Deformations of Maxwell gauge field theory

by

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A THESIS SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in

The Faculty of Mathematics and Sciences

Department of Physics

BROCK UNIVERSITY

Abstract

Deformations of Maxwell gauge theory are studied in 2+1 dimensions. Unlike in previous work in the literature, no Lagrangian structure is assumed for possible deformations, and instead the requirement of gauge invariance of the deformed field equations under the deformed gauge symmetry is used. The results yield a new nonlinear generalization of Maxwell gauge theory. A non-abelian extension of this theory is also obtained.

Contents

Acknowledgements

I would like to thank Professor Stephen Anco for his guidance, my love Nikki for her support and my parents for instilling a life long interest in learning.

Chapter 1

Introduction

An interesting problem in classical field theory is the study of nonlinear deformations of linear abelian gauge theories [1, 2, 3, 4]. Deformations in this setting consist of adding linear and higher power local terms to the abelian gauge symmetry, while also adding quadratic and higher power local terms to the linear field equation. This is done such that a locally gauge invariant field equation exists which is not equivalent to the undeformed linear theory by nonlinear field redefinitions. The condition of gauge invariance can be formulated in various ways [1, 3, 4, 5, 6] that yield determining equations to find all allowed deformation terms order-by-order in powers of the fields, without the need to assume any special structure a priori for the deformed field equation, the deformed gauge symmetry, or the gauge group (as determined by its commutators).

Of particular interest are deformations that have minimal differential degree. These theories have the least possible number of derivatives of the fields in the deformation terms.

One important example is provided by the structure of non-abelian Yang-Mills theory as a nonlinear Lagrangian gauge theory of 1-form fields with an infinitesimal gauge group based on any semi-simple Lie algebra (where the dimension of the algebra equals the number of fields) [7, 8]. In four and higher spacetime dimensions, this structure is the only deformation of linear Yang-Mills theory in which the deformed Lagrangian is Lorentz covariant and has minimal differential degree, while the gauge group is deformed so it becomes non-abelian [1, 9, 10].

Somewhat surprisingly, in three spacetime dimensions there is a different deformation of abelian Yang-Mills theory [11, 12], featuring a Lorentz covariant form with minimal differential degree. The deformation comes from specializing a $p+2$ dimensional nonlinear gauge theory of p-form gauge fields [13] to the case $p = 1$, as described by a dual formulation of principal chiral models based on any non-abelian Lie group. In contrast with Yang-Mills theory, the commutators of the deformed gauge symmetry vanish on the solutions of the deformed field equation, so in this Lagrangian based deformation, the gauge group remains abelian.

There has also been interest in theories that do not have a Lagrangian [27, 28, 29]. These theories have equations of motion that cannot be derived from Euler-Lagrange equations applied to a Lagrangian without introducing ambiguities or extra degrees of freedom in the Lagrangian.

In this thesis, a study of minimal derivative, general deformations of Maxwell gauge theory in 2+1 spacetime dimensions is undertaken without requiring the existence of a Lagrangian or any other structure. Here the deformation is determined by requiring that the variation of the field equation is a linear combination of the field equation and its derivatives. This is the most general way on-shell gauge invariance can be satisfied.

Chapter 2 covers how Maxwell's equations are written in special relativity and what gauge symmetry they possess in potential form.

Chapter 3 carries out the deformation analysis of Maxwell gauge theory.

Chapter 4 presents the main results, which consist of a new nonlinear generalization of Maxwell gauge theory and its non-abelian extension.

Chapter 5 states some conclusions.

An appendix contains the notations and identities that are used in this work.

Chapter 2

Maxwell Gauge Theory

In classical physics, the phenomenon of electromagnetism with specified sources is captured by the four familiar Maxwell equations for the electric field **E** and the magnetic field **B**

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho, \tag{2.1}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{2.4}$$

plus the continuity equation for the charge density ρ and current density $\mathbf{J}$

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}. \tag{2.5}$$

Note that ϵ_0 and μ_0 are the permittivity and permeability of free space respectively.

This form of the equations hides their invariance under Lorentz transformations. To bring out this feature, it is more convenient to write the electric and magnetic fields in terms of two other fields V and $\mathbf{A}$ called potentials:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \tag{2.6}$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{2.7}$$

This potential formulation automatically satisfies two of the Maxwell equations, (2.2) and (2.3), but it does not uniquely specify the potential fields. Specifically, V and $\mathbf{A}$ can be modified by

$$V' = V - \frac{\partial \zeta}{\partial t}, \tag{2.8}$$

$$\mathbf{A}' = \mathbf{A} + \nabla \zeta, \tag{2.9}$$

under which **E** and **B** remain unchanged, where ζ is an arbitrary scalar field. This invariance is called a gauge freedom or a gauge symmetry and the scalar field ζ is called the gauge parameter.

It is common to restrict the gauge freedom by imposing a condition on the potentials, such as in the Lorenz gauge or Coulomb gauge, but I will not do this, as it adds an unnecessary, additional condition.

The remaining two Maxwell equations (2.1) and (2.4) become

$$-\Delta V - \frac{\partial \nabla \cdot \mathbf{A}}{\partial t} = \frac{1}{\epsilon_0}\rho, \tag{2.10}$$

$$c^2 \nabla \times (\nabla \times \mathbf{A}) + \frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla \frac{\partial V}{\partial t} = \frac{1}{\epsilon_0}\mathbf{J}, \tag{2.11}$$

where $c^2 = 1/(\epsilon_0 \mu_0)$.

2.1 Relativistic formulation of Maxwell gauge theory

Lorentz invariance of the system of potential equations (2.6), (2.7), (2.10), (2.11) can be seen by defining a relativistic vector potential

$$A^\mu \equiv \begin{pmatrix} \frac{1}{c}V \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}, \tag{2.12}$$

and a relativistic field tensor

$$F^{\mu\nu} \equiv \frac{1}{2} \begin{pmatrix} 0 & \frac{1}{c}E^1 & \frac{1}{c}E^2 & \frac{1}{c}E^3 \\ -\frac{1}{c}E^1 & 0 & B^3 & -B^2 \\ -\frac{1}{c}E^2 & -B^3 & 0 & B^1 \\ -\frac{1}{c}E^3 & B^2 & -B^1 & 0 \end{pmatrix}, \tag{2.13}$$

where the spatial components of $\mathbf{A}$, $\mathbf{E}$ and $\mathbf{B}$ refer to standard Cartesian coordinates (x, y, z). Here the Greek indices are defined to be $\mu = 0, 1, 2, 3$, with "0" denoting the time index and "$1, 2, 3$" denoting the standard Cartesian indices.

Then the first two potential equations (2.6) and (2.7) can be expressed as

$$F^{\mu\nu} = \tfrac{1}{2}\big(\partial^\mu A^\nu - \partial^\nu A^\mu\big) \tag{2.14}$$

using the relativistic gradient operator

$$\partial^\mu \equiv \begin{pmatrix} -\frac{1}{c}\partial_t \\ \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}. \tag{2.15}$$

This is equivalent to the curl equation

$$\partial^\alpha F^{\mu\nu} + \partial^\mu F^{\nu\alpha} + \partial^\nu F^{\alpha\mu} = 0. \tag{2.16}$$

The last two potential equations (2.10) and (2.11) can be expressed as divergence equations

$$\partial_\mu J^\mu = 0, \tag{2.17}$$

$$\partial_\nu F^{\mu\nu} = \frac{\mu_0}{2} J^\mu, \tag{2.18}$$

where

$$J^\mu \equiv \begin{pmatrix} c\rho \\ J^1 \\ J^2 \\ J^3 \end{pmatrix} \tag{2.19}$$

defines the relativistic current density and

$$\partial_\mu \equiv \begin{pmatrix} \frac{1}{c}\partial_t \\ \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \tag{2.20}$$

is the relativistic partial derivative operator. In fact, the relativistic current continuity equation (2.17) is an automatic consequence of equation (2.18), since $\partial_\mu J^\mu = \frac{2}{\mu_0}\partial_\mu \partial_\nu F^{\mu\nu} = 0$ holds by antisymmetry of the field tensor $F^{\mu\nu} = -F^{\nu\mu}$ combined with commutativity of the partial derivatives (2.20).

Hence, the relativistic potential formulation of the Maxwell gauge theory (2.1)–(2.5) consists simply of equations (2.14) and (2.18).

It is straightforward to show that A^μ, J^μ, and ∂^μ each transform as 4-vectors under Lorentz transformations, while $F^{\mu\nu}$ transforms similarly as an antisymmetric 4-tensor. As a consequence, the Maxwell equations are Lorentz covariant.

The relativistic formulation of the gauge symmetry (2.8) and (2.9) now is very simple — since $F^{\mu\nu}$ is the relativistic curl of A^μ, it is invariant under the addition of a relativistic gradient to A^μ:

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \zeta. \tag{2.21}$$

The infinitesimal form of this gauge symmetry is given by

$$\delta_\zeta A^\mu = \partial^\mu \zeta. \tag{2.22}$$

Note that, since $F^{\mu\nu}$ is gauge invariant, so are the Maxwell equations (2.14) and (2.18).

The relativistic form of the Maxwell equations (2.16) and (2.18) without potentials has a dual formulation given in terms of the dual field strength tensor

$$*F_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} \tag{2.23}$$

where $\epsilon_{\mu\nu\alpha\beta}$ is the volume form which is a totally antisymmetric tensor with $\epsilon_{0123} = 1$. It is straightforward to show that the divergence of $*F_{\mu\nu}$ is equivalent to the curl of $F^{\alpha\beta}$:

$$\partial^\nu *F_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}\partial^\nu F^{\alpha\beta}. \tag{2.24}$$

Hence, by exchanging divergences and curls and simultaneously exchanging the field strength and its dual, the Maxwell equations (2.16) and (2.18) become

$$\partial_\mu {*F}^{\mu\nu} = 0, \tag{2.25}$$

$$\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\partial_\nu {*F}_{\alpha\beta} = -\frac{\mu_0}{2}J^\mu. \tag{2.26}$$

Using the identity

$$\epsilon^{\tau\gamma\alpha\beta}\epsilon_{\alpha\beta\mu\nu} = -2(\delta^\tau_\mu\delta^\gamma_\nu - \delta^\gamma_\mu\delta^\tau_\nu) \tag{2.27}$$

it follows that the dual of the dual field strength tensor is the negative field strength tensor:

$$\begin{aligned}\tfrac{1}{2}\epsilon^{\tau\gamma\alpha\beta}{*F}_{\alpha\beta} &= \tfrac{1}{4}\epsilon^{\tau\gamma\alpha\beta}\epsilon_{\alpha\beta\mu\nu}F^{\mu\nu}\\ &= -\tfrac{1}{2}(\delta^\tau_\mu\delta^\gamma_\nu - \delta^\gamma_\mu\delta^\tau_\nu)F^{\mu\nu}\\ &= -F^{\tau\gamma}.\end{aligned} \tag{2.28}$$

2.2 Maxwell gauge theory in 2+1 dimensions

There are two equivalent ways to reduce the above 3+1 dimensional Maxwell gauge theory to 2+1 dimensions.

2.2.1 Physical restriction

Since there are only two spacial dimensions, the charge density ρ and the current density $\mathbf{J}$ are restricted to the x, y-plane and there is no z coordinate on which anything can depend.

$$\mathbf{J} = \begin{pmatrix} J^1 \\ J^2 \end{pmatrix}. \tag{2.29}$$

The electric field still points radially towards or from point sources but now only has two components:

$$\mathbf{E} = \begin{pmatrix} E^1 \\ E^2 \end{pmatrix}. \tag{2.30}$$

The magnetic field more subtle. The magnetic field vector cannot be in the same plane as the current density $\mathbf{J}$, but since there is only one plane, the magnetic field cannot be a vector; it is actually a scalar. Since the magnetic field only exists in the x, y-plane, this scalar is the would be z component in $3 + 1$ dimensions. Therefore, we will write

$$\mathbf{B} = \begin{pmatrix} B^3 \end{pmatrix} \tag{2.31}$$

to relate back to the formalism from $3 + 1$ dimensions.

While the Maxwell equations (2.1)–(2.4) must be modified due to the lack of a proper curl [35], we can see that embedding the above planar theory in a third dimension where

$$\mathbf{E} = \begin{pmatrix} E^1 \\ E^2 \\ 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ B^3 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} J^1 \\ J^2 \\ 0 \end{pmatrix} \tag{2.32}$$

satisfies all four equations. The resulting equations are

$$\frac{\partial E^1}{\partial x} + \frac{\partial E^2}{\partial y} = \frac{1}{\epsilon_0}\rho, \tag{2.33}$$

$$\frac{\partial E^2}{\partial x} - \frac{\partial E^1}{\partial y} = -\frac{B^3}{\partial t}, \tag{2.34}$$

$$-\frac{\partial B^3}{\partial x} = \mu_0 J^2 + \epsilon_0\mu_0\frac{\partial E^2}{\partial t}, \tag{2.35}$$

$$\frac{\partial B^3}{\partial y} = \mu_0 J^1 + \epsilon_0\mu_0\frac{\partial E^1}{\partial t}, \tag{2.36}$$

$$\frac{\partial J^1}{\partial x} + \frac{\partial J^2}{\partial y} = -\frac{\partial \rho}{\partial t}. \tag{2.37}$$

Note that the divergence equation for the magnetic field is trivially true.

In the potential formulation (2.6) and (2.7), with V and $\mathbf{A}$ assumed to be invariant under translations in z, we see that $\mathbf{A}$ can be taken to have no z component, because these two equations imply $\frac{\partial A^3}{\partial t} = 0$, $\frac{\partial A^3}{\partial x} = 0$ and $\frac{\partial A^3}{\partial y} = 0$. Hence the vector potential has the form

$$\mathbf{A} = \begin{pmatrix} A^1 \\ A^2 \\ 0 \end{pmatrix}. \tag{2.38}$$

It is easy to write the fields and potentials in a relativistic formalism by omitting the z row and column in the 3+1 dimensional relativistic current density (2.19), relativistic vector potential (2.12), relativistic field tensor (2.13), relativistic gradient operator (2.15) and relativistic partial derivative operator (2.20):

$$J^\mu \equiv \begin{pmatrix} c\rho \\ J^1 \\ J^2 \end{pmatrix}, \tag{2.39}$$

$$A^\mu \equiv \begin{pmatrix} \frac{1}{c}V \\ A^1 \\ A^2 \end{pmatrix}, \tag{2.40}$$

$$F^{\mu\nu} \equiv \frac{1}{2}\begin{pmatrix} 0 & \frac{1}{c}E^1 & \frac{1}{c}E^2 \\ -\frac{1}{c}E^1 & 0 & B^3 \\ -\frac{1}{c}E^2 & -B^3 & 0 \end{pmatrix}, \tag{2.41}$$

and

$$\partial^\mu \equiv \begin{pmatrix} -\frac{1}{c}\partial_t \\ \partial_x \\ \partial_y \end{pmatrix}, \quad \partial_\mu \equiv \begin{pmatrix} \frac{1}{c}\partial_t \\ \partial_x \\ \partial_y \end{pmatrix}. \tag{2.42}$$

As in the 3+1 dimensional case, the two potential equations (2.6) and (2.7) are again contained in

$$F^{\mu\nu} = \tfrac{1}{2}\big(\partial^\mu A^\nu - \partial^\nu A^\mu\big), \tag{2.43}$$

and, similarly, equations (2.10) and (2.11) are contained in

$$\partial_\nu F^{\mu\nu} = \frac{\mu_0}{2} J^\mu. \tag{2.44}$$

The gauge symmetry is still given by the transformation (2.21) with the gauge parameter now having no dependence on z.

Thus the form of the relativistic potential formulation of the Maxwell equations looks the same in 2+1 dimensions as in 3+1 dimensions. However, the theory in 2+1 dimensions has 2 fewer degrees of freedom.

2.2.2 Mathematical reduction

An alternative method is to express the electric and magnetic fields in terms of the relativistic field tensor:

$$E^i = 2cF^{0i} \tag{2.45}$$

$$B^i = \epsilon^{0ijk} \eta_{jl} \eta_{km} F^{lm} \tag{2.46}$$

where $\eta_{\alpha\beta}$ is the Minkowski metric which is symmetric, diagonal, and has $\eta_{00} = -1$ and $\eta_{11} = \eta_{22} = \eta_{33} = 1$; $\epsilon^{\alpha\beta\mu\nu}$ is the Levi-Civita tensor which is totally antisymmetric and has $\epsilon^{0123} = -1$.

To reduce from 3+1 dimensions to 2+1 dimensions, the field tensor is taken to have only t, x, y components with no dependence on z: $F^{\mu 3} \equiv 0$ and $\dfrac{\partial F^{\mu\nu}}{\partial z} = 0$. Then the relations (2.45) and (2.46) show that $E^3 = 0$ and $B^1 = B^2 = 0$. The components of the Maxwell equations (2.16) and (2.18) then yield the 2+1 equations (2.33)–(2.37).

2.2.3 Dual formulation

Finally, in 2+1 dimensions the counterpart of the dual formulation (2.25) and (2.26) of the Maxwell equations can be obtained by introducing the dual field strength 1-form

$$*F_\mu \equiv \epsilon_{\mu\nu\tau} F^{\nu\tau} = \tfrac{1}{2}\epsilon_{\mu\nu\tau}(\partial^\nu A^\tau - \partial^\tau A^\nu) = \epsilon_{\mu\nu\tau}\partial^\nu A^\tau. \tag{2.47}$$

The components of this 1-form are

$$*F_\mu = \begin{pmatrix} B^3 & -\tfrac{1}{c}E^2 & \tfrac{1}{c}E^1 \end{pmatrix}. \tag{2.48}$$

Then the dual of the 2+1 dimensional Maxwell equations (2.16) and (2.18) consists of the equations

$$\partial_\mu {*F^\mu} = 0, \tag{2.49}$$

$$\tfrac{1}{2}\epsilon^{\mu\nu\alpha}\partial_\nu {*F_\alpha} = -\tfrac{1}{2}\mu_0 J^\mu. \tag{2.50}$$

Chapter 3

Analysis of non-Lagrangian deformations of Maxwell gauge theory

In 2+1 spacetime dimensions, Maxwell theory for the gauge field A_μ is given by the linear field equation

$$E_\mu = \partial^\nu \partial_{[\nu} A_{\mu]} = \tfrac{1}{2}\epsilon_\mu{}^{\nu\sigma}\partial_\nu {*F}_\sigma \tag{3.1}$$

and the abelian gauge symmetry

$$\delta_\zeta A_\mu = \partial_\mu \zeta \tag{3.2}$$

where

$$\delta_\zeta E_\mu = 0 \tag{3.3}$$

holds off-shell. Here ${*F}_\mu = \epsilon_\mu{}^{\alpha\beta}\partial_\alpha A_\beta$ is the dualized linear field strength, which satisfies

$$\partial^\mu {*F}_\mu = 0. \tag{3.4}$$

A nonlinear deformation consists of a field equation E_μ and gauge symmetry $\delta_\zeta A_\mu$ that reduce at linear order in powers of A_μ to Maxwell gauge theory, such that the nonlinear field equation is on-shell gauge invariant

$$\big(\delta_\zeta E_\mu\big)\big|_{E=0} = 0. \tag{3.5}$$

This thesis focuses on minimal-derivative deformations where each term in E_μ has at most two derivatives distributed as either $\partial^2 A$ or $(\partial A)^2$, while each $\delta_\zeta A_\mu$ term has at most one distributed as either ∂A or $\partial\zeta$. Such deformations have the general form

$$E_\mu = \partial^\nu \partial_{[\nu} A_{\mu]} + G_\mu(A) + G_\mu{}^{\nu\alpha}(A)\partial_\nu A_\alpha + G_\mu{}^{\nu\alpha\sigma\beta}(A)(\partial_\nu A_\alpha)\partial_\sigma A_\beta + G_\mu{}^{\nu\sigma\alpha}(A)\partial_\nu\partial_\sigma A_\alpha \tag{3.6}$$

and

$$\delta_\zeta A_\mu = \partial_\mu \zeta + \big(H_\mu(A) + H_\mu{}^{\nu\alpha}(A)\partial_\nu A_\alpha\big)\zeta + H_\mu{}^\nu(A)\partial_\nu\zeta. \tag{3.7}$$

From these considerations, one finds

$$\delta_\zeta E_\mu = \big(\mathcal{P}_\mu{}^\nu(A) + \mathcal{P}_\mu{}^{\nu\sigma\alpha}(A)\partial_\sigma A_\alpha\big)E_\nu\zeta + \mathcal{Q}_\mu{}^{\nu\alpha}(A)(\partial_\alpha E_\nu)\zeta + \mathcal{P}_\mu{}^{\sigma\nu}(A)E_\nu\partial_\sigma\zeta. \tag{3.8}$$

Lorentz covariance will also use to limit the possible forms of the terms. There will be no other restrictions imposed on possible deformations.

3.1 Redefinition freedoms in deformations

Several kinds of freedom need to be taken into account in the form of the deformations. First, there are invertible field redefinitions and field-dependent changes of gauge parameter:

$$\zeta \to \tilde{\zeta} = \zeta + U(A)\zeta, \tag{3.9}$$

$$A_\mu \to \tilde{A}_\mu = A_\mu + V_\mu(A), \tag{3.10}$$

with $U(0) = 0$ and $\dfrac{\partial V_\mu}{\partial A_\nu}(0) = 0$. This yields

$$\begin{aligned}\delta_{\tilde{\zeta}}\tilde{A}_\mu &= \delta_{\tilde{\zeta}}\big(A_\mu + V_\mu(A)\big)\\ &= \Big(\delta^\tau_\mu + \frac{\partial V_\mu(A)}{\partial A_\tau}\Big)\Big(\partial_\tau\big(\zeta + U(A)\zeta\big) + \big(H_\tau(A) + H_\tau{}^{\nu\alpha}(A)\partial_\nu A_\alpha\big)\big(\zeta + U(A)\zeta\big)\\ &\qquad + H_\tau{}^\nu(A)\partial_\nu\big(\zeta + U(A)\zeta\big)\Big)\\ &\equiv \partial_\mu\zeta + \tilde{H}_\mu{}^\nu(A)\partial_\nu\zeta + \big(\tilde{H}_\mu(A) + \tilde{H}_\mu{}^{\nu\tau}(A)\partial_\nu A_\tau\big)\zeta\end{aligned} \tag{3.11}$$

where

$$\tilde{H}_\mu{}^\nu(A) = \Big(\delta^\tau_\mu + \frac{\partial V_\mu(A)}{\partial A_\tau}\Big)\big(H_\tau{}^\nu(A) + U(A)\delta^\nu_\tau + H_\tau{}^\nu(A)U(A)\big) + \frac{\partial V_\mu(A)}{\partial A_\nu}, \tag{3.12}$$

$$\tilde{H}_\mu(A) = \Big(\delta^\nu_\mu + \frac{\partial V_\mu(A)}{\partial A_\nu}\Big)\big(H_\nu(A) + H_\nu(A)U(A)\big) \tag{3.13}$$

and

$$\begin{aligned}\tilde{H}_\mu{}^{\nu\tau}(A)\partial_\nu A_\tau = \Big(\delta^\tau_\mu + \frac{\partial V_\mu(A)}{\partial A_\tau}\Big)\Big(&H_\tau{}^{\nu\alpha}(A)\partial_\nu A_\alpha + \partial_\tau U(A)\\ &+ H_\tau{}^{\nu\alpha}(A)\partial_\nu A_\alpha U(A) + H_\tau{}^\nu(A)\partial_\nu U(A)\Big).\end{aligned} \tag{3.14}$$

Second, the nonlinear field equation can be invertibly transformed by

$$\tilde{E}_\mu = E_\mu + N_\mu{}^\nu(A)E_\nu \tag{3.15}$$

with $N_\mu{}^\nu(0) = 0$. Note that

$$\begin{aligned}\big(\delta_\zeta\tilde{E}_\mu\big)\big|_{\tilde{E}=0} &= \big(\delta_\zeta E_\mu + \delta_\zeta N_\mu{}^\nu E_\nu + N_\mu{}^\nu\delta_\zeta E_\nu\big)\big|_{\tilde{E}=0}\\ &= \big(\delta_\zeta E_\mu + \delta_\zeta N_\mu{}^\nu E_\nu + N_\mu{}^\nu\delta_\zeta E_\nu\big)\big|_{E=0}\\ &= 0\end{aligned} \tag{3.16}$$

due to $\tilde{E}_\mu = 0 \Leftrightarrow E_\mu = 0$, thus gauge invariance is preserved.

3.2 Deformation equations

The deformation equation (3.8) can be rewritten as

$$\delta_\zeta E_\mu = P_\mu{}^\nu(\zeta)E_\nu + Q_\mu{}^{\nu\alpha}(\zeta)\partial_\alpha E_\nu \tag{3.17}$$

where

$$P_\mu{}^\nu(\zeta) = \big(\mathcal{P}_\mu{}^\nu(A) + \mathcal{P}_\mu{}^{\nu\sigma\alpha}(A)\partial_\sigma A_\alpha\big)\zeta + \mathcal{P}_\mu{}^{\sigma\nu}(A)\partial_\sigma\zeta \tag{3.18}$$

and

$$Q_\mu{}^{\nu\tau}(\zeta) = \mathcal{Q}_\mu{}^{\nu\tau}(A)\zeta. \tag{3.19}$$

The commutator of two gauge symmetries

$$\begin{aligned}
[\delta_{\zeta_1},\delta_{\zeta_2}]A_\mu &= \delta_{\zeta_1}\big(\delta_{\zeta_2}A_\mu\big) - (\zeta_1\leftrightarrow\zeta_2)\\
&= \big(\delta_{\zeta_1}H_\mu(A) + \delta_{\zeta_1}H_\mu{}^{\nu\alpha}(A)\partial_\nu A_\alpha + H_\mu{}^{\nu\alpha}(A)\partial_\nu\delta_{\zeta_1}A_\alpha\big)\zeta_2\\
&\qquad + \delta_{\zeta_1}H_\mu{}^\nu(A)\partial_\nu\zeta_2 - (\zeta_1\leftrightarrow\zeta_2)\\
&= \bigg(\Big(\frac{\partial H_\mu{}^{\nu\alpha}(A)}{\partial A_\beta}\big(\delta^\sigma_\beta + H_\beta{}^\sigma(A)\big) + H_\mu{}^{\sigma\beta}(A)H_\beta{}^{\nu\alpha}(A)\\
&\qquad\qquad - \frac{\partial H_\mu{}^\sigma(A)}{\partial A_\beta}H_\beta{}^{\nu\alpha}(A)\Big)\partial_\nu A_\alpha\\
&\qquad + \frac{\partial H_\mu(A)}{\partial A_\beta}\big(\delta^\sigma_\beta + H_\beta{}^\sigma(A)\big) + H_\mu{}^{\sigma\beta}(A)H_\beta(A)\\
&\qquad + H_\mu{}^{\nu\beta}(A)\partial_\nu H_\beta{}^\sigma(A) - \frac{\partial H_\mu{}^\sigma(A)}{\partial A_\beta}H_\beta(A)\bigg)\zeta_2\partial_\sigma\zeta_1 - (\zeta_1\leftrightarrow\zeta_2)\\
&\qquad + H_\mu{}^{\sigma\beta}(A)\big(\delta^\nu_\beta + H_\beta{}^\nu(A)\big)\zeta_2\partial_{\sigma\nu}\zeta_1 - (\zeta_1\leftrightarrow\zeta_2)\\
&\qquad + \frac{\partial H_\mu{}^\nu(A)}{\partial A_\beta}\big(\delta^\sigma_\beta + H_\beta{}^\sigma(A)\big)(\partial_\nu\zeta_2)\partial_\sigma\zeta_1 - (\zeta_1\leftrightarrow\zeta_2)
\end{aligned} \tag{3.20}$$

defines a possibly-new gauge symmetry involving the two gauge parameters. This gauge symmetry must similarly satisfy a deformation equation

$$[\delta_{\zeta_1},\delta_{\zeta_2}]E_\mu = R_\mu{}^\nu(\zeta_1,\zeta_2)E_\nu + S_\mu{}^{\nu\alpha}(\zeta_1,\zeta_2)\partial_\alpha E_\nu. \tag{3.21}$$

A useful identity is given by

$$[\delta_{\zeta_1},\delta_{\zeta_2}]E_\mu = \delta_{\zeta_1}\big(\delta_{\zeta_2}E_\mu\big) - (\zeta_1\leftrightarrow\zeta_2) = \big(\delta_{\zeta_1}\delta_{\zeta_2}\big)E_\mu - (\zeta_1\leftrightarrow\zeta_2). \tag{3.22}$$

This can be proved by first using expression (3.6) to get

$$\begin{aligned}
\delta_{\zeta_2}E_\mu &= \partial^\nu\partial_{[\nu}\delta_{\zeta_2}A_{\mu]} + \delta_{\zeta_2}G_\mu(A) + \delta_{\zeta_2}G_\mu{}^{\nu\alpha}(A)\partial_\nu A_\alpha + G_\mu{}^{\nu\alpha}(A)\partial_\nu\delta_{\zeta_2}A_\alpha\\
&\quad + \delta_{\zeta_2}G_\mu{}^{\nu\alpha\sigma\beta}(A)(\partial_\nu A_\alpha)\partial_\sigma A_\beta + 2G_\mu{}^{\nu\alpha\sigma\beta}(A)(\partial_\nu A_\alpha)\partial_\sigma\delta_{\zeta_2}A_\beta\\
&\quad + \delta_{\zeta_2}G_\mu{}^{\nu\sigma\alpha}(A)\partial_\nu\partial_\sigma A_\alpha + G_\mu{}^{\nu\sigma\alpha}(A)\partial_\nu\partial_\sigma\delta_{\zeta_2}A_\alpha,
\end{aligned} \tag{3.23}$$

and next applying δ_{ζ_1}, which yields

$$\begin{aligned}
&\delta_{\zeta_1}(\delta_{\zeta_2}E_\mu) - (\zeta_1 \leftrightarrow \zeta_2)\\
&= \partial^\nu\partial_{[\nu}[\delta_{\zeta_1},\delta_{\zeta_2}]A_{\mu]} + [\delta_{\zeta_1},\delta_{\zeta_2}]G_\mu(A) + [\delta_{\zeta_1},\delta_{\zeta_2}]G_\mu{}^{\nu\alpha}(A)\partial_\nu A_\alpha + G_\mu{}^{\nu\alpha}(A)\partial_\nu[\delta_{\zeta_1},\delta_{\zeta_2}]A_\alpha\\
&\quad + [\delta_{\zeta_1},\delta_{\zeta_2}]G_\mu{}^{\nu\alpha\sigma\beta}(A)(\partial_\nu A_\alpha)\partial_\sigma A_\beta + 2G_\mu{}^{\nu\alpha\sigma\beta}(A)(\partial_\nu A_\alpha)\partial_\sigma[\delta_{\zeta_1},\delta_{\zeta_2}]A_\beta\\
&\quad + [\delta_{\zeta_1},\delta_{\zeta_2}]G_\mu{}^{\nu\sigma\alpha}(A)\partial_\nu\partial_\sigma A_\alpha + G_\mu{}^{\nu\sigma\alpha}(A)\partial_\nu\partial_\sigma[\delta_{\zeta_1},\delta_{\zeta_2}]A_\alpha\\
&\quad + \big(\delta_{\zeta_2}G_\mu{}^{\nu\alpha}(A)\partial_\nu\delta_{\zeta_1}A_\alpha + \delta_{\zeta_1}G_\mu{}^{\nu\alpha}(A)\partial_\nu\delta_{\zeta_2}A_\alpha\big) - (\zeta_1 \leftrightarrow \zeta_2)\\
&\quad + 2\big(G_\mu{}^{\nu\alpha\sigma\beta}(A)(\partial_\nu\delta_{\zeta_1}A_\alpha)\partial_\sigma\delta_{\zeta_2}A_\beta\big) - (\zeta_1 \leftrightarrow \zeta_2)\\
&\quad + 2\big(\delta_{\zeta_1}G_\mu{}^{\nu\alpha\sigma\beta}(A)(\partial_\nu A_\alpha)\partial_\sigma\delta_{\zeta_2}A_\beta + \delta_{\zeta_2}G_\mu{}^{\nu\alpha\sigma\beta}(A)(\partial_\nu A_\alpha)\partial_\sigma\delta_{\zeta_1}A_\beta\big) - (\zeta_1 \leftrightarrow \zeta_2)\\
&\quad + \big(\delta_{\zeta_1}G_\mu{}^{\nu\sigma\alpha}(A)\partial_\nu\partial_\sigma\delta_{\zeta_2}A_\alpha + \delta_{\zeta_2}G_\mu{}^{\nu\sigma\alpha}(A)\partial_\nu\partial_\sigma\delta_{\zeta_1}A_\alpha\big) - (\zeta_1 \leftrightarrow \zeta_2).
\end{aligned} \tag{3.24}$$

Clearly, all of the terms written with $(\zeta_1 \leftrightarrow \zeta_2)$ cancel out, leaving just the terms of $\big(\delta_{\zeta_1}\delta_{\zeta_2}\big)E_\mu - (\zeta_1 \leftrightarrow \zeta_2)$. This completes the proof.

Consequently, in the commutator deformation equation (3.21),

$$R_\mu{}^\tau(\zeta_1,\zeta_2) = \delta_{\zeta_1}P_\mu{}^\tau(\zeta_2) + P_\mu{}^\nu(\zeta_2)P_\nu{}^\tau(\zeta_1) + Q_\mu{}^{\nu\alpha}(\zeta_2)\partial_\alpha P_\nu{}^\tau(\zeta_1) - (\zeta_1 \leftrightarrow \zeta_2) \tag{3.25}$$

and

$$\begin{aligned}
S_\mu{}^{\tau\alpha}(\zeta_1,\zeta_2) = \delta_{\zeta_1}Q_\mu{}^{\tau\alpha}(\zeta_2) + P_\mu{}^\nu(\zeta_2)Q_\nu{}^{\tau\alpha}(\zeta_1) + Q_\mu{}^{\nu\alpha}(\zeta_2)P_\nu{}^\tau(\zeta_1)\\
+ Q_\mu{}^{\nu\beta}(\zeta_2)\partial_\beta Q_\nu{}^{\tau\alpha}(\zeta_1) - (\zeta_1 \leftrightarrow \zeta_2).
\end{aligned} \tag{3.26}$$

3.3 Power series expansion

We will expand both the deformation equation (3.17) and the commutator deformation equation (3.21) in powers of A_μ. This allows us to solve order-by-order for the terms in the unknowns $G_\mu(A)$, $G_\mu{}^{\nu\alpha}(A)$, $G_\mu{}^{\nu\sigma\alpha}(A)$, $G_\mu{}^{\nu\alpha\sigma\beta}(A)$, $H_\mu(A)$, $H_\mu{}^\nu(A)$ and $H_\mu{}^{\nu\alpha}(A)$, as well as $P_\mu{}^\nu(\zeta)$ and $Q_\mu{}^{\nu\tau}(\zeta)$. These unknowns partially decouple when equations (3.17) and (3.21) are solved in the following particular sequence:

Step (1): solve the commutator deformation equation (3.21) at zeroth order for the linear terms in the gauge symmetry.

Step (2): put $\zeta = \text{const.}$ in the deformation equation (3.17) at linear order, which gives a consistency condition on the gauge symmetry terms and partly determines the coefficients at zeroth order in the deformation equation.

Step (3): return to arbitrary ζ and solve the deformation equation (3.17) at linear order for the quadratic terms in the field equation, using the redefinition freedoms to eliminate terms.

Step (4): solve the commutator deformation equation (3.21) at linear order for the quadratic terms in the gauge symmetry, using the redefinition freedoms to eliminate terms.

Step (5): put ζ = const. in the deformation equation (3.17) at quadratic order, which gives a consistency condition on the gauge symmetry terms and partly determines the coefficients at linear order in the deformation equation.

Step (6): return to arbitrary ζ and solve the deformation equation (3.17) at quadratic order for the cubic terms in the field equation, using the redefinition freedoms to eliminate terms.

Step (7): seek a general pattern for the deformation terms and guess a solution to all orders, which will then be checked.

3.4 Linear theory

As stated above, at lowest order, the undeformed field equation and the undeformed gauge symmetry are given by Maxwell gauge theory:

$$\overset{(1)}{E}_{\mu} = \partial^{\nu}\partial_{[\nu}A_{\mu]} = \frac{1}{2}\epsilon_{\mu}{}^{\nu\tau\alpha}\partial_{\nu\tau}A_{\alpha} = \frac{1}{2}\epsilon_{\mu}{}^{\nu\sigma}\partial_{\nu}{*F}_{\sigma} \tag{3.27}$$

and

$$\overset{(0)}{\delta}_{\zeta}A_{\mu} = \partial_{\mu}\zeta, \tag{3.28}$$

with the undeformed gauge invariance condition (3.5) holding off-shell

$$\overset{(0)}{\delta}_{\zeta}\overset{(1)}{E}_{\mu} = 0. \tag{3.29}$$

3.5 Step 1: Linear-order gauge symmetry

From the general form of the deformed gauge symmetry (3.7),

$$\overset{(1)}{\delta}_{\zeta}A_{\mu} = \big(\overset{(0)}{H}_{\mu}{}^{\nu\alpha}\partial_{\nu}A_{\alpha} + \overset{(1)}{H}_{\mu}(A)\big)\zeta + \overset{(1)}{H}_{\mu}{}^{\nu}(A)\partial_{\nu}\zeta. \tag{3.30}$$

Expressions for the unknowns $\overset{(0)}{H}_{\mu}{}^{\nu\alpha}$, $\overset{(1)}{H}_{\mu}(A)$, $\overset{(1)}{H}_{\mu}{}^{\nu}(A)$ are determined by Lorentz covariance, which yields

$$\overset{(0)}{H}_{\mu}{}^{\nu\alpha} = c_1\epsilon_{\mu}{}^{\nu\tau}, \quad \overset{(1)}{H}_{\mu}(A) = c_2A_{\mu}, \quad \overset{(1)}{H}_{\mu}{}^{\nu}(A) = c_3\epsilon_{\mu}{}^{\nu\tau}A_{\tau}, \tag{3.31}$$

where c_1, c_2, c_3 are arbitrary constants. Thus,

$$\overset{(1)}{\delta}_{\zeta}A_{\mu} = (c_2A_{\mu} + c_1{*F}_{\mu})\zeta + c_3\epsilon_{\mu}{}^{\nu\tau}A_{\nu}\partial_{\tau}\zeta. \tag{3.32}$$

The commutator deformation equation (3.21) at order 0

$$[\delta_{\zeta_1}, \delta_{\zeta_2}]\overset{(0)}{}\overset{(1)}{E}_{\mu} = 0 \tag{3.33}$$

implies that the linear field equation (3.27) has the gauge symmetry $[\delta_{\zeta_1}, \overset{(0)}{\delta_{\zeta_2}}]A_\mu$. However, its gauge invariance consists only of the abelian gauge symmetry (3.28), and so

$$[\delta_{\zeta_1}, \overset{(0)}{\delta_{\zeta_2}}]A_\mu = \overset{(0)}{\delta_{\zeta_3}} A_\mu = \partial_\mu \zeta_3 \tag{3.34}$$

for some gauge parameter ζ_3 which is a function of ζ_1, ζ_2, and their first derivatives. Since expression (3.34) is a gradient, its curl must vanish,

$$0 = \epsilon_\beta{}^{\alpha\mu}\partial_\alpha\big([\delta_{\zeta_1}, \overset{(0)}{\delta_{\zeta_2}}]A_\mu\big) \tag{3.35}$$

where

$$\begin{aligned}
[\delta_{\zeta_1}, \overset{(0)}{\delta_{\zeta_2}}]A_\mu &= \overset{(0)}{\delta_{\zeta_1}}\big(\overset{(1)}{\delta_{\zeta_2}} A_\mu\big) - (\zeta_1 \leftrightarrow \zeta_2) \\
&= \big(c_2 \overset{(0)}{\delta_{\zeta_1}} A_\mu + c_1 \overset{(0)}{\delta_{\zeta_1}} {*F}_\mu\big)\zeta_2 + c_3 \epsilon_\mu{}^{\nu\tau} \overset{(0)}{\delta_{\zeta_1}} A_\nu \partial_\tau \zeta_2 - (\zeta_1 \leftrightarrow \zeta_2) \\
&= c_2 (\partial_\mu \zeta_1)\zeta_2 + c_3 \epsilon_\mu{}^{\nu\tau}(\partial_\nu \zeta_1)\partial_\tau \zeta_2 - (\zeta_1 \leftrightarrow \zeta_2).
\end{aligned} \tag{3.36}$$

Substituting this expression into the integrability equation (3.35) gives

$$\begin{aligned}
0 &= \epsilon_\beta{}^{\alpha\mu}\partial_\alpha\big(c_2(\partial_\mu\zeta_1)\zeta_2 + c_3\epsilon_\mu{}^{\nu\tau}(\partial_\nu\zeta_1)\partial_\tau\zeta_2\big) - (\zeta_1 \leftrightarrow \zeta_2) \\
&= 2c_2\epsilon_\beta{}^{\alpha\mu}(\partial_\mu\zeta_1)\partial_\alpha\zeta_2 + 4c_3\big((\partial^\nu\partial_{[\nu}\zeta_1)\partial_{\beta]}\zeta_2 + (\partial_{[\nu}\zeta_1)\partial^\nu\partial_{\beta]}\zeta_2\big).
\end{aligned} \tag{3.37}$$

Since each of the terms are linearly independent,

$$c_2 = 0, \quad c_3 = 0. \tag{3.38}$$

Therefore, the gauge symmetry at first order is given by

$$\overset{(1)}{\delta_\zeta} A_\mu = c_1 {*F}_\mu \zeta \tag{3.39}$$

where c_1 remains an arbitrary constant. This yields $\zeta_3 = 0$, and hence

$$[\delta_{\zeta_1}, \overset{(0)}{\delta_{\zeta_2}}]A_\mu = 0. \tag{3.40}$$

3.6 Step 2: Consistency condition

The deformation equation (3.17) at order 1 is given by

$$\overset{(0)}{\delta_\zeta}\overset{(2)}{E}_\mu + \overset{(1)}{\delta_\zeta}\overset{(1)}{E}_\mu = \overset{(0)}{P}_\mu{}^\nu(\zeta)\overset{(1)}{E}_\nu + \overset{(0)}{Q}_\mu{}^{\nu\tau}(\zeta)\partial_\tau \overset{(1)}{E}_\nu \tag{3.41}$$

where

$$\overset{(1)}{\delta_\zeta}\overset{(1)}{E}_\mu = \frac{1}{2}c_1\epsilon_\mu{}^{\nu\tau\gamma}\big(\partial_{\nu\tau}{*F}_\gamma\zeta + 2(\partial_{(\nu}{*F}_{|\gamma|})\partial_{\tau)}\zeta + {*F}_\gamma\partial_{\nu\tau}\zeta\big), \tag{3.42}$$

and

$$\overset{(0)}{P}_{\mu}{}^{\nu}(\zeta) = \overset{(0)}{\mathcal{P}}_{\mu}{}^{\nu}\zeta + \overset{(0)}{\mathcal{P}}_{\mu}{}^{\nu\tau}\partial_{\tau}\zeta, \tag{3.43}$$

$$\overset{(0)}{Q}_{\mu}{}^{\nu\tau}(\zeta) = \overset{(0)}{\mathcal{Q}}_{\mu}{}^{\nu\tau}\zeta, \tag{3.44}$$

with

$$\overset{(0)}{\mathcal{P}}_{\mu}{}^{\nu} = p_1\delta^{\nu}_{\mu}, \quad \overset{(0)}{\mathcal{P}}_{\mu}{}^{\nu\tau} = p_2\epsilon_{\mu}{}^{\nu\tau}, \quad \overset{(0)}{\mathcal{Q}}_{\mu}{}^{\nu\tau} = q_1\epsilon_{\mu}{}^{\nu\tau}, \tag{3.45}$$

through Lorentz covariance, where p_1, p_2 and q_1 are arbitrary constants.

Setting ζ constant in this equation yields

$$\begin{aligned} p_1\overset{(1)}{E}_{\mu} + q_1\epsilon_{\mu}{}^{\nu\tau}\partial_{\tau}\overset{(1)}{E}_{\nu} &= \frac{1}{2}c_1\epsilon_{\mu}{}^{\nu\tau\gamma}\partial_{\nu\tau}{*F}_{\gamma} \\ &= -c_1\epsilon_{\mu}{}^{\nu\tau}\partial_{\tau}\overset{(1)}{E}_{\nu} \end{aligned} \tag{3.46}$$

through the curl expression for the linear field equation. Since $\overset{(1)}{E}_{\nu}$ and $\partial_{\tau}\overset{(1)}{E}_{\nu}$ are linearly independent, their coefficients must vanish, which forces

$$p_1 = 0, \quad q_1 = -c_1. \tag{3.47}$$

Thus,

$$\overset{(0)}{P}_{\mu}{}^{\nu}(\zeta) = p_2\epsilon_{\mu}{}^{\nu\tau}\partial_{\tau}\zeta, \tag{3.48}$$

$$\overset{(0)}{Q}_{\mu}{}^{\nu\tau}(\zeta) = -c_1\epsilon_{\mu}{}^{\nu\tau}\zeta. \tag{3.49}$$

3.7 Step 3: Quadratic-order field equation

From the general form of the deformed field equation (3.6),

$$\overset{(2)}{E}_{\mu} = \overset{(2)}{G}_{\mu}(A) + \overset{(1)}{G}_{\mu}{}^{\nu\alpha}(A)\partial_{\nu}A_{\alpha} + \overset{(0)}{G}_{\mu}{}^{\nu\alpha\sigma\beta}(\partial_{\nu}A_{\alpha})\partial_{\sigma}A_{\beta} + \overset{(1)}{G}_{\mu}{}^{\nu\sigma\alpha}(A)\partial_{\nu\sigma}A_{\alpha}, \tag{3.50}$$

where expressions for the unknowns $\overset{(0)}{G}_{\mu}{}^{\nu\alpha\sigma\beta} = \overset{(0)}{G}_{\mu}{}^{\sigma\beta\nu\alpha}$, $\overset{(1)}{G}_{\mu}{}^{\nu\sigma\alpha}(A) = \overset{(1)}{G}_{\mu}{}^{(\nu\sigma)\alpha}(A)$ $\overset{(1)}{G}_{\mu}{}^{\nu\alpha}(A)$, $\overset{(2)}{G}_{\mu}(A)$ are determined by Lorentz covariance. Rather than write down all possible terms, which comprise 21 in total, it will be much easier to solve for $\overset{(2)}{E}_{\mu}$ directly.

In the deformation equation (3.41) at order 1, all of the remaining terms contain $\partial\zeta$. These can be turned into $\overset{(0)}{\delta}_{\zeta}A$'s such that

$$\overset{(0)}{\delta}_{\zeta}\overset{(2)}{E}_{\mu} = -\frac{1}{2}c_1\epsilon_{\mu}{}^{\nu\tau\gamma}(2\partial_{(\nu}{*F}_{|\gamma|}\partial_{\tau)}\zeta + {*F}_{\gamma}\partial_{\nu\tau}\zeta) + p_2\epsilon_{\mu}{}^{\nu\tau}\partial_{\tau}\zeta\overset{(1)}{E}_{\nu} \tag{3.51}$$

$$= \overset{(0)}{\delta}_{\zeta}\big(-\frac{1}{2}c_1\epsilon_{\mu}{}^{(\nu\tau)\gamma}(2\partial_{\nu}{*F}_{\gamma}A_{\tau} + {*F}_{\gamma}\partial_{\nu}A_{\tau}) + p_2\epsilon_{\mu}{}^{\nu\tau}A_{\tau}\overset{(1)}{E}_{\nu}\big). \tag{3.52}$$

This yields

$$\overset{(2)}{E}_\mu = -\frac{1}{2}c_1\epsilon_\mu{}^{(\nu\tau)\gamma}(2\partial_\nu{*F}_\gamma A_\tau + {*F}_\gamma\partial_\nu A_\tau) + p_2\epsilon_\mu{}^{\nu\tau}A_\tau\overset{(1)}{E}_\nu + \overset{(2)}{G}_\mu(*F) \tag{3.53}$$

where $\overset{(2)}{G}_\mu(*F) = \overset{(0)}{G}_\mu{}^{\nu\tau}{*F}_\nu{*F}_\tau$ is the integration term satisfying

$$\overset{(0)}{\delta}_\zeta\big(\overset{(0)}{G}_\mu{}^{\nu\tau}{*F}_\nu{*F}_\tau\big) = 0. \tag{3.54}$$

Applying Lorentz covariance leaves only

$$\overset{(0)}{G}_\mu{}^{\nu\tau} = g\epsilon_\mu{}^{\nu\tau}, \tag{3.55}$$

where g is a constant. But then $\overset{(0)}{G}_\mu{}^{\nu\tau}{*F}_\nu{*F}_\tau$ vanishes.

Therefore, after some factoring,

$$\overset{(2)}{E}_\mu = -\frac{1}{2}c_1\epsilon_\mu{}^{\beta\nu\tau}\partial_\beta(A_\nu{*F}_\tau) - (c_1 + p_2)\epsilon_\mu{}^{\nu\tau}A_\nu\overset{(1)}{E}_\tau. \tag{3.56}$$

Use of redefinition freedom

This quadratic part of the field equation can be simplified using the transformation freedom (3.15).

At lowest order, this transformation is

$$\overset{(1)}{N}_\mu{}^\nu(A) = n_1\epsilon_\mu{}^{\nu\tau}A_\tau \tag{3.57}$$

by Lorentz covariance, where n_1 is an arbitrary constant. Thus $\overset{(2)}{E}_\mu$ can be transformed by adding the term

$$\overset{(1)}{N}_\mu{}^\nu(A)\overset{(1)}{E}_\nu = n_1\epsilon_\mu{}^{\nu\tau}A_\tau\overset{(1)}{E}_\nu. \tag{3.58}$$

This freedom allows us to put

$$p_2 = -c_1 \tag{3.59}$$

so the quadratic part of the field equation becomes simply

$$\overset{(2)}{E}_\mu = -\frac{1}{2}c_1\epsilon_\mu{}^{\beta\nu\tau}\partial_\beta(A_\nu{*F}_\tau). \tag{3.60}$$

In addition, substituting relation (3.59) into expression (3.48) gives

$$\overset{(0)}{P}_\mu{}^\nu(\zeta) = -c_1\epsilon_\mu{}^{\nu\tau}\partial_\tau\zeta. \tag{3.61}$$

3.8 Step 4: Quadratic-order gauge symmetry

Rather than write down an expression for $\overset{(2)}{\delta}_\zeta A_\mu$ from the general form of the gauge symmetry (3.7), which will contain numerous terms, the commutator deformation equation (3.21) at order 1 will be solved to directly obtain $\overset{(2)}{\delta}_\zeta A_\mu$. Thus by (3.40), we start from

$$[\delta_{\zeta_1}, \delta_{\zeta_2}]\overset{(1)}{}\overset{(1)}{E}_\mu = \overset{(0)}{R}_\mu{}^\nu(\zeta_1, \zeta_2)\overset{(1)}{E}_\nu + \overset{(0)}{S}_\mu{}^{\nu\tau}(\zeta_1, \zeta_2)\partial_\tau \overset{(1)}{E}_\nu \tag{3.62}$$

where

$$\overset{(1)}{[\delta_{\zeta_1}, \delta_{\zeta_2}]}\overset{(1)}{E}_\mu = [\overset{(0)}{\delta}_{\zeta_1}, \overset{(2)}{\delta}_{\zeta_2}]\overset{(1)}{E}_\mu + [\overset{(1)}{\delta}_{\zeta_1}, \overset{(1)}{\delta}_{\zeta_2}]\overset{(1)}{E}_\mu \tag{3.63}$$

with

$$\begin{aligned}
[\overset{(1)}{\delta}_{\zeta_1}, \overset{(1)}{\delta}_{\zeta_2}]A_\mu &= \overset{(1)}{\delta}_{\zeta_1}\big(\overset{(1)}{\delta}_{\zeta_2}A_\mu\big) - (\zeta_1 \leftrightarrow \zeta_2) \\
&= c_1\, \overset{(1)}{\delta}_{\zeta_1} {*F}_\mu \zeta_2 - (\zeta_1 \leftrightarrow \zeta_2) \\
&= {c_1}^2 \epsilon_\mu{}^{\nu\tau} {*F}_\tau(\zeta_2\partial_\nu\zeta_1 - \zeta_1\partial_\nu\zeta_2),
\end{aligned} \tag{3.64}$$

and where, through expressions (3.49) and (3.61),

$$\begin{aligned}
\overset{(0)}{R}_\mu{}^\nu(\zeta_1, \zeta_2) &= \overset{(0)}{P}_\mu{}^\tau(\zeta_2)\overset{(0)}{P}_\tau{}^\nu(\zeta_1) + \overset{(0)}{Q}_\mu{}^{\tau\alpha}(\zeta_2)\partial_\alpha \overset{(0)}{P}_\tau{}^\nu(\zeta_1) + \overset{(0)}{\delta}_{\zeta_1}\overset{(1)}{P}_\mu{}^\nu(\zeta_2) - (\zeta_1 \leftrightarrow \zeta_2) \\
&= {c_1}^2\epsilon_\mu{}^{\tau\sigma\nu}\big((\partial_\tau\zeta_2)\partial_\sigma\zeta_1 + \zeta_2\partial_{\tau\sigma}\zeta_1\big) + \overset{(0)}{\delta}_{\zeta_1}\overset{(1)}{P}_\mu{}^\nu(\zeta_2) - (\zeta_1 \leftrightarrow \zeta_2),
\end{aligned} \tag{3.65}$$

$$\begin{aligned}
\overset{(0)}{S}_\mu{}^{\tau\alpha}(\zeta_1, \zeta_2) &= \overset{(0)}{P}_\mu{}^\nu(\zeta_2)\overset{(0)}{Q}_\nu{}^{\tau\alpha}(\zeta_1) + \overset{(0)}{Q}_\mu{}^{\nu\alpha}(\zeta_2)\overset{(0)}{P}_\nu{}^\tau(\zeta_1) \\
&\qquad + \overset{(0)}{Q}_\mu{}^{\nu\beta}(\zeta_2)\partial_\beta \overset{(0)}{Q}_\nu{}^{\tau\alpha}(\zeta_1) + \overset{(0)}{\delta}_{\zeta_1}\overset{(1)}{Q}_\mu{}^{\tau\alpha}(\zeta_2) - (\zeta_1 \leftrightarrow \zeta_2) \\
&= {c_1}^2\big(\epsilon_\mu{}^{\alpha\sigma\tau} - \epsilon_\mu{}^{\sigma\tau\alpha}\big)\zeta_2\partial_\sigma\zeta_1 - {c_1}^2\epsilon_\mu{}^{\sigma\tau\alpha}\zeta_1\partial_\sigma\zeta_2 \\
&\qquad + \overset{(0)}{\delta}_{\zeta_1}\overset{(1)}{Q}_\mu{}^{\tau\alpha}(\zeta_2) - (\zeta_1 \leftrightarrow \zeta_2) \\
&= {c_1}^2\epsilon_\mu{}^{\alpha\sigma\tau}\zeta_2\partial_\sigma\zeta_1 + \overset{(0)}{\delta}_{\zeta_1}\overset{(1)}{Q}_\mu{}^{\tau\alpha}(\zeta_2) - (\zeta_1 \leftrightarrow \zeta_2)
\end{aligned} \tag{3.66}$$

with

$$\overset{(1)}{P}_\mu{}^\nu(\zeta) = \big(\overset{(1)}{\mathcal{P}}_\mu{}^\nu(A) + \overset{(0)}{\mathcal{P}}_\mu{}^{\nu\alpha\beta}\partial_\beta A_\alpha\big)\zeta + \overset{(1)}{\mathcal{P}}_\mu{}^{\nu\beta}(A)\partial_\beta\zeta, \tag{3.67}$$

$$\overset{(1)}{Q}_\mu{}^{\nu\tau}(\zeta) = \overset{(1)}{\mathcal{Q}}_\mu{}^{\nu\tau}(A)\zeta. \tag{3.68}$$

Lorentz covariance could be used to write down $\overset{(0)}{\mathcal{P}}_\mu{}^{\nu\alpha\beta}$, $\overset{(1)}{\mathcal{P}}_\mu{}^\nu(A)$, $\overset{(1)}{\mathcal{P}}_\mu{}^{\nu\beta}(A)$, and $\overset{(1)}{\mathcal{Q}}_\mu{}^{\nu\tau}(A)$ explicitly, but this generates a cumbersome number of terms, and it will be easier to solve for them directly.

Equation (3.62) can be solved by the following three steps.

Go to on-shell $\overset{(1)}{E}_\mu = 0$

This eliminates the entire right hand side of equation (3.62), forcing

$$\Big([\overset{(0)}{\delta}_{\zeta_1}, \overset{(2)}{\delta}_{\zeta_2}]\overset{(1)}{E}_\mu + [\overset{(1)}{\delta}_{\zeta_1}, \overset{(1)}{\delta}_{\zeta_2}]\overset{(1)}{E}_\mu\Big)\Big|_{\overset{(1)}{E}=0} = 0. \tag{3.69}$$

By using expression (3.27) for the linear part of the field equation, and expression (3.64) for the commutator of the linear gauge part of the gauge symmetry, this becomes

$$\begin{aligned} 0 &= \Big(\frac{1}{2}\epsilon_\mu{}^{\nu\tau\alpha}\partial_{\nu\tau}\big(\overset{(0)}{\delta}_{\zeta_1}(\overset{(2)}{\delta}_{\zeta_2}A_\alpha) + c_1{}^2\epsilon_\alpha{}^{\beta\gamma}{*F}_\gamma\zeta_2\partial_\beta\zeta_1\big) - (\zeta_1 \leftrightarrow \zeta_2)\Big)\Big|_{\overset{(1)}{E}=0} \\ &= \Big(\frac{1}{2}\epsilon_\mu{}^{\nu\tau\alpha}\partial_{\nu\tau}\overset{(0)}{\delta}_{\zeta_1}\big(\overset{(2)}{\delta}_{\zeta_2}A_\alpha + c_1{}^2\epsilon_\alpha{}^{\beta\gamma}{*F}_\gamma A_\beta\zeta_2\big) - (\zeta_1 \leftrightarrow \zeta_2)\Big)\Big|_{\overset{(1)}{E}=0} \end{aligned} \tag{3.70}$$

since $\overset{(0)}{\delta}_\zeta {*F}_\mu = 0$. This motivates putting

$$\overset{(2)}{\delta}_\zeta A_\mu = -c_1{}^2\epsilon_\mu{}^{\nu\tau}A_\nu{*F}_\tau\zeta + \big(\overset{(2)}{H'}_\mu(A) + \overset{(1)}{H'}_\mu{}^{\nu\alpha}(A)\partial_\nu A_\alpha\big)\zeta + \overset{(2)}{H'}_\mu{}^\nu(A)\partial_\nu\zeta \tag{3.71}$$

based on the general form of the gauge symmetry (3.7). From expression (3.71) for $\overset{(2)}{\delta}_\zeta A_\mu$, then commutator expression (3.63) becomes

$$\begin{aligned} [\overset{(1)}{\delta}_{\zeta_1}, \overset{(1)}{\delta}_{\zeta_2}]\overset{(1)}{E}_\mu &= \frac{1}{2}\epsilon_\mu{}^{\nu\tau\alpha}\partial_{\nu\tau}\overset{(0)}{\delta}_{\zeta_1}\big(\big(\overset{(2)}{H'}_\alpha(A) + \overset{(1)}{H'}_\alpha{}^{\sigma\beta}(A)\partial_\sigma A_\beta\big)\zeta_2 + \overset{(2)}{H'}_\alpha{}^\sigma(A)\partial_\sigma\zeta_2\big) - (\zeta_1 \leftrightarrow \zeta_2) \\ &= \frac{1}{2}\epsilon_\mu{}^{\nu\tau\alpha}\partial_{\nu\tau}\Big(\big(\overset{(0)}{\delta}_{\zeta_1}\overset{(2)}{H'}_\alpha(A) + \overset{(0)}{\delta}_{\zeta_1}\overset{(1)}{H'}_\alpha{}^{\sigma\beta}(A)\partial_\sigma A_\beta + \overset{(1)}{H'}_\alpha{}^{\sigma\beta}(A)\partial_{\sigma\beta}\zeta_1\big)\zeta_2 \\ &\qquad + \overset{(0)}{\delta}_{\zeta_1}\overset{(2)}{H'}_\alpha{}^\sigma(A)\partial_\sigma\zeta_2\Big) - (\zeta_1 \leftrightarrow \zeta_2), \end{aligned} \tag{3.72}$$

and thus

$$\begin{aligned} &\frac{1}{2}\epsilon_\mu{}^{\nu\tau\alpha}\partial_{\nu\tau}\Big(\big(\overset{(0)}{\delta}_{\zeta_1}\overset{(2)}{H'}_\alpha(A) + \overset{(0)}{\delta}_{\zeta_1}\overset{(1)}{H'}_\alpha{}^{\sigma\beta}(A)\partial_\sigma A_\beta + \overset{(1)}{H'}_\alpha{}^{\sigma\beta}(A)\partial_{\sigma\beta}\zeta_1\big)\zeta_2 + \overset{(0)}{\delta}_{\zeta_1}\overset{(2)}{H'}_\alpha{}^\sigma(A)\partial_\sigma\zeta_2\Big)\Big|_{\overset{(1)}{E}=0} \\ &\qquad - (\zeta_1 \leftrightarrow \zeta_2) = 0. \end{aligned} \tag{3.73}$$

The following will show how $\overset{(2)}{H'}_\alpha(A)$, $\overset{(2)}{H'}_\alpha{}^\sigma(A)$ and $\overset{(1)}{H'}_\alpha{}^{\sigma\beta}(A)$ each vanish modulo the redefinition freedom.

In equation (3.73), all terms having $\partial^4\zeta_1$ and ζ_2 are given by

$$\frac{1}{2}\epsilon_\mu{}^{\nu\tau\alpha}\overset{(1)}{H'}_\alpha{}^{\sigma\beta}(A)\zeta_2\,\partial_{\nu\tau\sigma\beta}\zeta_1 \tag{3.74}$$

which must vanish on-shell $\overset{(1)}{E} = 0$. This implies

$$\epsilon_\mu{}^{(\nu\tau|\alpha}\overset{(1)}{H'}_\alpha{}^{|\sigma\beta)}(A) = 0 \tag{3.75}$$

from which it can be shown that

$$\overset{(1)}{H'}{}_{\alpha}{}^{(\sigma\beta)}(A) = c_{1,1}\delta_{\alpha}^{(\beta}A^{\sigma)} \tag{3.76}$$

where $c_{1,1}$ is a constant. Lorentz covariance then determines

$$\overset{(1)}{H'}{}_{\alpha}{}^{\sigma\beta}(A) = c_{1,1}\delta_{\alpha}^{(\beta}A^{\sigma)} + c_{1,2}\delta_{\alpha}^{[\sigma}A^{\beta]} \tag{3.77}$$

where $c_{1,2}$ is a constant. Moreover, Lorentz covariance also determines

$$\overset{(2)}{H'}{}_{\alpha}(A) = c_{1,3}\epsilon_{\alpha}{}^{\sigma\tau}A_{\sigma}A_{\tau} = 0 \tag{3.78}$$

and

$$\overset{(2)}{H'}{}_{\alpha}{}^{\sigma}(A) = \big(c_{1,4}\delta_{\alpha}^{\sigma}\eta^{\tau\beta} + c_{1,5}\delta_{\alpha}^{\tau}\eta^{\beta\sigma}\big)A_{\beta}A_{\tau} = c_{1,4}\delta_{\alpha}^{\sigma}A^{\tau}A_{\tau} + c_{1,5}A_{\alpha}A^{\sigma}, \tag{3.79}$$

where $c_{1,3}$, $c_{1,4}$ and $c_{1,5}$ are constants. As a consequence, the commutator expression (3.72) becomes

$$\begin{aligned}
[\overset{(1)}{\delta}_{\zeta_1}, \delta_{\zeta_2}]\overset{(1)}{E}_{\mu} &= \frac{1}{2}\epsilon_{\mu}{}^{\nu\tau\alpha}\partial_{\nu\tau}\Big(\big(c_{1,1}\overset{(0)}{\delta}_{\zeta_1}(A^{\beta}A_{\alpha\beta}) - \frac{1}{2}c_{1,2}\epsilon_{\alpha}{}^{\sigma\beta}\overset{(0)}{\delta}_{\zeta_1}(A_{\sigma}{*}F_{\beta})\big)\zeta_2 \\
&\qquad + 2c_{1,4}A^{\beta}(\overset{(0)}{\delta}_{\zeta_1}A_{\beta})\partial_{\alpha}\zeta_2 + c_{1,5}\overset{(0)}{\delta}_{\zeta_1}(A_{\alpha}A^{\sigma})\partial_{\sigma}\zeta_2\Big) - (\zeta_1 \leftrightarrow \zeta_2) \\
&= \frac{1}{2}\epsilon_{\mu}{}^{\nu\tau\alpha}\partial_{\nu\tau}\Big(\big(c_{1,1}A_{\alpha\sigma} - \frac{1}{2}c_{1,2}\epsilon_{\alpha\sigma}{}^{\beta}{*}F_{\beta}\big)\zeta_2\partial^{\sigma}\zeta_1 + c_{1,1}A^{\beta}\zeta_2\partial_{\alpha\beta}\zeta_1 \\
&\qquad + (2c_{1,4} - c_{1,5})A^{\beta}(\partial_{\beta}\zeta_1)\partial_{\alpha}\zeta_2\Big) - (\zeta_1 \leftrightarrow \zeta_2).
\end{aligned} \tag{3.80}$$

Next, all terms in equation (3.73) containing $\partial\zeta_2$ and $\partial^3\zeta_1$ are given by

$$\begin{aligned}
&c_{1,1}\epsilon_{\mu}{}^{\nu\tau\alpha}A^{\beta}(\partial_{(\nu}\zeta_2)\partial_{\tau)\alpha\beta}\zeta_1 + \frac{1}{2}(2c_{1,4} - c_{1,5})\epsilon_{\mu}{}^{\nu\tau\alpha}A^{\beta}\big((\partial_{\nu\tau\beta}\zeta_1)\partial_{\alpha}\zeta_2 - (\partial_{\nu\tau\alpha}\zeta_1)\partial_{\beta}\zeta_2\big) \\
&= \frac{1}{2}(2c_{1,4} - c_{1,5} - c_{1,1})\epsilon_{\mu}{}^{\nu\tau\alpha}A^{\beta}(\partial_{\nu\tau\beta}\zeta_1)\partial_{\alpha}\zeta_2
\end{aligned} \tag{3.81}$$

due to $\epsilon_{\mu}{}^{(\nu\tau\alpha)} = 0$. Since this term must vanish on-shell $\overset{(1)}{E} = 0$,

$$2c_{1,4} = c_{1,5} + c_{1,1}, \tag{3.82}$$

and hence

$$\overset{(2)}{H'}{}_{\alpha}{}^{\sigma}(A) = \frac{1}{2}(c_{1,1} + c_{1,5})\delta_{\alpha}^{\sigma}A^{\tau}A_{\tau} + c_{1,5}A_{\alpha}A^{\sigma}. \tag{3.83}$$

Thus, the commutator expression (3.80) becomes

$$\begin{aligned}
[\overset{(1)}{\delta}_{\zeta_1}, \delta_{\zeta_2}]\overset{(1)}{E}_{\mu} &= \frac{1}{2}\epsilon_{\mu}{}^{\nu\tau\alpha}\partial_{\nu\tau}\Big(\big(c_{1,1}A_{\alpha\sigma} - \frac{1}{2}c_{1,2}\epsilon_{\alpha\sigma}{}^{\beta}{*}F_{\beta}\big)\zeta_2\partial^{\sigma}\zeta_1 + c_{1,1}A^{\beta}\zeta_2\partial_{\alpha\beta}\zeta_1 \\
&\qquad + c_{1,1}A^{\beta}(\partial_{\beta}\zeta_1)\partial_{\alpha}\zeta_2\Big) - (\zeta_1 \leftrightarrow \zeta_2).
\end{aligned} \tag{3.84}$$

All terms in equation (3.73) containing $\partial^2\zeta_1$ and $\partial^2\zeta_2$ cancel, so to continue, consider all terms having $\partial^3\zeta_1$ and ζ_2:

$$
\begin{aligned}
&\frac{1}{2}\epsilon_{\mu}{}^{\nu\tau\alpha}\big(c_{1,1}\eta^{\sigma\beta}A_{\alpha\beta} - \frac{1}{2}c_{1,2}\epsilon_{\alpha}{}^{\sigma\beta}{*F}_{\beta}\big)\zeta_2\partial_{\nu\tau\sigma}\zeta_1 + \epsilon_{\mu}{}^{\nu\tau\alpha}\big(c_{1,1}\partial_{(\nu}A^{\beta}\big)\zeta_2\partial_{\tau)\alpha\beta}\zeta_1 \\
&= \frac{1}{2}\epsilon_{\mu}{}^{\nu\tau\alpha}\big(c_{1,1}\eta^{\sigma\beta}A_{\alpha\beta} - \frac{1}{2}c_{1,2}\epsilon_{\alpha}{}^{\sigma\beta}{*F}_{\beta}\big)\zeta_2\partial_{\nu\tau\sigma}\zeta_1 \\
&\qquad + \epsilon_{\mu}{}^{\nu\tau\alpha}\big(c_{1,1}\eta^{\beta\sigma}A_{\beta(\nu} - \frac{1}{2}c_{1,1}\epsilon_{(\nu}{}^{\sigma\beta}{*F}_{|\beta|}\big)\zeta_2\partial_{\tau)\alpha\sigma}\zeta_1 \\
&= \frac{1}{2}c_{1,1}\eta^{\sigma\beta}A_{\alpha\beta}\big(\epsilon_{\mu}{}^{\nu\tau\alpha} + 2\epsilon_{\mu}{}^{(\alpha\tau)\nu}\big)\zeta_2\partial_{\nu\tau\sigma}\zeta_1 + \frac{1}{2}\epsilon_{\alpha}{}^{\sigma\beta}{*F}_{\beta}\big(-\frac{1}{2}c_{1,2}\epsilon_{\mu}{}^{\nu\tau\alpha} - c_{1,1}\epsilon_{\mu}{}^{(\alpha\tau)\nu}\big)\zeta_2\partial_{\nu\tau\sigma}\zeta_1 \\
&= \frac{1}{2}c_{1,1}\eta^{\sigma\beta}A_{\alpha\beta}\big(\epsilon_{\mu}{}^{\nu\tau\alpha} - \epsilon_{\mu}{}^{\tau\nu\alpha}\big)\zeta_2\partial_{\nu\tau\sigma}\zeta_1 + \frac{1}{4}\epsilon_{\alpha}{}^{\sigma\beta}{*F}_{\beta}\big(-c_{1,2}\epsilon_{\mu}{}^{\nu\tau\alpha} + c_{1,1}\epsilon_{\mu}{}^{\tau\nu\alpha}\big)\zeta_2\partial_{\nu\tau\sigma}\zeta_1
\end{aligned}
\tag{3.85}
$$

after use of the identity (A.29) and $\epsilon_{\mu}{}^{\nu(\tau\alpha)} = 0$. The terms with $A_{\alpha\beta}$ vanish due to the symmetry of $\partial_{\nu\tau\sigma}\zeta_1$, whereas the terms with ${*F}_{\beta}$ are non-vanishing due to $\epsilon_{\alpha}{}^{\beta(\sigma}\epsilon_{\mu}{}^{\nu\tau)\alpha} = \epsilon_{\mu}{}^{\beta(\nu}\eta^{\sigma\tau)} \neq 0$ through the identity (A.27). Hence, these terms reduce to

$$
\frac{1}{4}(c_{1,1} - c_{1,2})\epsilon_{\mu}{}^{\nu\beta}\eta^{\sigma\tau}{*F}_{\beta}\zeta_2\partial_{\nu\tau\sigma}\zeta_1 \tag{3.86}
$$

which must vanish on-shell $\overset{(1)}{E} = 0$. This yields

$$
c_{1,2} = c_{1,1}, \tag{3.87}
$$

and consequently

$$
\overset{(1)}{H'}{}_{\alpha}{}^{\sigma\beta}(A) = c_{1,1}\delta^{\sigma}_{\alpha}A^{\beta}. \tag{3.88}
$$

Thus, from expressions (3.88), (3.83) and (3.78), combined with expression (3.71),

$$
\overset{(2)}{\delta}_{\zeta}A_{\mu} = \big(c_{1,1}A^{\tau}A_{\mu\tau} - (\frac{1}{2}c_{1,1} + {c_1}^2)\epsilon_{\mu}{}^{\nu\tau}A_{\nu}{*F}_{\tau}\big)\zeta + \frac{1}{2}(c_{1,1} + c_{1,5})A^{\tau}A_{\tau}\partial_{\mu}\zeta + c_{1,5}A_{\mu}A^{\tau}\partial_{\tau}\zeta. \tag{3.89}
$$

Use of redefinition freedom

The quadratic part of the gauge symmetry (3.89) can be simplified using the redefinition freedoms (3.9)–(3.10).

At lowest order, these two freedoms are trivial due to Lorentz covariance:

$$
\overset{(1)}{U}(A) = 0, \quad \overset{(2)}{V}_{\mu}(A) = v\epsilon_{\mu}{}^{\sigma\tau}A_{\sigma}A_{\tau} = 0, \tag{3.90}
$$

where v is an arbitrary constant. At the next order, there are two nontrivial terms:

$$
\overset{(2)}{U}(A) = u_1\eta^{\sigma\tau}A_{\sigma}A_{\tau} = u_1A^{\nu}A_{\nu}, \quad \overset{(3)}{V}_{\mu}(A) = v_1\delta^{(\alpha}_{\mu}\eta^{\beta\gamma)}A_{\alpha}A_{\beta}A_{\gamma} = v_1A_{\mu}A^{\beta}A_{\beta}, \tag{3.91}
$$

where u_1 and v_1 are arbitrary constants. To use these freedoms, the expressions are substituted into equations (3.12)–(3.14), yielding the terms

$$\frac{\partial \overset{(3)}{V}_\mu(A)}{\partial A_\nu}\partial_\nu\zeta + \partial_\mu\big(\overset{(2)}{U}(A)\zeta\big) = 2u_1(A^\nu\partial_\mu A_\nu)\zeta + (u_1+v_1)A^\nu A_\nu\partial_\mu\zeta + 2v_1 A_\mu A^\nu\partial_\nu\zeta, \tag{3.92}$$

which can be added to expression (3.89) for $\overset{(2)}{\delta}_\zeta A_\mu$. After using the identity (A.29), this gives

$$\begin{aligned}\overset{(2)}{\delta}_\zeta A_\mu = \big(-(\tfrac{1}{2}c_{1,1}+u_1+{c_1}^2)\epsilon_\mu{}^{\nu\tau}A_\nu{*F}_\tau + (c_{1,1}+2u_1)A^\nu A_{\mu\nu}\big)\zeta \\ +\big(\tfrac{1}{2}(c_{1,1}+c_{1,5})+u_1+v_1\big)A^\tau A_\tau\partial_\mu\zeta + (c_{1,5}+2v_1)A_\mu A^\tau\partial_\tau\zeta.\end{aligned} \tag{3.93}$$

We can put $v_1 = -\frac{1}{2}c_{1,5}$ and $u_1 = -\frac{1}{2}c_{1,1}$, which removes the terms with $c_{1,1}$ and $c_{1,5}$ in expression (3.89). Hence, modulo the redefinition freedoms,

$$c_{1,1} = 0, \quad c_{1,5} = 0, \tag{3.94}$$

whereby

$$\overset{(2)}{\delta}_\zeta A_\mu = -{c_1}^2\epsilon_\mu{}^{\nu\tau}A_\nu{*F}_\tau\zeta. \tag{3.95}$$

Return to off-shell $\overset{(1)}{E}_\mu$

Using relations (3.94) makes the commutator expression (3.84) vanish,

$$[\delta_{\zeta_1},\overset{(1)}{\delta}_{\zeta_2}]\overset{(1)}{E}_\mu = 0. \tag{3.96}$$

Consequently, the entire left hand side of equation (3.62) will vanish, leaving

$$\overset{(0)}{R}_\mu{}^\nu(\zeta_1,\zeta_2)\overset{(1)}{E}_\nu + \overset{(0)}{S}_\mu{}^{\nu\tau}(\zeta_1,\zeta_2)\partial_\tau\overset{(1)}{E}_\nu = 0. \tag{3.97}$$

Since $\overset{(1)}{E}_\nu$ and $\partial_\tau\overset{(1)}{E}_\nu$ are linearly independent, their coefficients must vanish,

$$\overset{(0)}{R}_\mu{}^\nu(\zeta_1,\zeta_2) = 0, \quad \overset{(0)}{S}_\mu{}^{\nu\tau}(\zeta_1,\zeta_2) = 0. \tag{3.98}$$

Therefore, using expressions (3.65) and (3.67) gives

$$\begin{aligned}0 = \big(\overset{(0)}{\delta}_{\zeta_1}\overset{(1)}{\mathcal{P}}_\mu{}^\nu(A)\big)\zeta_2 + \big(\overset{(0)}{\delta}_{\zeta_1}\overset{(1)}{\mathcal{P}}_\mu{}^{\nu\beta}(A) + {c_1}^2\epsilon_\mu{}^{\beta\alpha\nu}\partial_\alpha\zeta_1\big)\partial_\beta\zeta_2 \\ +\big(\overset{(0)}{\mathcal{P}}_\mu{}^{\nu\alpha\beta} + {c_1}^2\epsilon_\mu{}^{\alpha\beta\nu}\big)\zeta_2\partial_{\alpha\beta}\zeta_1 - (\zeta_1\leftrightarrow\zeta_2),\end{aligned} \tag{3.99}$$

and using expressions (3.66) and (3.68) gives

$$0 = \big({c_1}^2\epsilon_\mu{}^{\alpha\sigma\tau}\partial_\sigma\zeta_1 + \overset{(0)}{\delta}_{\zeta_1}\overset{(1)}{\mathcal{Q}}_\mu{}^{\tau\alpha}(A)\big)\zeta_2 - (\zeta_1\leftrightarrow\zeta_2). \tag{3.100}$$

Terms in these two equations containing a different number of derivatives on ζ_1 or ζ_2 are linearly independent. Hence, we have

$$\overset{(0)}{\delta}_{\zeta_1}\overset{(1)}{\mathcal{P}}_{\mu}{}^{\nu}(A) = 0, \tag{3.101}$$

$$\big(\overset{(0)}{\delta}_{\zeta_1}\overset{(1)}{\mathcal{P}}_{\mu}{}^{\nu\beta}(A) + {c_1}^2\epsilon_{\mu}{}^{\beta\alpha\nu}\partial_\alpha\zeta_1\big)\partial_\beta\zeta_2 - (\zeta_1 \leftrightarrow \zeta_2) = 0, \tag{3.102}$$

$$\big(\overset{(0)}{\mathcal{P}}_{\mu}{}^{\nu\alpha\beta} + {c_1}^2\epsilon_{\mu}{}^{\alpha\beta\nu}\big)\partial_{\alpha\beta}\zeta_1 = 0, \tag{3.103}$$

$${c_1}^2\epsilon_{\mu}{}^{\alpha\sigma\tau}\partial_\sigma\zeta_1 + \overset{(0)}{\delta}_{\zeta_1}\overset{(1)}{\mathcal{Q}}_{\mu}{}^{\tau\alpha}(A) = 0. \tag{3.104}$$

First, consider equation (3.101). By Lorentz covariance,

$$\overset{(1)}{\mathcal{P}}_{\mu}{}^{\nu}(A) = p_{1,1}\epsilon_{\mu}{}^{\nu\tau}A_\tau, \tag{3.105}$$

where $p_{1,1}$ is a constant. This yields $\overset{(0)}{\delta}_{\zeta_1}\overset{(1)}{\mathcal{P}}_{\mu}{}^{\nu}(A) = p_{1,1}\epsilon_{\mu}{}^{\nu\tau}\partial_\tau\zeta_1$, and therefore we must have $p_{1,1} = 0$. Thus,

$$\overset{(1)}{\mathcal{P}}_{\mu}{}^{\nu}(A) = 0. \tag{3.106}$$

Second, equation (3.103) directly yields $\overset{(0)}{\mathcal{P}}_{\mu}{}^{\nu(\alpha\beta)} = -{c_1}^2\epsilon_{\mu}{}^{(\alpha\beta)\nu}$ by symmetry of $\partial_{\alpha\beta}\zeta_1$. Then Lorentz covariance implies $\overset{(0)}{\mathcal{P}}_{\mu}{}^{\nu[\alpha\beta]} = p_{1,2}\epsilon_{\mu}{}^{\nu\alpha\beta}$ where $p_{1,2}$ is a constant. Hence,

$$\overset{(0)}{\mathcal{P}}_{\mu}{}^{\nu\alpha\beta} = -{c_1}^2\epsilon_{\mu}{}^{(\alpha\beta)\nu} + p_{1,2}\epsilon_{\mu}{}^{\nu\alpha\beta}. \tag{3.107}$$

Next, consider equation (3.104). It can be expressed as

$$\overset{(0)}{\delta}_{\zeta_1}\big({c_1}^2\epsilon_{\mu}{}^{\alpha\sigma\tau}A_\sigma + \overset{(1)}{\mathcal{Q}}_{\mu}{}^{\tau\alpha}(A)\big) = 0, \tag{3.108}$$

which yields

$$\overset{(1)}{\mathcal{Q}}_{\mu}{}^{\tau\alpha}(A) = -{c_1}^2\epsilon_{\mu}{}^{\alpha\sigma\tau}A_\sigma \tag{3.109}$$

since $\overset{(1)}{\mathcal{Q}}_{\mu}{}^{\tau\alpha}(A)$ cannot contain $*F$.

Last, consider equation (3.102). Similarly to equation (3.109), we have

$$\overset{(0)}{\delta}_{\zeta_1}\big(\overset{(1)}{\mathcal{P}}_{\mu}{}^{\nu\beta}(A) + {c_1}^2\epsilon_{\mu}{}^{\beta\alpha\nu}A_\alpha\big)\partial_\beta\zeta_2 - (\zeta_1 \leftrightarrow \zeta_2) = 0. \tag{3.110}$$

Note this equation is antisymmetric in $\partial_\alpha\zeta_1$ and $\partial_\beta\zeta_2$. Then Lorentz covariance implies that

$$\overset{(1)}{\mathcal{P}}_{\mu}{}^{\nu\beta}(A) = \big(p_{1,3}\delta_\mu^\nu\eta^{\alpha\beta} + p_{1,4}\delta_\mu^{(\alpha}\eta^{\beta)\nu} - {c_1}^2\epsilon_{\mu}{}^{\beta\alpha\nu}\big)A_\alpha \tag{3.111}$$

where $p_{1,3}$ and $p_{1,4}$ are constants.

Substituting expressions (3.111), (3.109), (3.107) and (3.106) into expressions (3.67)–(3.68), we have

$$\begin{aligned}\overset{(1)}{P}_{\mu}{}^{\nu}(\zeta) = & -\big({c_1}^2\epsilon_{\mu}{}^{\alpha\beta\nu}A_{\alpha\beta} + p_{1,2}\epsilon_{\mu}{}^{\nu\sigma}*F_\sigma\big)\zeta \\ & - {c_1}^2\epsilon_{\mu}{}^{\beta\alpha\nu}A_\alpha\partial_\beta\zeta + p_{1,3}\delta_\mu^\nu A^\alpha\partial_\alpha\zeta + p_{1,4}\eta^{\nu\beta}A_{(\mu}\partial_{\beta)}\zeta\end{aligned} \tag{3.112}$$

and

$$\overset{(1)}{Q}{}_{\mu}{}^{\nu\tau}(\zeta) = -{c_1}^2 \epsilon_{\mu}{}^{\tau\sigma\nu} A_\sigma \zeta \tag{3.113}$$

after use of identity (A.28).

This completes the solution of the commutator deformation equation (3.62).

3.9 Step 5: Consistency condition

The deformation equation (3.17) at order 2 is

$$\overset{(0)}{\delta}_\zeta \overset{(3)}{E}_\mu + \overset{(1)}{\delta}_\zeta \overset{(2)}{E}_\mu + \overset{(2)}{\delta}_\zeta \overset{(1)}{E}_\mu = \overset{(0)}{P}{}_\mu{}^\nu(\zeta)\overset{(2)}{E}_\nu + \overset{(1)}{P}{}_\mu{}^\nu(\zeta)\overset{(1)}{E}_\nu + \overset{(0)}{Q}{}_\mu{}^{\nu\tau}(\zeta)\partial_\tau\overset{(2)}{E}_\nu + \overset{(1)}{Q}{}_\mu{}^{\nu\tau}(\zeta)\partial_\tau\overset{(1)}{E}_\nu \tag{3.114}$$

with $\overset{(0)}{P}{}_\mu{}^\nu(\zeta)$ and $\overset{(0)}{Q}{}_\mu{}^{\nu\tau}(\zeta)$ given by expressions (3.48)–(3.49); $\overset{(1)}{P}{}_\mu{}^\nu(\zeta)$ and $\overset{(1)}{Q}{}_\mu{}^{\nu\tau}(\zeta)$ given by expressions (3.112)–(3.113); $\overset{(2)}{E}_\mu$ and $\overset{(2)}{\delta}_\zeta A_\mu$ given by expressions (3.60) and (3.95). The primary unknown is $\overset{(3)}{E}_\mu$.

We set ζ to be constant in this equation, which eliminates the $\overset{(0)}{\delta}_\zeta \overset{(3)}{E}_\mu$ and $\overset{(0)}{P}{}_\mu{}^\nu(\zeta)\overset{(2)}{E}_\nu$ terms, while the rest become

$$\overset{(1)}{\delta}_\zeta \overset{(2)}{E}_\mu = -{c_1}^2 \epsilon_\mu{}^{\beta\nu\tau}\zeta\Big(\big(A_{\beta\nu} - \frac{1}{2}\epsilon_{\beta\nu}{}^{\sigma} {*F}_\sigma\big)\overset{(1)}{E}_\tau + A_\nu \partial_\beta \overset{(1)}{E}_\tau\Big), \tag{3.115}$$

$$\overset{(2)}{\delta}_\zeta \overset{(1)}{E}_\mu = -\frac{1}{2}{c_1}^2 \epsilon_\mu{}^{\alpha\beta\tau\theta}\zeta\partial_{\alpha\beta}\big(A_\tau {*F}_\theta\big), \tag{3.116}$$

$$\overset{(1)}{P}{}_\mu{}^\nu(\zeta)\overset{(1)}{E}_\nu = -\big({c_1}^2\epsilon_\mu{}^{\alpha\beta\nu}A_{\alpha\beta} + p_{1,2}\epsilon_\mu{}^{\nu\tau}{*F}_\tau\big)\zeta\overset{(1)}{E}_\nu, \tag{3.117}$$

$$\overset{(0)}{Q}{}_\mu{}^{\nu\tau}(\zeta)\partial_\tau\overset{(2)}{E}_\nu = -\frac{1}{2}{c_1}^2 \epsilon_\mu{}^{\alpha\beta\tau\theta}\zeta\partial_{\alpha\beta}\big(A_\tau {*F}_\theta\big), \tag{3.118}$$

$$\overset{(1)}{Q}{}_\mu{}^{\nu\tau}(\zeta)\partial_\tau\overset{(1)}{E}_\nu = -{c_1}^2\epsilon_\mu{}^{\tau\alpha\nu}A_\alpha\zeta\partial_\tau\overset{(1)}{E}_\nu. \tag{3.119}$$

Note that $\overset{(2)}{\delta}_\zeta \overset{(1)}{E}_\mu$ and $\overset{(0)}{Q}{}_\mu{}^{\nu\tau}(\zeta)\partial_\tau\overset{(2)}{E}_\nu$ immediately cancel leaving only three linearly independent types of terms: $A_\mu\partial_\nu\overset{(1)}{E}_\tau$, $A_{\mu\nu}\overset{(1)}{E}_\tau$ and ${*F}_\mu\overset{(1)}{E}_\nu$.

We find that the $A_\mu\partial_\nu\overset{(1)}{E}_\tau$ and $A_{\mu\nu}\overset{(1)}{E}_\tau$ coefficients vanish, while the ${*F}_\mu\overset{(1)}{E}_\nu$ terms require

$$p_{1,2} = -\frac{1}{2}{c_1}^2. \tag{3.120}$$

Thus (3.112) becomes

$$\begin{aligned}\overset{(1)}{P}{}_\mu{}^\nu(\zeta) = &-{c_1}^2\big(\epsilon_\mu{}^{\alpha\beta\nu}A_{\alpha\beta} - \frac{1}{2}\epsilon_\mu{}^{\nu\sigma}{*F}_\sigma\big)\zeta \\ &- {c_1}^2\epsilon_\mu{}^{\beta\alpha\nu}A_\alpha\partial_\beta\zeta + p_{1,3}\delta^\nu_\mu A^\alpha\partial_\alpha\zeta + p_{1,4}\eta^{\nu\beta}A_{(\mu}\partial_{\beta)}\zeta.\end{aligned} \tag{3.121}$$

3.10 Step 6: Cubic-order field equation

In the deformation equation at order 2 (3.114), all of the remaining terms contain $\partial\zeta$ or $\partial^2\zeta$:

$$\overset{(1)}{\delta}_\zeta \overset{(2)}{E}_\mu = -\frac{1}{2}{c_1}^2\epsilon_\mu{}^{\beta\nu\tau}\Big(\epsilon_\tau{}^{\theta\gamma}\big(*F_\gamma\partial_\theta\zeta\big(A_{\beta\nu}-\frac{1}{2}\epsilon_{\beta\nu}{}^\sigma *F_\sigma\big) + A_\nu\partial_\theta\zeta\big(*F_{\beta\gamma}-\epsilon_{\beta\gamma}{}^\sigma\overset{(1)}{E}_\sigma\big) + A_\nu *F_\gamma\partial_{\beta\theta}\zeta\big) + 2A_\nu\partial_\beta\zeta\overset{(1)}{E}_\tau\Big), \tag{3.122}$$

$$\overset{(2)}{\delta}_\zeta \overset{(1)}{E}_\mu = -\frac{1}{2}{c_1}^2\epsilon_\mu{}^{(\alpha\beta)\tau\theta}\Big(2*F_\theta\partial_\alpha\zeta\big(A_{\beta\tau}-\frac{1}{2}\epsilon_{\beta\tau}{}^\sigma *F_\sigma\big) + A_\tau *F_\theta\partial_{\alpha\beta}\zeta + 2A_\tau\partial_\alpha\zeta\big(*F_{\beta\theta}-\epsilon_{\beta\theta}{}^\sigma\overset{(1)}{E}_\sigma\big)\Big), \tag{3.123}$$

$$\overset{(0)}{P}_\mu{}^\nu\overset{(2)}{E}_\nu = -\frac{1}{2}{c_1}^2\epsilon_\mu{}^{\tau\sigma\alpha\beta}\partial_\tau\zeta\Big(*F_\beta\big(A_{\sigma\alpha}-\frac{1}{2}\epsilon_{\sigma\alpha}{}^\gamma *F_\gamma\big) + A_\alpha\big(*F_{\sigma\beta}-\epsilon_{\sigma\beta}{}^\gamma\overset{(1)}{E}_\gamma\big)\Big), \tag{3.124}$$

$$\overset{(1)}{P}_\mu{}^\nu\overset{(1)}{E}_\nu = \big(-{c_1}^2\epsilon_\mu{}^{\beta\alpha\nu}A_\alpha\partial_\beta\zeta + p_{1,3}\delta^\nu_\mu A^\alpha\partial_\alpha\zeta + p_{1,4}\eta^{\nu\beta}A_{(\mu}\partial_{\beta)}\zeta\big)\overset{(1)}{E}_\nu. \tag{3.125}$$

Now $\overset{(3)}{E}_\mu$ can be found using the same method as for finding $\overset{(2)}{E}_\mu$. Omitting details, we get

$$\overset{(3)}{E}_\mu = \frac{1}{2}p_{1,3}A^\nu A_\nu\overset{(1)}{E}_\mu + \frac{1}{2}p_{1,4}A_\mu A^\nu\overset{(1)}{E}_\nu + \frac{1}{2}{c_1}^2\epsilon_\mu{}^{\sigma\alpha\beta\nu}\partial_\sigma\big(A_\alpha A_\beta *F_\nu\big). \tag{3.126}$$

Note that there is no integration term of the form $\overset{(3)}{G}_\mu(*F) = \overset{(0)}{G}_\mu{}^{\nu\tau\alpha}*F_\nu *F_\tau *F_\alpha$ since it contains too many derivatives compared to the general form of the deformed field equation (3.6).

Use of redefinition freedom

Simplifying the cubic part of the field equation through the transformation freedom (3.15) requires adding the terms $\overset{(2)}{N}_\mu{}^\nu(A)\overset{(1)}{E}_\nu$ and $\overset{(1)}{N}_\mu{}^\nu(A)\overset{(2)}{E}_\nu$ to $\overset{(3)}{E}_\nu$, where $\overset{(1)}{N}_\mu{}^\nu(A)$ is given by expression (3.57).

By Lorentz covariance,

$$\overset{(2)}{N}_\mu{}^\nu(A) = n_{1,1}\delta^\nu_\mu A^\tau A_\alpha + n_{1,2}A_\mu A^\nu \tag{3.127}$$

where $n_{1,1}$ and $n_{1,2}$ are constants. Thus, the added terms have the form

$$\overset{(2)}{N}_\mu{}^\nu(A)\overset{(1)}{E}_\nu + \overset{(1)}{N}_\mu{}^\nu(A)\overset{(2)}{E}_\nu = n_{1,1}\overset{(1)}{E}_\mu A^\tau A_\alpha + n_{1,2}A_\mu A^\nu\overset{(1)}{E}_\nu + n_1\epsilon_\mu{}^{\nu\tau}A_\tau\overset{(2)}{E}_\nu. \tag{3.128}$$

This freedom shows that we can put

$$p_{1,3}=0, \quad p_{1,4}=0, \tag{3.129}$$

whereby

$$\overset{(1)}{P}{}_{\mu}{}^{\nu}(\zeta) = -{c_1}^2\big(\epsilon_{\mu}{}^{\alpha\beta\nu}A_{\alpha\beta} - \frac{1}{2}\epsilon_{\mu}{}^{\nu\sigma}{*F}_{\sigma}\big)\zeta - {c_1}^2\epsilon_{\mu}{}^{\beta\alpha\nu}A_{\alpha}\partial_{\beta}\zeta \tag{3.130}$$

and

$$\overset{(3)}{E}_{\mu} = \frac{1}{2}{c_1}^2\epsilon_{\mu}{}^{\sigma\alpha\beta\nu}\partial_{\sigma}\big(A_{\alpha}A_{\beta}{*F}_{\nu}\big). \tag{3.131}$$

3.11 Step 7: General pattern to higher orders

So far we have determined

$$\overset{(1)}{E}_{\mu} = \frac{1}{2}\epsilon_{\mu}{}^{\nu\tau}\partial_{\nu}{*F}_{\tau}, \tag{3.132a}$$

$$\overset{(2)}{E}_{\mu} = -\frac{1}{2}c_1\epsilon_{\mu}{}^{\beta\nu\tau}\partial_{\beta}\big(A_{\nu}{*F}_{\tau}\big), \tag{3.132b}$$

$$\overset{(3)}{E}_{\mu} = \frac{1}{2}{c_1}^2\epsilon_{\mu}{}^{\sigma\alpha\beta\nu}\partial_{\sigma}\big(A_{\alpha}A_{\beta}{*F}_{\nu}\big), \tag{3.132c}$$

and

$$\overset{(0)}{\delta}_{\zeta}A_{\mu} = \partial_{\mu}\zeta, \tag{3.133a}$$

$$\overset{(1)}{\delta}_{\zeta}A_{\mu} = c_1{*F}_{\mu}\zeta, \tag{3.133b}$$

$$\overset{(2)}{\delta}_{\zeta}A_{\mu} = -{c_1}^2\epsilon_{\mu}{}^{\nu\tau}A_{\nu}{*F}_{\tau}\zeta, \tag{3.133c}$$

as well as

$$\overset{(0)}{P}{}_{\mu}{}^{\nu}(\zeta) = -c_1\epsilon_{\mu}{}^{\nu\tau}\partial_{\tau}\zeta, \tag{3.134}$$

$$\overset{(1)}{P}{}_{\mu}{}^{\nu}(\zeta) = -{c_1}^2\big(\epsilon_{\mu}{}^{\alpha\beta\nu}A_{\alpha\beta} - \frac{1}{2}\epsilon_{\mu}{}^{\nu\sigma}{*F}_{\sigma}\big)\zeta - {c_1}^2\epsilon_{\mu}{}^{\beta\alpha\nu}A_{\alpha}\partial_{\beta}\zeta, \tag{3.135}$$

$$\overset{(0)}{Q}{}_{\mu}{}^{\nu\tau}(\zeta) = -c_1\epsilon_{\mu}{}^{\nu\tau}\zeta, \tag{3.136}$$

$$\overset{(1)}{Q}{}_{\mu}{}^{\nu\tau}(\zeta) = -{c_1}^2\epsilon_{\mu}{}^{\tau\sigma\nu}A_{\sigma}\zeta, \tag{3.137}$$

where c_1, is an arbitrary constant.

Determining the pattern

From the terms in the series expansion that have been determined, a pattern can be guessed and checked.

We begin by introducing new notations. First, we define a deformation of the identity tensor:

$$Y^{\nu}_{\mu} \equiv \delta^{\nu}_{\mu} + c_1\epsilon_{\mu}{}^{\sigma\nu}A_{\sigma}. \tag{3.138}$$

Its inverse $Y^{-1\nu}_{\ \ \mu}$ is determined by

$$Y^{-1\nu}_{\ \ \mu}Y^{\tau}_{\nu} = Y^{\nu}_{\mu}Y^{-1\tau}_{\ \ \nu} = \delta^{\tau}_{\mu}, \tag{3.139}$$

which yields

$$Y^{-1}{}^{\nu}_{\ \mu} = \delta^{\nu}_{\mu} - c_1 \epsilon_{\mu}{}^{\sigma\nu} A_\sigma + c_1{}^2 \epsilon_{\mu}{}^{\sigma\tau\nu} A_\sigma A_\tau - \cdots . \tag{3.140}$$

Second, we define a deformed dualized field strength:

$$K_\mu \equiv Y^{-1}{}^{\nu}_{\ \mu} {*F}_\nu. \tag{3.141}$$

Inverting this gives

$$*F_\mu = Y^{\nu}_{\ \mu} K_\nu. \tag{3.142}$$

Note that by combining this relation (3.142) and the divergence identity (3.4) on $*F_\mu$, we obtain

$$\begin{aligned} 0 &= \partial^\mu \left(Y^{\nu}_{\ \mu} K_\nu \right) \\ &= \partial^\mu K_\mu + c_1 \epsilon^{\mu\sigma\nu} \partial_\mu \left(A_\sigma K_\nu \right) \\ &= \partial^\mu K_\mu + c_1 {*F^\nu} K_\nu + c_1 \epsilon^{\mu\sigma\nu} \partial_\mu K_\nu A_\sigma \\ &= \partial^\mu K_\mu + c_1 K^\nu K_\nu + c_1 \epsilon^{\mu\sigma\nu} \partial_\mu K_\nu A_\sigma . \end{aligned} \tag{3.143}$$

Hence, K_μ satisfies the nonlinear divergence identity

$$\eta^{\mu\nu} \left(\partial_\mu K_\nu + c_1 K_\mu K_\nu \right) = c_1 \epsilon^{\mu\nu\sigma} \partial_\mu K_\nu A_\sigma = 2 c_1 E^\sigma A_\sigma . \tag{3.144}$$

Expressions (3.132)–(3.133) fit the following pattern:

$$\delta_\zeta A_\mu = \partial_\mu \zeta + c_1 K_\mu \zeta, \tag{3.145}$$

$$E_\mu = \frac{1}{2} \epsilon_{\mu}{}^{\sigma\tau} \partial_\sigma K_\tau . \tag{3.146}$$

To check this pattern, we expand in powers of A, giving

$$\begin{aligned} \delta_\zeta A_\mu &= \partial_\mu \zeta + c_1 K_\mu \zeta \\ &= \underbrace{\partial_\mu \zeta}_{\overset{(0)}{\delta_\zeta A_\mu}} + \underbrace{c_1 {*F_\mu} \zeta}_{\overset{(1)}{\delta_\zeta A_\mu}} \underbrace{- c_1{}^2 \epsilon_{\mu}{}^{\nu\tau} A_\nu {*F_\tau} \zeta}_{\overset{(2)}{\delta_\zeta A_\mu}} + \cdots \end{aligned} \tag{3.147}$$

and

$$\begin{aligned} E_\mu &= \frac{1}{2} \epsilon_{\mu}{}^{\sigma\tau} \partial_\sigma K_\tau \\ &= \underbrace{\frac{1}{2} \epsilon_{\mu}{}^{\sigma\tau} \partial_\sigma {*F_\tau}}_{\overset{(1)}{E_\mu}} \underbrace{- \frac{1}{2} c_1 \epsilon_{\mu}{}^{\sigma\alpha\nu} \partial_\sigma \left(A_\alpha {*F_\nu} \right)}_{\overset{(2)}{E_\mu}} + \underbrace{\frac{1}{2} c_1{}^2 \epsilon_{\mu}{}^{\sigma\alpha\beta\nu} \partial_\sigma \left(A_\alpha A_\beta {*F_\nu} \right)}_{\overset{(3)}{E_\mu}} - \cdots . \end{aligned} \tag{3.148}$$

We will now check that the nonlinear gauge symmetry (3.145) and the nonlinear field equation (3.146) satisfy the full deformation equation (3.17). Using (3.141), we have

$$\begin{aligned} Y^{\nu}_{\ \mu}\delta_\zeta K_{\nu} &= Y^{\nu}_{\ \mu}\delta_\zeta\big(Y^{-1\tau}{}_{\nu}{*F}_{\tau}\big) \\ &= Y^{\nu}_{\ \mu}\Big(-Y^{-1\alpha}{}_{\nu}\big(\delta_\zeta Y^{\beta}_{\ \alpha}\big)Y^{-1\tau}{}_{\beta}{*F}_{\tau} + Y^{-1\tau}{}_{\nu}\epsilon_{\tau}{}^{\alpha\beta}\partial_{\alpha}\delta_\zeta A_\beta\Big) \\ &= -c_1\epsilon_{\mu}{}^{\alpha\beta}\delta_\zeta A_\alpha K_\beta + \epsilon_{\mu}{}^{\alpha\beta}\partial_\alpha\big(\partial_\beta\zeta + c_1 K_\beta\zeta\big) \\ &= c_1\zeta\epsilon_{\mu}{}^{\alpha\beta}\partial_\alpha K_\beta, \end{aligned} \tag{3.149}$$

which through (3.139) implies

$$\delta_\zeta K_\nu = 2c_1 Y^{-1\mu}{}_{\nu}E_\mu\zeta. \tag{3.150}$$

This yields

$$\delta_\zeta E_\mu = c_1\epsilon_{\mu}{}^{\sigma\tau}\partial_\sigma\big(Y^{-1\nu}{}_{\tau}E_\nu\zeta\big), \tag{3.151}$$

therefore,

$$\big(\delta_\zeta E_\mu\big)\big|_{E=0} = 0 \tag{3.152}$$

where c_1 is still an arbitrary constant. Thus, the full on-shell gauge invariance condition is satisfied.

Chapter 4

Main results

First, the nonlinear deformation of Maxwell gauge theory with a non-trivial nonlinear form for the gauge symmetry will be summarized. Its dual form is obtained and shown to be equivalent to a linear scalar field theory under a field transformation.

Next, a non-abelian extension of the nonlinear gauge theory is described, which involves an interesting internal algebraic structure. This theory is a different deformation of abelian Yang-Mills theory than the familiar non-abelian Yang-Mills theory based on semi-simple Lie algebras.

4.1 Single-field theory

The deformed gauge symmetry and deformed field equation are given by

$$\delta_\zeta A_\mu = \partial_\mu \zeta + cK_\mu \zeta \tag{4.1}$$

$$E_\mu = \frac{1}{2}\epsilon_\mu{}^{\sigma\nu}\partial_\sigma K_\nu \tag{4.2}$$

where

$$K_\mu = Y^{-1}{}^\nu_\mu {*F}_\nu \tag{4.3}$$

is the deformed field strength defined in terms of the deformed identity tensor

$$Y^\nu_\mu = \delta^\nu_\mu + c\epsilon_\mu{}^{\sigma\nu}A_\sigma . \tag{4.4}$$

Here c denotes an arbitrary constant. When $c = 0$, this theory reduces to Maxwell gauge theory.

On-shell gauge invariance of the deformed theory is shown by

$$\delta_\zeta E_\mu = c\epsilon_\mu{}^{\nu\sigma}\partial_\nu\big(Y^{-1}{}^\tau_\sigma \zeta E_\tau\big). \tag{4.5}$$

4.2 Dual formulation

The nonlinear field strength satisfies the divergence identity

$$\partial^\nu K_\nu = 2cE_\nu A^\nu - cK_\nu K^\nu . \tag{4.6}$$

On solutions of the field equations $E_\mu = \frac{1}{2}\epsilon_\mu{}^{\sigma\nu}\partial_\sigma K_\nu = 0$, the nonlinear field strength is curl-free and thus it can be expressed as a gradient of a scalar field,

$$K_\mu = \partial_\mu \phi . \tag{4.7}$$

The divergence identity (4.6) then yields a nonlinear field equation on ϕ:

$$\Box\phi + c\partial_\nu\phi\partial^\nu\phi = 0. \tag{4.8}$$

However, the change of field variable

$$\Psi = \exp(c\phi) \tag{4.9}$$

leads to

$$\begin{aligned}\Box\Psi &= \partial^\nu\left(c\partial_\nu\phi e^{c\phi}\right)\\ &= \left(c\partial^\nu\partial_\nu\phi + c^2\partial^\nu\phi\partial_\nu\phi\right)e^{c\phi}\\ &= 0.\end{aligned} \tag{4.10}$$

Therefore, this single field dual theory is equivalent to the wave equation. Although the wave equation has a Lagrangian, there is no corresponding local Lagrangian for the nonlinear gauge theory.

4.3 Multi-field theory

The easiest way to extend the nonlinear gauge theory (4.1)–(4.4) to the case of $n > 1$ gauge fields is by introducing an internal algebraic structure that will replace scalar multiplication on A_μ and ζ:

$$A_\mu \to A^a_\mu, \tag{4.11}$$

$$\zeta \to \zeta^a, \tag{4.12}$$

$$c \to c^a{}_{bc}, \tag{4.13}$$

where latin indices $a, b, c, \ldots$ running over $1, 2, \ldots, n$ are used to label the set of gauge fields, with $n > 1$ being an arbitrary whole number.

Then the nonlinear deformation (4.4) of the identity tensor δ^ν_μ and the nonlinear deformation (4.3) of the dual field strength $*F_\mu$ are given by

$$Y^{a\nu}_{\mu b} \equiv \delta^\nu_\mu\delta^a_b + c^a{}_{bc}\epsilon_\mu{}^{\sigma\nu}A^c_\sigma, \tag{4.14}$$

$$K^a_\mu \equiv Y^{-1a\nu}_{\mu b}{*F^b_\nu}, \tag{4.15}$$

where

$$Y^{-1a\nu}_{\mu b} = \delta^\nu_\mu\delta^a_b - c^a{}_{bc}\epsilon_\mu{}^{\sigma\nu}A^c_\sigma + c^a{}_{ec}c^e{}_{bd}\epsilon_\mu{}^{\sigma\tau\nu}A^c_\sigma A^d_\tau - \cdots \tag{4.16}$$

is determined by

$$Y^{a\nu}_{\mu b}Y^{-1b\tau}_{\nu c} = Y^{-1a\nu}_{\mu b}Y^{b\tau}_{\nu c} = \delta^\tau_\mu\delta^a_c. \tag{4.17}$$

Note this allows expressing

$$*F^a_\mu = Y^{a\nu}_{\mu b}K^b_\nu. \tag{4.18}$$

Likewise, the corresponding extensions of the nonlinear gauge symmetry (4.1) and the nonlinear field equation (4.2) have the form

$$E^a_\mu = \frac{1}{2}\epsilon_\mu{}^{\sigma\tau}\partial_\sigma K^a_\tau, \tag{4.19}$$

$$\delta_\zeta A^a_\mu = \partial_\mu \zeta^a + c^a{}_{bc} K^b_\mu \zeta^c. \tag{4.20}$$

To show that this theory is gauge invariant, we apply the gauge symmetry (4.20) to the relation (4.15):

$$\begin{aligned} Y^{a\nu}_{\mu b}\delta_\zeta K^b_\nu &= Y^{a\nu}_{\mu b}\Big(-Y^{-1b\alpha}_{\nu d}\big(\delta_\zeta Y^{d\beta}_{\alpha e}\big)Y^{-1e\tau}_{\beta c}{*F}^c_\tau + Y^{-1b\tau}_{\nu c}\epsilon_\tau{}^{\alpha\beta}\partial_\alpha\delta_\zeta A^c_\beta\Big) \\ &= -c^a{}_{ef}\epsilon_\mu{}^{\alpha\beta}\delta_\zeta A^f_\alpha K^e_\beta + \epsilon_\mu{}^{\alpha\beta}\partial_\alpha\big(\partial_\beta\zeta^a + c^a{}_{bc}K^b_\beta\zeta^c\big) \\ &= -c^a{}_{ef}c^f{}_{bc}\epsilon_\mu{}^{\alpha\beta}K^b_\alpha K^e_\beta\zeta^c + c^a{}_{bc}\zeta^c\epsilon_\mu{}^{\alpha\beta}\partial_\alpha K^b_\beta \\ &= -c^a{}_{[d|e}c^e{}_{|b]c}\epsilon_\mu{}^{\alpha\beta}K^b_\alpha K^d_\beta\zeta^c + 2c^a{}_{bc}E^b_\mu\zeta^c. \end{aligned} \tag{4.21}$$

This will correspond to the form of the gauge symmetry relation (3.150) on the nonlinear field strength if the first term vanishes which requires:

$$c^a{}_{[b|e}c^e{}_{|d]c} = 0. \tag{4.22}$$

Then the gauge invariance holds on-shell

$$\big(\delta_\zeta E^a_\mu\big)\big|_{E=0} = c^d{}_{bc}\epsilon_\mu{}^{\sigma\tau}\partial_\sigma\big(Y^{-1a\nu}_{\tau d}E^b_\nu\zeta^c\big)\big|_{E=0} = 0. \tag{4.23}$$

Moreover, the divergence identity $\partial^\mu{*F}^a_\mu = 0$ yields

$$\begin{aligned} 0 &= \partial^\mu\big(Y^{a\nu}_{\mu b}K^b_\nu\big) \\ &= \partial^\mu K^a_\mu + c^a{}_{bc}\epsilon^{\mu\sigma\nu}\partial_\mu\big(A^c_\sigma K^b_\nu\big) \\ &= \partial^\mu K^a_\mu + c^a{}_{bc}{*F}^{c\nu}K^b_\nu + c^a{}_{bc}\epsilon^{\mu\sigma\nu}\partial_\mu K^b_\nu A^c_\sigma \\ &= \partial^\mu K^a_\mu + c^a{}_{bc}K^{c\nu}K^b_\nu + c^a{}_{bc}c^c{}_{de}\epsilon^{\nu\tau\sigma}A^e_\tau K^d_\sigma K^b_\nu - c^a{}_{bc}\epsilon^{\mu\nu\sigma}A^c_\sigma\partial_\mu K^b_\nu. \end{aligned} \tag{4.24}$$

which with relation (4.22) gives

$$\eta^{\mu\nu}(\partial_\mu K^a_\nu + c^a{}_{bc}K^b_\mu K^c_\nu) = 2c^a{}_{bc}\eta^{\mu\nu}E^b_\mu A^c_\nu. \tag{4.25}$$

The internal structure of this gauge theory is given by the structure constants $c^a{}_{bc}$ together with the relation (4.22), which defines a left-commutative algebra (also called left permutable) [33, 34] on $\mathbb{R}^n$. Its abstract structure consists of $x(yz) = y(xz)$ for all elements x, y, z in the algebra. Note this algebra is, in general, non-commutative and non-associative. It differs significantly from a Lie algebra.

4.4 Dual formulation of multi-field gauge theory

The nonlinear field strength can be shown to satisfy the divergence identity

$$\partial^\nu K^a_\nu = c^a{}_{bc}\big(2E^b_\nu A^{c\nu} - K^b_\nu K^{c\nu}\big). \tag{4.26}$$

Similarly to the single-field gauge theory, this field strength is curl-free on solutions of the field equations $E^a_\mu = \frac{1}{2}\epsilon_\mu{}^{\sigma\nu}\partial_\sigma K^a_\nu = 0$, and thus it can be expressed as a gradient of a set of scalar fields,

$$K^a_\mu = \partial_\mu \phi^a. \tag{4.27}$$

The divergence identity (4.26) thereby yields a nonlinear field equation on ϕ^a:

$$\partial_\nu \partial^\nu \phi^a + c^a{}_{bc}\partial_\nu \phi^b \partial^\nu \phi^c = 0. \tag{4.28}$$

Unlike in the single-field case, there is no local change of field variable that can be used to transform the nonlinear theory to a linear theory.

Therefore, the multi-field gauge theory is genuinely nonlinear.

Chapter 5

Conclusions

Previous methods for finding deformations of linear gauge field theories have always assumed some structure besides the field equations to formulate determining equations. In contrast, the general method in Chapter 3 only assumes minimal derivatives and Lorentz covariance. This yields a non-trivial generalization of Maxwell gauge theory in 2+1 dimensions. Its multi-field extension presented in Chapter 4 is very interesting and has a quite different structure compared to non-abelian Yang-Mills theory.

Given this new result, it will be interesting to apply the same method to the linear (abelian) Yang-Mills theory. This will be a lengthy calculation. Nevertheless, the multi-field extension of the deformation of Maxwell gauge theory indicates that the calculation will yield something non-trivial besides Yang-Mills theory itself.

Appendix A

Tensors and index calculations

This section contains an overview of tensors and how to manipulate them in expressions in 2+1 dimensional Minkowski spacetime, which is all that is needed to understand my work. Like most work in theoretical physics that uses tensors, I will use a component formulation based on indices.

A.1 Einstein Summation Notation

When working with tensors in component form, it quickly becomes tedious to write out sums of components. This can be avoided by employing the Einstein summation notation in which a summation is implied over an upper index and a lower index that have the same symbol. To avoid ambiguities, only one pair of up/down indices in each tensor expression can use the same symbol. The repeated index can be any symbol and is often changed when substituting terms.

A.2 Various tensors

A.2.1 Identity Tensor

$$\delta^{\nu}_{\mu} = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases}. \tag{A.1}$$

This tensor is also known as the Kronecker delta.

Its trace is the spacetime dimension:

$$\delta^{\nu}_{\nu} = 3. \tag{A.2}$$

A.2.2 Minkowski metric

The Minkowski metric is a symmetric diagonal matrix

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{A.3}$$

The inverse metric is the same matrix

$$\eta^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{A.4}$$

and satisfies

$$\eta_{\alpha\beta}\eta^{\beta\mu} = \delta^{\alpha}_{\mu}. \tag{A.5}$$

Indices on tensors are raised and lowered by contraction with the metric or the inverse metric.

A.2.3 Volume tensor

$$\epsilon_{\mu\nu\alpha} \equiv \begin{cases} +1 & \text{if } \{\mu,\nu,\alpha\} \text{ is an even permutation of } \{0,1,2\} \\ -1 & \text{if } \{\mu,\nu,\alpha\} \text{ is an odd permutation of } \{0,1,2\} \\ 0 & \text{else} \end{cases} \tag{A.6}$$

A.3 Tensor identities

Here I list all of the tensor identities that are used in my work.

$$\epsilon_{\mu}{}^{\nu\alpha}\epsilon_{\alpha}{}^{\tau\gamma} = -\left(\delta^{\tau}_{\mu}\eta^{\nu\gamma} - \delta^{\gamma}_{\mu}\eta^{\nu\tau}\right) \tag{A.7}$$

where

$$\epsilon_{\mu}{}^{\nu\tau} \equiv \epsilon_{\mu\alpha\beta}\eta^{\alpha\nu}\eta^{\beta\tau}. \tag{A.8}$$

A.4 Derivatives

Throughout my work I use spacetime Cartesian coordinates

$$x^{\mu} \equiv \begin{pmatrix} t \\ x \\ y \end{pmatrix}. \tag{A.9}$$

The partial derivative operator associated with these coordinates is denoted

$$\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}} \equiv \begin{pmatrix} \partial_t \\ \partial_x \\ \partial_y \end{pmatrix} \tag{A.10}$$

where

$$\partial_t \equiv \frac{\partial}{\partial t}, \quad \partial_x \equiv \frac{\partial}{\partial x}, \quad \partial_y \equiv \frac{\partial}{\partial y} \tag{A.11}$$

are abbreviations for the standard partial derivatives.

A.4.1 Grad, div, curl

The Minkowski gradient on scalar functions f is

$$\partial_{\mu} f = \begin{pmatrix} \partial_t f \\ \partial_x f \\ \partial_y f \end{pmatrix}. \tag{A.12}$$

The Minkowski divergence of vector functions g^μ is

$$\partial_\mu g^\mu = \partial_t g^0 + \partial_x g^x + \partial_y g^y. \tag{A.13}$$

The Minkowski curl of dual-vector functions h_τ is

$$\epsilon_\mu{}^{\nu\tau}\partial_\nu h_\tau = \begin{pmatrix} \partial_x h_y - \partial_y h_x \\ \partial_y h_t - \partial_t h_y \\ \partial_t h_x - \partial_x h_t \end{pmatrix}. \tag{A.14}$$

Notations and Identities

The following notations and identities are used throughout the paper.

A.4.2 Notations

$$\delta^\nu_\mu \equiv \eta^{\nu\tau}\eta_{\tau\mu} \tag{A.15}$$
$$\partial^\tau \equiv \eta^{\tau\nu}\partial_\nu \tag{A.16}$$
$$\partial_{\mu\nu} \equiv \partial_\mu\partial_\nu \tag{A.17}$$
$$\partial_{\mu\nu\tau} \equiv \partial_\mu\partial_\nu\partial_\tau \tag{A.18}$$
$$\epsilon_\mu{}^{\alpha\beta\gamma} \equiv \epsilon_\mu{}^{\alpha\nu}\epsilon_\nu{}^{\beta\gamma} \tag{A.19}$$
$$\epsilon_\mu{}^{\alpha\beta\gamma\sigma} \equiv \epsilon_\mu{}^{\alpha\nu}\epsilon_\nu{}^{\beta\tau}\epsilon_\tau{}^{\gamma\sigma} \tag{A.20}$$
$$A_{\mu\nu} \equiv \partial_{(\mu}A_{\nu)} \tag{A.21}$$
$$A_{\mu\nu\tau} \equiv \partial_{(\mu}A_{\nu\tau)} \tag{A.22}$$
$$*F_\mu \equiv \epsilon_\mu{}^{\nu\tau}F_{\nu\tau} = \epsilon_\mu{}^{\nu\tau}\partial_\nu A_\tau \tag{A.23}$$
$$*F_{\mu\nu} \equiv \partial_{(\mu}*F_{\nu)} \tag{A.24}$$
$$*F_{\mu\nu\tau} \equiv \partial_{(\mu}*F_{\nu\tau)} \tag{A.25}$$

Also, $\overset{(n)}{B}_m$ with $B \in (P, Q, R, S, E, G, U, V, N)$ represents the term with n powers of A_μ where m is used to distinguish terms with the same power of A_μ.

A.4.3 Identities

$$\eta^{\mu\nu}\eta_{\mu\nu} = 3 \tag{A.26}$$

$$\epsilon_{\mu}{}^{\nu\tau}\epsilon_{\tau}{}^{\alpha\beta} = 2\delta_{\mu}^{[\beta}\eta^{\alpha]\nu} \tag{A.27}$$

$$\epsilon_{\mu}{}^{[\sigma\tau]\nu} = \frac{1}{2}\epsilon_{\mu}{}^{\nu\sigma\tau} \tag{A.28}$$

$$\partial_{\mu}A_{\nu} = A_{\mu\nu} - \frac{1}{2}\epsilon_{\mu\nu}{}^{\tau}{*F}_{\tau} \tag{A.29}$$

$$\partial_{\mu}{*F}_{\nu} = {*F}_{\mu\nu} - \epsilon_{\mu\nu}{}^{\tau}\overset{(1)}{E}_{\tau} \tag{A.30}$$

$$\partial^{\nu}{*F}_{\nu} = \eta^{\mu\nu}{*F}_{\mu\nu} = 0 \tag{A.31}$$

$$\eta^{\mu\nu}{*F}_{\mu\nu\tau} = \frac{1}{3}\partial^{\nu}\partial_{\nu}{*F}_{\tau} = \frac{2}{3}\epsilon_{\tau}{}^{\alpha\beta}\partial_{\alpha}\overset{(1)}{E}_{\beta} \tag{A.32}$$

$$\partial_{\mu}A_{\nu\tau} = A_{\mu\nu\tau} - \frac{1}{3}\big(\epsilon_{\mu(\nu}{}^{\alpha}{*F}_{\tau)\alpha} + \eta_{\nu\tau}\overset{(1)}{E}_{\mu} - \eta_{\mu(\nu}\overset{(1)}{E}_{\tau)}\big) \tag{A.33}$$

$$\partial_{\mu\nu}A_{\tau} = A_{\mu\nu\tau} - \frac{2}{3}\big(- \epsilon_{\tau(\mu}{}^{\alpha}{*F}_{\nu)\alpha} - \eta_{\mu\nu}\overset{(1)}{E}_{\tau} + \eta_{\tau(\mu}\overset{(1)}{E}_{\nu)}\big) \tag{A.34}$$

$$\partial_{\mu}{*F}_{\nu\tau} = {*F}_{\mu\nu\tau} - \frac{2}{3}\epsilon_{\mu(\nu}{}^{\alpha}\partial_{\tau)}\overset{(1)}{E}_{\alpha} \tag{A.35}$$

$$\partial_{\mu\nu}{*F}_{\tau} = {*F}_{\mu\nu\tau} + \frac{4}{3}\epsilon_{\tau(\mu}{}^{\alpha}\partial_{\nu)}\overset{(1)}{E}_{\alpha} \tag{A.36}$$

$$\epsilon_{\mu}{}^{\tau\alpha}A_{\alpha\nu}A_{\tau}{*F}^{\nu} = \epsilon_{\mu}{}^{\tau\alpha}A_{\alpha\nu}A^{\nu}{*F}_{\tau} + \big(\epsilon_{\mu}{}^{\tau\nu}A^{\alpha}_{\alpha} + \epsilon^{\tau\alpha\nu}A_{\mu\alpha}\big)A_{\tau}{*F}_{\nu} \tag{A.37}$$

$$\epsilon^{\nu\alpha\tau}A_{\nu}{*F}_{\mu\tau}\partial_{\alpha}\zeta = \epsilon_{\mu}{}^{\alpha\tau}\big(A^{\nu}{*F}_{\nu\tau}\partial_{\alpha}\zeta - A_{\alpha}{*F}_{\nu\tau}\partial^{\nu}\zeta\big) \tag{A.38}$$

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