  $v \in V(G)$ . Thus,  $G$  satisfies Lemma 1 to 5. Since  $G$  is a planar graph, then; using Euler's formula,  $G$  has the characteristic that  $|V(G)| - |E(G)| + |F(G)| = 2$ . With some arithmetic manipulation, the relation yields  $(4|E| - 6|V|) + (2|E| - 6|F|) = -12$ . When summing the degree of the vertices and the degrees of the faces of  $G$ , we can derive the following identities:  $\sum d(v) = 2|E(G)|$  and  $\sum d(f) = 2|E(G)|$ , since each

edge is counted twice in each summation.

$$\begin{aligned}
 & \sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F} (d(f) - 6) \\
 &= 4|E| - 6|V| + 2|E| - 6|F| \\
 &= 6(|E| - |V| - |F|) = -12.
 \end{aligned}$$

If  $G$  is assigned charges of  $2d(v) - 6$  to each vertex  $v \in V(G)$  and is assigned charges of  $d(f) - 6$  to each face  $f \in F(G)$ , where  $F(G)$  is the set of faces  $G$ , then the total charge assigned to  $G$  is  $-12$ . We define a charge function  $ch$  by  $ch(v) = 2d(v) - 6$  for all  $v \in V(G)$ , and  $ch(f) = d(f) - 6$  if  $f \in F(G)$ . It follows from the above identity that the total sum of charges  $ch(G) = -12$ . We design appropriate discharging rules and redistribute charges accordingly. Once the discharging is finished, a new charge function  $ch^*$  is produced. However, the total sum of charges is kept fixed when the discharging is in process. Nevertheless, after the discharging is complete, the new charge function  $ch^*(x) \geq 0$  for all  $x \in V(G) \cup F(G)$ . This leads to the following obvious contradiction,

$$0 \leq \sum_{x \in V(G) \cup F(G)} ch^*(x) = \sum_{x \in V(G) \cup F(G)} ch(x) = -12.$$

For  $x, y \in V(G) \cup F(G)$ , let  $\tau(x \rightarrow y)$  denote the amount of charges transferred from  $x$  to  $y$ . Suppose that  $f = [v_1 v_2 v_3]$  is a 3-face with  $d(v_1) \leq$

$d(v_2) \leq d(v_3)$ . We use  $(d(v_1), d(v_2), d(v_3)) \rightarrow (c_1, c_2, c_3)$  to denote that the vertex  $v_i$  gives  $f$  the amount of charge  $c_i$  for  $i = 1, 2, 3$ . For example,  $R2(d)$  in section 4.3 states as follows:

$$(4^+, 4^+, 4^+) \rightarrow (1, 1, 1).$$

This discharging rule states that each  $4^+$ -vertex gives 1 charge to each incident 3-face. In other words, each 3-face having all incident vertices with degrees at least 4 receives 1 charge from each of these vertices. Our discharging rules are as follows:

### 4.3 Discharging Rules

R1 Every  $5^+$ -vertex  $v$  gives

- (a) 1 charge to each adjacent 2-vertex
- (b)  $\frac{1}{2}$  charge to each adjacent pendant light 3-vertex.

R2 Let  $f = [v_1v_2v_3]$  be a 3-face with  $d(v_1) \leq d(v_2) \leq d(v_3)$ . Then

- (a)  $(3, 3, 6^+) \rightarrow (\frac{1}{2}, \frac{1}{2}, 2)$ ;
- (b)  $(3, 4, 5^+) \rightarrow (\frac{1}{2}, 1, \frac{3}{2})$ ;
- (c)  $(3, 5^+, 5^+) \rightarrow (\frac{1}{2}, \frac{5}{4}, \frac{5}{4})$ ;
- (d)  $(4^+, 4^+, 4^+) \rightarrow (1, 1, 1)$ .

R3 Let  $f = [v_1v_2v_3v_4]$  be a weak 4-face with  $w$  being a weak 6-vertex and  $d(x) = 2$ . By Definition 1,  $f$  has two possible configurations. Then

(a)  $(6, 2, 6^+, 2) \rightarrow (1, 0, 1, 0)$ ;

(b)  $(6, 2, 5^+, 4^+) \rightarrow (\frac{1}{2}, 0, \frac{3}{4}, \frac{3}{4})$ ;

R4 Let  $f$  be a  $4^+$ -face in  $G$  with boundary vertices  $v_1, \dots, v_k$ , for  $k = d(f)$ . Let  $\Gamma(f)$  be the set of vertices  $\{v_1, \dots, v_k\} \setminus \{v_i\}$  such that  $v_i$  is a 2-, 3- or weak 6-vertex in the boundary of  $f$ . Let  $\gamma(f) = |\Gamma(f)|$ . Then each vertex  $v \in \Gamma(f)$  gives

(a)  $\frac{2}{\gamma(f)}$  to each 4-face which is not weak;

(b)  $\frac{1}{\gamma(f)}$  to each 5-face.

## 4.4 Proof of Theorem 1

For each  $x \in V(G) \cup F(G)$  we will show that  $ch^*(x) \geq 0$ , which produces a contradiction.

### *Proof.* 4.4.1 Faces

$d(f) = 3$  An initial charge of  $ch(f) = -3$  is assigned to all 3-faces  $f = [xyz] \in G$ .

By (C9.1), a 2-vertex is not incident to a 3-face. By (C9.2),  $G$  does not contain a  $(3, 3, 5^-)$ -face. So all 3-faces with  $n_3(f) = 2$  are  $(3, 3, 6^+)$ -faces.

By R2(a),  $ch^*(f) = ch(f) + \tau((x, y, z) \rightarrow (f)) = -3 + \frac{1}{2} + \frac{1}{2} + 2 = 0$ .



By (C9.2),  $G$  does not contain a  $(3, 4, 4)$ -face. If  $d(x) = 3$  and  $d(y) = 4$ , then  $z$  is a  $5^+$ -vertex. By R2(b),  $ch^*(f) = ch(f) + \tau((x, y, z) \rightarrow (f)) = -3 + \frac{1}{2} + 1 + \frac{3}{2} = 0$ .

If  $f$  is a  $(3, 5^+, 5^+)$ -face, then by R2(c),  $ch^*(f) = ch(f) + \tau((x, y, z) \rightarrow (f)) = -3 + \frac{1}{2} + \frac{5}{4} + \frac{5}{4} = 0$ .

Otherwise,  $f$  is a  $(4^+, 4^+, 4^+)$ -face. By R2(d),  $ch^*(f) = ch(f) + \tau((x, y, z) \rightarrow (f)) = -3 + 1 + 1 + 1 = 0$ .

$$d(f) = 4$$

**Observation 1.** *Let  $f = [wxyz]$  be a 4-face. If  $f$  is not weak then by (C2) and (C3.1),  $n_2(f) + n_3(f) \leq 2$  and hence  $\Gamma(f) \geq 2$ . Otherwise, if  $f$  is weak, then consider the following two cases. If  $n_2(f) = 2$ , then  $\Gamma(f) \leq 1$ . If  $n_2(f) = 1$ , then by (C2) and Lemma 1,  $\Gamma(f) = 2$ .*

An initial charge of  $ch(f) = -2$  is assigned to all 4-faces  $f = [wxyz] \in G$ . Consider first if  $f$  is weak. If  $\gamma(f) = 0$  or  $1$ , then by R3(a),  $ch^*(f) = ch(f) + \tau((w, x, y, z) \rightarrow (f)) = -2 + 1 + 0 + 1 + 0 = 0$ . Otherwise, by Observation 1,  $\gamma(f) = 2$ . By R3(b)  $ch^*(f) = ch(f) + \tau((w, x, y, z) \rightarrow (f)) = -2 + \frac{1}{2} + 0 + \frac{3}{4} + \frac{3}{4} = 0$ . If,  $f$  is not weak, then by R4(a),  $ch^*(f) = ch(f) + \tau(\Gamma(f) \rightarrow (f)) = -2 + \frac{2}{\gamma(f)} * \gamma(f) = 0$ .

$$d(f) = 5$$

**Observation 2.** *Let  $f = [vwxyz]$  be a 5-face. If  $f$  is not weak, then by (C2),  $n_2(f) \leq 2$ . If  $n_2(f) = 2$ , then  $\Gamma(f) = 3$ . If  $n_2(f) \leq 1$ ,*

then by (C3.1),  $\Gamma(f) \geq 2$ . If  $w$  is a weak 6-vertex then by (C6.2) and Definition 1, then vertices  $v$  and  $x$  are not 2-, 3-, or weak 6-vertices. Hence,  $\gamma(f) \neq 0$ .

An initial charge  $ch(f) = -1$  is assigned to all 5-faces  $f \in G$ . Since by Observation 2 and by R4(b),  $ch^*(f) = ch(f) + \tau(\Gamma(f) \rightarrow (f)) = -1 + \frac{1}{\gamma(f)}\gamma(f) = 0$ .

$d(f) \geq 6$  An initial charge  $ch(f) \geq 0$  is assigned to all 6<sup>+</sup>-faces  $f \in G$ . Then  $ch^*(f) = ch(f) \geq 0$ , as the discharging rules do not transfer charges from faces to either vertices or faces.

#### 4.4.2 Vertices

$d(v) = 2$  An initial charge  $ch(v) = -2$  is assigned to all 2-vertices  $v \in G$ . By (C2), if  $u$  and  $w$  are the two neighbours of  $v$  with  $d(u) \leq d(w)$ , then  $5 \leq d(u) \leq d(w)$ . By R1(a), both  $u$  and  $w$  give 1 charge to  $v$ . Hence,  $ch^*(v) = ch(v) + \tau((u, w) \rightarrow v) = -2 + 1 + 1 = 0$ .

$d(v) = 3$  An initial charge  $ch(v) = 0$  is assigned to all 3-vertices  $v \in G$ . If  $v$  is a pendant light 3-vertex of a vertex  $u$ , and is incident to a 3-face  $f$ , then by R1(b) and R2(a),(b),(c),  $ch^*(v) = ch(v) + \tau(u \rightarrow v) - \tau(v \rightarrow f) = 0 + \frac{1}{2} - \frac{1}{2} = 0$ . Otherwise,  $v$  is not incident to a 3-face, hence  $ch^*(v) = ch(v) = 0$ .

**Observation 3.** *By Observation 1, R3 and R4(a), every 4<sup>+</sup>-vertex  $v \in G$  gives at most 1 charge to each incident 4-face. Since, by assumption,*

a 4-face is not adjacent to a 3-, 4- or 5-face, then every  $4^+$ -vertex  $v \in G$ , on average, gives at most  $\frac{1}{2}$  charge to each of two adjacent faces  $f_1$  and  $f_2$  is one of these faces is a 4-face. By Observation 2 and R4(b), since  $\gamma \geq 2$  for all 5-faces in  $G$ , then every  $v \in G$  gives at most  $\frac{1}{2}$  charges to each incident 5-face. Similarly, since a 3-face is not adjacent to a 4- or 5-face, then a pendant light 3-vertex  $v$  of  $u$  is incident to one 3-face and two  $6^+$ -faces. And, like the 5-face, a pendant light 3-vertex requires only  $\frac{1}{2}$  charges from its neighbour  $u$ . Hence, a vertex having  $\frac{d(v)}{2}$  charges available to distribute to its incident faces has sufficient charge if  $v$  is not incident a 3-face.

If, however,  $v$  is incident to 3-faces, then for any  $6^+$ -vertex,  $v$  distributes up to 2 charges to a 3-face by R3. Since, by the assumption that a 3-face is not adjacent to a 3-, 4- or 5-face, then every  $v \in G$ , on average, at most 1 to each of two adjacent faces  $f_1$  and  $f_2$  is one of these faces is a 3-face. Hence, a vertex  $v$  which has at least 1 charge available to distribute to each of  $d(v)$  incident faces has sufficient charges for any configuration which contains  $\lfloor \frac{d(v)}{2} \rfloor$  3-faces.

Lastly, since a 2-vertex is not incident to a 3-face by (C9.1), then we consider a vertex  $v$  which is incident a 4-face containing two 2-vertices. By R1(a), R3(a) and R4(a), every  $6^+$ -vertex  $v$  distributes 1 charge to each 2-vertex and 4-face. Since, by assumption a 4-face is not adjacent to a 3-, 4- or 5-face, then a vertex with  $\lfloor \frac{3d(v)}{2} \rfloor$  has sufficient charge for every possible configuration.

In the following lemmas we define  $N_2(v)$  as the set of vertices  $v_i \in N(v)$  such that  $d(v_i) = 2$ . Similarly,  $N_3(v)$  as the set of vertices  $v_i \in N(v)$  such that  $v_i$  is a pendant light 3-vertex of  $v$ . Lastly, we define  $\{f_1, \dots, f_k\}$  is the set of faces incident with one common vertex  $v$ , where  $k = d(v)$ .

$d(v) = 4$  An initial charge  $ch(v) = 2$  is assigned to all 4-vertices  $v \in G$ . By (C2) and (C4),  $v$  is not adjacent to a 2-vertex or a pendant light 3-vertex. By (C9.3),  $v$  is not incident a (3, 4, 4)-face. By Observation 3, R2(b)(d) and R4,  $ch^*(v) \geq ch(v) - \tau(v \rightarrow (f_1, \dots, f_k)) = 2 - 4(\frac{1}{2}) = 0$ .

$d(v) = 5$  An initial charge  $ch(v) = 4$  is assigned to all 5-vertices  $v \in G$ . By (C5.1),  $v$  is adjacent to at most one 2-vertex. Consider the following two cases:

$n_2(v) = 1$  By (C5.2), if  $v$  is adjacent to a 2-vertex  $u$ , then  $v$  is not adjacent to a pendant light 3-vertex. Since, by (C5.3),  $v$  distributes one charge to each incident 3-face as per R3(d). Hence, as in Observation 3,  $ch^*(v) \geq ch(v) - \tau(v \rightarrow u) - \tau(v \rightarrow (f_1, \dots, f_k)) = 4 - 1(1) - 5(\frac{1}{2}) = \frac{1}{2}$  by R1(a), R4 and by Observation 2.

$n_2(v) = 0$  If  $v$  is not incident to a 3 or 4-face, then  $v$  is incident to at most five 5-faces which require charge, meaning  $ch^*(v) \geq ch(v) - \tau(v \rightarrow (f_1, \dots, f_k)) = ch(v) - 5(\frac{1}{2}) = \frac{3}{2}$  by R4(b) and Observation 2. If  $v$  is incident to one 4-face, then  $v$  is incident to at most three 5-faces or three pendant light 3-vertices by (C5.4). By R1(b) and R4(a)(b),  $ch^*(v) \geq ch(v) - \tau(v \rightarrow (f_1, \dots, f_k)) = ch(v) - 1(1) - 3(\frac{1}{2}) = \frac{3}{2}$ .

If  $v$  is incident to two 4-faces, then  $v$  is not incident to a 5-face, but adjacent to at most one pendant light 3-vertex. By R1(b) and R4(a),  $ch^*(v) \geq ch(v) - \tau(v \rightarrow (f_1, \dots, f_k)) - \tau(v \rightarrow N_3(v)) = ch(v) - 2(1) - 1(\frac{1}{2}) = \frac{3}{2}$ . If  $v$  is incident to one 3-face  $f_1$ , then  $v$  is incident to at most three 5-faces or at most two pendant light 3-vertices by (C5.5) and (C5.6). Since by (C9.2)  $f_1 \neq (3, 3, 5)$ -face, then by R1(b), R2(b)(c)(d) and R4(b),  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_3(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = ch(v) - 1(\frac{3}{2}) - 3(\frac{1}{2}) = 1$ . If  $v$  is incident to two 3-faces, then  $v$  is not incident to a 5-face, but adjacent to at most one pendant light 3-vertex. Hence,  $ch^*(v) \geq ch(v) - \tau(v \rightarrow (f_1, \dots, f_k)) - \tau(v \rightarrow N_3(v)) = ch(v) - 2(\frac{3}{2}) - 1(\frac{1}{2}) = \frac{1}{2}$ .

$d(v) = 6$  An initial charge  $ch(v) = 6$  is assigned to all 6-vertices  $v \in G$ . By (C6.1),  $v$  is adjacent to at most four 2-vertex. Consider the following cases:

$n_2(v) = 4$  If  $v$  is weak, then by Definition 1, it is incident to either three 4-faces or to two 4-faces and one 5-face. By R1(a) and R3(a), either  $ch^*(v) = ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow \{f_1, \dots, f_k\}) = 6 - 4 - 1 - 1 = 0$  if every vertex in  $N_2(v)$  are contained in only two 4-faces. Otherwise by R1(a) and R3(b),  $ch^*(v) = ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow \{f_1, \dots, f_k\}) = 6 - 4 - 1 - \frac{1}{2} - \frac{1}{2} = 0$ . If  $v$  is not weak, then by R1, it distributes one to each incident 2-

vertex, leaving two charges to distribute amongst its incident faces. As a consequence of (C6.2) and by (C6.3),  $p_3(v) = 0$  and  $v$  is not incident to a 3-face. By Definition 1,  $v$  is incident to at most two 4-faces (else, it would be weak), so consider the following cases: If  $v$  is incident to two 4-faces, then  $ch^*(v) = ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow \{f_1, \dots, f_k\}) = 6 - 4 - 1 - 1 = 0$  if  $v$  is incident to four  $6^+$ -faces. Otherwise,  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow \{f_1, \dots, f_k\}) = 6 - 4 - 1(1) - 1(\frac{2}{3}) - 1(\frac{1}{3}) = 0$  If  $v$  is incident to a 5-face. If  $v$  is incident one 4-face then by assumption,  $v$  is incident to at most three 5-faces. By (C2), (C3.1) and Definition 1, each of these 5-faces has  $\Gamma \geq 3$ . By R1(a) and R4(b),  $ch^*(v) \geq ch(v) - 4(1) - 1(1) - 3(\frac{1}{3}) = 0$ .

$n_2(v) = 3$  Then by (C6.8) and (C9.1),  $v$  is incident to at most one 3-face which is not a  $(3, 4^-, 6)$ -face. Consider the following two cases:

If  $v$  is incident to a  $(3, 5^+, 6)$ -face, then by (C6.4),  $p_3(v) = 0$ . If  $v$  is not incident to a 4-face, then  $v$  is incident to at most three 5-faces. Hence  $ch^* \geq ch(v) - 3(1) - 1(\frac{5}{4}) - 3(\frac{1}{2}) = \frac{1}{4}$  by R2(a) and R4(b). Otherwise,  $v$  is incident to at most two 4-faces. By Lemma 4, R2 and R4,  $ch^* \geq ch(v) - 3(1) - 1(\frac{5}{4}) - 1(1) - 1(\frac{2}{3}) = \frac{1}{12}$ .

$n_2(v) = 2$  By R1(a),  $v$  has four charges to distribute to its adjacent pendant light 3-vertices and its incident faces. By (C6.5) and (C6.6) and by lemma 4,  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow \{f_1, \dots, f_k\}) = 6 - 2(1) - 2(\frac{5}{4}) - 2(\frac{1}{2}) = \frac{1}{2}$  since by assumption, a 3-face is not

adjacent to a 3-, 4- or 5-face.

$n_2(v) = 1$  Then by R1(a), R2, R4, and Observation 3 then  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow \{f_1, \dots, f_k\}) = 6 - 1(1) - 6(\frac{1}{2}) = 2$  if  $v$  is not incident to a 3-face and  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow \{f_1, \dots, f_k\}) = 6 - 1(1) - 4(1) - 2(\frac{1}{2}) = 0$ .

$n_2(v) = 0$  If  $n_2(v) = 0$ , then  $v$  has at least one charge to give to each incident face. By the assumption that  $G$  contains no adjacent small cycles and by R2 and R4, this is sufficient charge. Hence  $ch^*(v) \geq 0$  for every 6-vertex in  $G$ .

$d(v) = 7$  An initial charge of  $ch(v) = 8$  is assigned to each 7-vertex in  $G$ . By (C7.1), since  $n_2(v) \leq 5$ , consider the following cases:

$n_2(v) = 5$  By (C7.3),  $n_3(v) = 0$  and  $t(v) = 0$ . If  $v$  is not incident to a 4-face then  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 8 - 5(1) - 7(\frac{1}{3}) = \frac{1}{3}$ , as per Observation 2. Otherwise,  $v$  is incident to at most three 4-faces. Hence,  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 8 - 5(1) - 3(1) = 0$  since by assumption no 4-face is adjacent to a 3-, 4- or 5-face.

$n_2(v) = 4$  By (C7.2) and (C7.4), consider the following three cases. Since  $v$  is incident to at most one 3-face, then if  $v$  is incident to a  $(3, 3, 7)$ -face, it is not adjacent to a pendant light 3-vertex. Hence,  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 8 - 4(1) - 1(2) - 4(\frac{1}{2}) = 0$  by Observation 3. If  $t(v) = 0$ , then by R1(a)(b) and

R4(a)(b),  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 8 - 7(\frac{1}{2}) = \frac{1}{2}$  if  $v$  is adjacent to pendant light 3-vertices, and incident to  $5^-$ -faces.

$n_2(v) = 3$  By (C9.1),  $v$  is incident to at most two 3-faces. If  $t(v) = 2$  then  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 8 - 3(1) - 2(2) - 2(\frac{1}{2}) = 0$  since by assumption, a 3-face is not incident a 3-, 4- or 5-face. if  $t(v) = 1$ , then  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 8 - 3(1) - 1(2) - 6(\frac{1}{2}) = 0$ . Hence, by Observation 3,  $v$  has sufficient charge to distribute to at most one adjacent pendant light 3-vertex (as per C7.5), and any configuration of  $5^-$ -faces.

$n_2(v) = 2$  By R1(a)(b) and R4, if  $v$  is not incident a 3-face, then  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 8 - 2(1) - 7(\frac{1}{2}) = \frac{5}{2}$  by Observation 3. Otherwise, by (C9.1),  $t(v) \leq 2$ . If  $t(v) = 2$ , then  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 8 - 2(1) - 2(2) - 2(\frac{1}{2}) = 1$ , by R1(a), R2(c) and R4, and by the assumption that a 3-face is not adjacent to a 3-, 4- or 5-face. If  $t(v) = 1$ , then  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 8 - 2(1) - 1(2) - 4(\frac{1}{2}) = 2$ .

$n_2(v) \leq 1$  If  $n_2(v) \leq 1$ , then  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 8 - 1(1) - 7(1) = 0$ . Since  $v$  has at least one charge to give to each incident face, then by Observation 3,  $v$  has sufficient charge.



$d(v) = 8$  An initial charge of  $ch(v) = 10$  is assigned to each 8-vertex in  $G$ . By (C8.1),  $n_2(v) \leq 6$ . If  $n_2(v) = 6$ , then by Observation 3, R1(a), R2(c) and R4(a)(b),  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 10 - 6(1) - 8(\frac{1}{2}) = 0$ , since  $v$  is not incident a 3-face by (C8.2). If  $n_2(v) = 5$ , then  $t(v) \leq 1$ . If  $v$  is incident to a 3-face then  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 10 - 5(1) - 1(2) - 5(\frac{1}{2}) = \frac{1}{2}$  since, by assumption, a 3-face is not adjacent to a 3-, 4- or 5-face. Otherwise,  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 10 - 5(1) - 8(\frac{1}{2}) = 1$  if  $v$  is not incident to a 3-face. If  $n_2(v) = 4$ , then  $t(v) \leq 2$ . If  $v$  is incident to two 3-faces then  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 10 - 4(1) - 2(2) - 4(\frac{1}{2}) = 0$  since, by assumption, a 3-face is not adjacent to a 3-, 4- or 5-face and not every incident face of  $v$  requires charge distribution. If  $v$  is incident to at most one 3-face then  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 10 - 4(1) - 1(2) - 7(\frac{1}{2}) = \frac{1}{2}$ .

$d(v) = 9$  An initial charge of  $ch(v) = 12$  is assigned to each 9-vertex in  $G$ . By Lemma 5,  $n_2(v) \leq 7$ . By Observation 3, R1(a), R2(c) and R4, if  $n_2(v) = 7$ , then  $v$  is incident to at most one 3-face by (C9.1). If  $v$  is incident to a 3-face then  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 12 - 7(1) - 1(2) - 6(\frac{1}{2}) = 0$  since a 3-face is not adjacent to a 4- or 5-face. Otherwise, if  $v$  is not incident a 3-face, then  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 12 - 7(1) - 9(\frac{1}{2}) = \frac{1}{2}$ . If  $n_2(v) = 6$ , then  $v$  is still incident to at most one 3-face, so consider that

$n_2(v) = 5$ . Then  $v$  is incident to at most two 3-faces, hence  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 12 - 5(1) - 2(2) - 4(\frac{1}{2}) = 1$  since at least three faces incident to  $v$  are  $6^+$ -faces. If  $n_2(v) = 4$  then again  $v$  is incident to at most two 3-faces. By R2,  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 12 - 4(1) - 2(2) - 7(\frac{1}{2}) = \frac{1}{2}$ . Finally, if  $n_2(v) \leq 3$  then by Observation 3, R1(a), R2 and R4,  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 12 - 3(1) - 9(1) = 12 - 12 = 0$ .

$d(v) = 10$  An initial charge of  $ch(v) = 14$  is assigned to each 10-vertex in  $G$ . By Lemma 5,  $n_2(v) \leq 8$ . By Observation 3, R1(a), R2(c) and R4, if  $n_2(v) = 8$ , then  $v$  is incident to at most one 3-face by (C9.1). If  $v$  is incident to a 3-face then  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 14 - 8(1) - 1(2) - 7(\frac{1}{2}) = \frac{1}{2}$  since a 3-face is not adjacent to a 4- or 5-face. Otherwise, if  $v$  is not incident a 3-face, then  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 14 - 8(1) - 10(\frac{1}{2}) = 1$ . If  $n_2(v) = 7$ , then  $v$  is still incident to at most one 3-face, so consider that  $n_2(v) = 6$ . Then  $v$  is incident to at most two 3-faces, hence  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 14 - 6(1) - 2(2) - 8(\frac{1}{2}) = 0$ . If  $n_2(v) = 5$  then again  $v$  is incident to at most two 3-faces. By R2,  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 14 - 5(1) - 2(2) - 8(\frac{1}{2}) = 1$ . Finally, if  $n_2(v) \leq 4$  then by Observation 3, R1(a), R2 and R4,  $ch^*(v) \geq ch(v) - 4(1) - 10(1) = 14 - 14 = 0$ .

$d(v) = 11$  An initial charge of  $ch(v) = 16$  is assigned to each 11-vertex in  $G$ . By

Lemma 5,  $n_2(v) \leq 9$ . By Observation 3, R1(a), R2(c) and R4, if  $v$  is incident to a 3-face then  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 16 - 9(1) - 1(2) - 10(\frac{1}{2}) = 0$ . If  $v$  is not incident to a 3-face then  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 16 - 9(1) - 11(\frac{1}{2}) = \frac{3}{2}$ . If  $n_2(v) = 8$ , then  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 16 - 8(1) - 1(2) - 10(\frac{1}{2}) = \frac{3}{2}$  if  $v$  is incident to one (3, 3, 11)-face. If  $n_2(v) = 7$ , then  $v$  is incident to at most two 3-faces. Hence, if all other incident faces are 4- or 5-faces, then by Observation 3,  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 16 - 7(1) - 2(2) - 9\frac{1}{2} = \frac{1}{2}$ . If  $n_2(v) = 6$ , then again  $v$  is incident to at most two 3-faces. Hence, if all other incident faces are 4- or 5-faces, then by Observation 3,  $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \dots, f_k)) = 16 - 6(1) - 2(2) - 9\frac{1}{2} = \frac{3}{2}$ . If  $n_2(v) \leq 5$ , then  $v$  has at least one charge to give to each incident face. By Observation 3,  $v$  has sufficient charge to distribute to its incident faces.

$d(v) \geq 12$  An initial charge of  $ch(v) \geq 18$  is assigned to each  $12^+$ -vertex in  $G$ . By Observation 3,  $ch^*(v) \geq ch(v) - \left\lfloor \frac{3d(v)}{2} \right\rfloor \geq 18 - \left\lfloor \frac{3(12)}{2} \right\rfloor = 0$ .

□

It was discussed earlier that

$$\sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F} (d(f) - 6) = -12.$$

is the total charge on the graph  $G$  because of the fact that  $G$  is planar. We have shown through careful redistribution of charges demonstrated in Lemmas 4.4.1 to 4.4.2 yields the desired contradiction that

$$0 \leq \sum_{x \in V(G) \cup F(G)} ch^*(x) = \sum_{x \in V(G) \cup F(G)} ch(x) = -12.$$

This last equation clearly shows that the assumption of the existence of a minimum counterexample  $G$  was incorrect. Without the existence of such a graph, we have succeeded in proving that every planar graph which does not contain a 3- or 4-cycle adjacent to a 3-, 4- and 5-cycle is acyclically 5-choosable.

# Chapter 5

## Conclusion

An improvement on this work will need further knowledge of the structure of  $G$ , especially the understanding of if and when triangles and 4-cycles can intersect, as most previous work does not allow intersections of such configurations. It was recently asked by Borodin and Ivanova [Bor13]

Is it true that every planar graph satisfies  $\chi(G) = \chi^l(G)$ ?

One might then ask about the relationship between  $\chi^l(G)$  and  $\chi_a^l(G)$ .

What, if any, is the connection between the  $k$ -choosability of planar graphs and general graphs?

Some progress has also been made in finding a bound for the  $k$ -choosability of general graphs for  $3 \leq k \leq 5$ . In this case, the maximum average degree,  $Mad(G)$  of a graph  $G$  is considered [MOR06]. Some recent results have found bounds of

- Every graph  $G$  with  $Mad(G) < \frac{8}{3}$  is acyclically 3-choosable

- Every graph  $G$  with  $Mad(G) < \frac{19}{6}$  is acyclically 4-choosable
- Every graph  $G$  with  $Mad(G) < \frac{24}{7}$  is acyclically 5-choosable

where

$$Mad(G) = \{max \frac{2|E(H)|}{|V(H)|}, \forall H \subseteq G\}.$$

These answers is still far off but strides are still being made. I myself have found many results on the restrictions of intersecting triangles and the restrictions on the degrees of the vertices of a 4-cycle in an attempt to allow a 3- and 4-cycle to exist adjacent a 5-cycle. There has been some difficulty in disproving the existence of some suns, i.e., a polygon with  $n$  sides adjacent to  $n$  triangles but early discharging attempts look promising.

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# Appendices

## Appendix A

### List of Symbols

Symbol	Description
$v$	The end point of an edge.
$k - vertex$	A vertex with exactly $k$ neighbours.
$k^+ - vertex$	A vertex with at least $k$ neighbours.
$k^- - vertex$	A vertex with at most $k$ neighbours.
$N(v)$	The set of neighbours adjacent to $v$ .
$n_i(v)$	The number neighbours of $v$ having degree $i$ .
$t(v)$	The number of 3-faces incident to a vertex $v$ .
$f$	A polygon enclosed by its outer edges.

---

$k$ – face	A face with exactly $k$ edges in its boundary.
$k^+$ – face	A face with at least $k$ edges in its boundary.
$k^-$ – face	A face with at most $k$ edges in its boundary.
$n_j(f)$	The number of incident vertices with degree $j$ .
$(a_1, a_2, a_3)$ -face	A 3-face with vertices having degrees $a_1, a_2$ and $a_3$ respectively.
$p_3(u)$	The number of pendant light 3-vertices of a vertex $u$ .
$(\alpha, \beta)$ -path	A path coloured with two alternating colours $\alpha$ and $\beta$ .
$w_6(f)$	The number of weak 6-vertices incident a 4- or 5-face.
$N_2(v)$	The set of 2-vertices in the neighbourhood of $v$ .
$N_3(v)$	The set of pendant light 3-vertices in the neighbourhood of $v$ .
$\tau(x \rightarrow y)$	The amount of charge redistributed from $x \in V(G) \cup F(G)$ to $y \in V(G) \cup F(G)$ .

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