Travelling Wave Solutions on a Non-zero Background for the Generalized Korteweg-de Vries Equation

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Abstract

In presenting this thesis, we try to find all non-periodic travelling waves of the generalized Korteweg-de Vries (gKdV) equation

$$u_t + \alpha u^p u_x + \beta u_{xxx} = 0$$

using an energy analysis method. Since the power p in the gKdV equation is arbitrary, we consider positive integer values for p. We first check the method for two cases where $p = 1$ and $p = 2$ which are known as the KdV and the mKdV equations, respectively. Then, we look at the general case where $p \geq 3$ is arbitrary.

By applying the energy analysis method on the KdV and the mKdV equations, we will find an explicit form of solitary waves on a non-zero background. Afterwards, we reparametrize the derived solutions in terms of speed and the background size to interpret these solutions physically. We also look at some limiting cases in which heavy-tailed and kink waves arise in the mKdV equation.

At last, we split up the gKdV equation into two cases of odd and even p powers and apply a similar derivation. In each case, the implicit solutions are introduced and characterized by their features.

Acknowledgements

I would like to thank my supervisor, Prof. Stephen Anco, for his patience and support; the opportunities he has provided me with, and the skills and knowledge I have learned from him.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

Introduction

Nonlinear wave equations describe a vast number of physical phenomena including water waves [1], propagation of magnetic flux [2], optical fiber signals [2], dynamics of quasi one-dimensional ferromagnets [2], and many other applications in physics and biophysics.

One of the well-known nonlinear wave equations, which has numerous applications in physics and has been studied enthusiastically, is the Korteweg-de Vries (KdV) equation

$$u_t + \alpha u u_x + \beta u_{xxx} = 0, \tag{1.1}$$

where t is time, x is position, u is wave amplitude, and α, β are constants. This equation was first derived to study shallow water waves in 1834 [3]. Today it is known to describe ion-acoustic waves in a plasma [1], pressure waves in a liquid gas [1], and blood pressure waves [4].

Another interesting nonlinear wave equation is the modified Korteweg- de Vries (mKdV) equation

$$u_t + \alpha u^2 u_x + \beta u_{xxx} = 0. \tag{1.2}$$

This equation was originally introduced in an effort to understand conservation laws of the KdV equation [5], and it describes many physical systems such as anharmonic lattices [6], electromagnetic waves in size-quantized films [7], elastic media [7], and plasma physics [8]. Solutions $u(t, x)$ behave differently depending on the sign of the nonlinearity term $u^2 u_x$: $\alpha > 0$ is known as the focussing case, and $\alpha < 0$ is the defocussing case.

A generalization of both the KdV and mKdV equations is given by

$$u_t + \alpha u^p u_x + \beta u_{xxx} = 0 \tag{1.3}$$

which is called the gKdV equation, where p is an arbitrary power. Physically, this equation describes dispersive wave phenomena with higher order nonlinearity and exhibits interesting different behaviour for solutions $u(t, x)$, depending on p.

It is difficult to analyze nonlinear wave equations. One important type of solution is a travelling wave. These solutions consist of a wave of translation which keeps its shape and moves at a constant speed,

$$u = U(\xi), \quad \xi = x - ct, \quad c = \text{const.} \tag{1.4}$$

A stable localized travelling wave is called a *solitary wave* [1]. For a travelling wave to be localized, it should decay to zero exponentially in $|x|$. This type of solution arises from a balance between nonlinear and dispersive terms in a nonlinear wave equation. In contrast, a travelling wave that decays to zero as a power of x is called a *heavy-tailed wave* since it is not exponentially localized.

For special nonlinearities, solitary waves can preserves their speed and shape after interaction with other waves. Solitary waves with these features are called *solitons* [1].

The main motivation in this thesis is to study the other types of travelling wave solutions of the gKdV equation (1.3), specifically solitary waves on a non-zero background and kink waves. A *solitary wave on a background* is a travelling wave that decays to a non-zero value exponentially in x. A *kink wave* is a travelling wave that exponentially approaches two different values for large positive and negative x.

First, all non-periodic travelling waves for the KdV and the mKdV equations are derived in a systematic way by applying an energy analysis method which is well known in the physics literature [9]. Next, a similar derivation is applied to the gKdV equation. Because the power p is arbitrary, the derivation becomes more complicated and so only the cases of even and odd (integer) powers will be considered.

The energy analysis method involves the following steps:

(1) Obtain first integrals $\Psi(U, U', U'', U''') = C = \text{const}$ of the differential equation for $U(\xi)$. This can be done as shown in Ref. [10, 11] by using known translation-invariant conservation laws of the nonlinear wave equation, such as mass, momentum, and energy.

(2) Through the first integrals, reduce the differential equation for $U(\xi)$ into a first-order equation of the form

$$\tfrac{1}{2}U'^2 + V(U) = E \tag{1.5}$$

where $V(U)$ is a nonlinear function of U, and E is a constant of integration. This equation is like a nonlinear oscillator equation in mechanics [9], with $\frac{1}{2}U'^2$ being the kinetic energy, $V(U)$ being the potential energy, and E being the total (conserved) energy. Every solution $U(\xi)$ will correspond to an oscillator motion in which U goes between the turning points given by roots of $V(U) = E$ where U' is zero.

(3) Determine the shape of the potential well(s) defined by $V(U)$: If $V(U)$ has an absolute minimum, find the largest absolute maximums of $V(U)$ to the left and right of the minimum point, and take $E_{\text{max}} \geq V(U)$ such that E_{max} is the smaller of two maximums. This defines the potential well, where for any energy E between V_{min} and E_{max}, all solutions $U(\xi)$ are bounded. If $V(U)$ has no absolute minimum but has at least one local minimum, start with the smallest local minimum of $V(U)$ and apply the preceding construction to define a local potential well. Then remove this potential well and repeat the process until all local potential wells have been found. Finally, if $V(U)$ has no local minimums, then no potential well exists.

(4) For a given potential well, find the critical points of $V(U)$ in the interior and on the rim(s). Solitary wave solutions will arise when one turning point of $E_{\text{max}} - V(U)$ coincides with a local maximum of $V(U)$; kink wave solutions will arise when two turning points of $E_{\text{max}} - V(U)$ coincide with a pair of local maxima of $V(U)$ with the same height. For both types of solution, the turning points give the minimum and maximum wave amplitudes, U_{min} and U_{max}, of $U(\xi)$.

When U_{max} is a peak of the wave $U(\xi)$, then the background on which the wave propagates is

$b = U_{\text{min}}$; when U_{min} is a trough (inverted peak) of the wave $U(\xi)$, then the background on which the wave propagates is $b = U_{\text{max}}$. A standard terminology in physics and applied math literature for these two types of solitary waves are, respectively, *bright* and *dark*. The height/depth of the wave is $h = U_{\text{max}} - U_{\text{min}}$.

(5) Each solution $U(\xi)$ is implicitly given by the quadrature

$$\int_{U_0}^{U} \frac{dU}{\sqrt{E - V(U)}} = \pm\sqrt{2}(\xi - \xi_0) \tag{1.6}$$

where $U_0 = U(\xi_0)$ is an arbitrary constant between U_{min} and U_{max}, and where ξ_0 can be chosen to be 0 by translation invariance. A convenient choice of U_0 is, for solitary waves, the turning point at which $V(U_0)$ is a local maximum, and for kink waves, the point at which $V'(U_0)$ is a local maximum.

A nice advantage of the energy analysis method is that all possible types of travelling wave solutions can be found just from knowing the shape of $V(U)$, without having to evaluate the integral (1.6) to find $U(\xi)$.

This method is applied the KdV equation (1.1) in chapter 2 and to the mKdV equation (1.2) in chapter 3. Finally, the method is used for the gKdV equation with even and odd integer powers p in chapter 4. A few conclusions are stated in chapter 5.

Several main results are obtained in the study.

- For the KdV equation, the explict form of solitary waves on a non-zero background is obtained in terms of the wave speed and the background size; these waves are shown to have the same width-height scaling relation as the KdV soliton.

- For the focussing mKdV equation, the explict form of solitary waves on a non-zero background is obtained in terms of the wave speed and the background size; in a special limit, heavy-tailed waves on a non-zero background are found.

- For the defocussing mKdV equation, the explict form of solitary waves on a non-zero background and kink waves are obtained in terms of the wave speed and the background size; these waves propagate in the opposite direction compared to the focussing case.
- For the gKdV equation, generalizations of the KdV solitary waves and the mKdV solitary waves, kinks, and heavy-tailed waves are obtained in the case of odd powers p and even powers p, respectively.

The main physical features of all of these travelling waves are discussed.

Chapter 2

Non-periodic travelling wave solutions of KdV equation

In the KdV equation (1.1), by scaling t and x we can put $\beta = 1$; and by scaling u, we can further put $\alpha = 1$. This yields the scaled form of the KdV equation

$$u_t + uu_x + u_{xxx} = 0. \tag{2.1}$$

It will be useful to recall the well-known conservation laws $D_tT + D_x\Phi = 0$ for mass, momentum, energy, and Galilean momentum:

$$T = u, \quad \Phi = \tfrac{1}{2}u^2 + u_{xx} \tag{2.2}$$

$$T = \tfrac{1}{2}u^2, \quad \Phi = \tfrac{1}{3}u^3 + uu_{xx} - \tfrac{1}{2}u_x^2 \tag{2.3}$$

$$T = -\tfrac{1}{2}u_x^2 + \tfrac{1}{6}u^3, \quad \Phi = \tfrac{1}{2}(u_{xx} + \tfrac{1}{2}u^2)^2 + u_tu_x \tag{2.4}$$

$$T = -\tfrac{1}{2}tu^2 + xu, \quad \Phi = t(-uu_{xx} + \tfrac{1}{2}u_x^2 - \tfrac{1}{3}u^3) + x(u_{xx} + \tfrac{1}{2}u^2) - u_x \tag{2.5}$$

Here T denotes a conserved density, Φ denotes a conserved flux, and D is directional derivative.

Consider travelling wave solutions (1.4) of the scaled KdV equation (2.1). These solutions satisfy the third-order ODE

$$U''' + (U - c)U' = 0 \tag{2.6}$$

It has first integrals

$$U'' + \tfrac{1}{2}U^2 - cU = J = \text{const} \tag{2.7}$$

$$UU'' - \tfrac{1}{2}U'^2 + \tfrac{1}{3}U^3 - \tfrac{1}{2}cU^2 = E = \text{const} \tag{2.8}$$

These integrals can be derived directly from the conservation laws for mass (2.2), momentum (2.3), and energy (2.4), by the method described in Ref. [10, 11]. Specifically, each of these conservation laws applied to travelling waves (1.4) yields a first integral $\Psi = T - c\Phi = C = \text{const}$ for the travelling wave ODE (2.6). The first integral has the physical meaning that it describes the conserved flux in a reference frame moving with the travelling wave. This yields $J, E, cJ + \frac{1}{2}E^2$, of which only the first two are functionally independent.

When the two first integrals (2.7) and (2.8) are combined, they yield a reduction of the travelling wave ODE to a first-order separable differential equation

$$\tfrac{1}{2}U'^2 + \tfrac{1}{6}U^3 - \tfrac{1}{2}cU^2 - JU - E = 0 \tag{2.9}$$

This ODE (2.9) has the form of a nonlinear oscillator equation (1.5) where

$$V(U) = \tfrac{1}{6}U^3 - \tfrac{1}{2}cU^2 - JU \tag{2.10}$$

is the potential and E is the oscillator energy.

Because $V(U)$ is cubic, a potential well exists whenever the critical points of $V(U)$ consist of a local minimum and a local maximum. The potential well will have a rim on one side which is given by the local maximum point. See Fig. 2.1

The critical points of $V(U)$ are determined by $0 = V'(U) = \frac{1}{2}U^2 - cU - J$, which yields

$$U = c \pm \sqrt{c^2 + 2J}, \quad J \geq -\tfrac{1}{2}c^2 \tag{2.11}$$

When $J = -\frac{1}{2}c^2$, these critical points coalesce into a single point $U = c$ which is an inflection point, since $V''(c) = 0$.

Hence for $J \leq -\frac{1}{2}c^2$, there is no potential well and so all travelling wave solutions are unbounded.

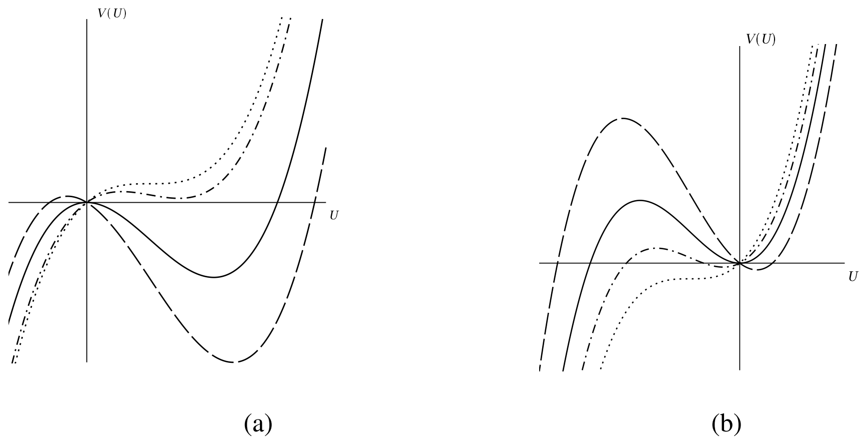


Figure 2.1: KdV potential for (a) $c > 0$ and (b) $c < 0$

Hereafter we consider $J > -\frac{1}{2}c^2$, which is the condition for existence of a potential well. Solitary waves on zero/non-zero backgrounds will arise for $E = V_{\text{max}}$. No kink waves exist because $V(U)$ never has a pair of local maximums.

2.1 KdV solitary waves on a background

For $J > -\frac{1}{2}c^2$, the local maximum and local minimum of $V(U)$ are given by

$$V_{\text{max}} = \tfrac{1}{3}(\sqrt{c^2 + 2J}^3 - c(c^2 + 3J)), \quad U = c - \sqrt{c^2 + 2J} \tag{2.12}$$

and

$$V_{\text{min}} = -\tfrac{1}{3}(\sqrt{c^2 + 2J}^3 + c(c^2 + 3J)), \quad U = c + \sqrt{c^2 + 2J} \tag{2.13}$$

Note that the minimum point is located to the right of the maximum point. Thus, the potential well is defined by

$$V(U) = \tfrac{1}{6}U^3 - \tfrac{1}{2}cU^2 - JU \leq E_{\text{max}} \quad E_{\text{max}} = V_{\text{max}} \tag{2.14}$$

Its domain is

$$U_{-} \leq U < \infty \tag{2.15}$$

where

$$U_- = c - \sqrt{c^2 + 2J}, \tag{2.16}$$

is the maximum point.

Solitary wave solutions $U(\xi)$ are obtained by taking $E = V_{\max}$ in the nonlinear oscillator equation (1.5). Then the energy equation becomes

$$0 = V_{\max} - V(U) = -\tfrac{1}{6}U^3 + \tfrac{1}{2}cU^2 + JU + \tfrac{1}{3}(\sqrt{c^2+2J}^3 - c(c^2+3J)) = \tfrac{1}{6}(U-U_-)^2(U_+ - U) \tag{2.17}$$

where

$$U_+ = c + 2\sqrt{c^2 + 2J} \tag{2.18}$$

is a turning point, and U_- is an asymptotic turning point. Hence, the quadrature (1.6) for the solitary wave solution $U(\xi)$ is given by

$$\int_U^{U_+} \frac{dU}{(U-U_-)\sqrt{U_+ - U}} = \tfrac{1}{\sqrt{3}}|\xi|, \quad U_- \leq U \leq U_+ \tag{2.19}$$

It is straightforward to evaluate this integral explicitly, which yields

$$U(\xi) = (U_+ - U_-)\,\mathrm{sech}^2\big(\tfrac{1}{\sqrt{12}}\sqrt{U_+ - U_-}\,\xi\big) + U_- = 3\sqrt{c^2+2J}\,\mathrm{sech}^2\big(\tfrac{1}{2}\sqrt[4]{c^2+2J}\,\xi\big) + c - \sqrt{c^2+2J} \tag{2.20}$$

Here $J > -\frac{1}{2}c^2$, while there is no restriction on the sign of c.

This family of solutions, which is parameterized by (J, c), has the following features determined by the two turning points (2.18). The wave peak is U_+ and the background (asymptote) is $b = U_-$, and so the wave height is $h = U_+ - U_- = 3\sqrt{c^2+2J}$. The width of the wave is proportional to $w = \sqrt{12/(U_+ - U_-)} = 2/\sqrt[4]{c^2+2J}$.

The well-known KdV soliton is given by the solution that has a zero background: $b = 0$. This condition is $c = \sqrt{c^2 + 2J}$, which holds when $J = 0$ and $c > 0$, yielding

$$U(\xi) = 3c\,\text{sech}^2\big(\tfrac{1}{2}\sqrt{c}\,\xi\big) \tag{2.21}$$

The solutions (2.20) with a non-zero background are much less widely known and their properties have not been studied to-date.

To examine the features of the non-zero background solutions, it is useful to employ a physical parameterization given by replacing J in terms of b through inverting the relation $b = U_- = c - \sqrt{c^2 + 2J}$:

$$J = \tfrac{1}{2}b^2 - cb > -\tfrac{1}{2}c^2 \tag{2.22}$$

where $\sqrt{c^2 + 2J} = c - b > 0$. This leads to the expression

$$U(\xi) = 3(c - b)\text{sech}^2\big(\tfrac{1}{2}\sqrt{c - b}\,\xi\big) + b, \quad \text{sgn}(c - b) = 1 \tag{2.23}$$

for the KdV solitary wave with a background b and a speed c. Here the background b can be positive, negative, or zero, with the speed c being restricted by $c > b$. Alternatively, the speed can be taken to be positive, negative, or zero, with the background being restricted to be $b < c$.

It is most interesting that, unlike the soliton solution, the solutions (2.23) with a negative background $b < 0$ can propagate in the opposite direction, namely $0 > c > b$. In particular, in this case we can write the solution in the form $U = 3(|b| - |c|)\text{sech}^2\big(\tfrac{1}{2}\sqrt{|b| - |c|}\,\xi\big) - |b|$, with $|c| < |b|$. The wave peak is $2|b| - 3|c|$ which can have any sign, while the wave height is $h = 3(|b| - |c|) > 0$ and the width of the wave is proportional to $w = 2/\sqrt{|b| - |c|}$. See Fig. 2.2. Solutions with $c < 0$ do not exhaust all negative background solutions. There are solutions with the form $U = 3(|b| + c)\text{sech}^2\big(\tfrac{1}{2}\sqrt{|b| + c}\,\xi\big) - |b|$ which have positive speed, $c > 0$. The wave peak is $2|b| + 3c > 0$, so the wave height is $h = 3(|b| + c) > 0$ and the width of the wave is proportional to $w = 2/\sqrt{|b| + c}$. See Fig. 2.2.

(a) (b)

Figure 2.2: KdV solitary waves on a negative background for (a) $c > 0$ and (b) $c < 0$

Physically, we can interpret both types of solutions as being either a *bright solitary wave on a negative background* or a *bright solitary hole in a negative background*. As the size $|b|$ of the background increases, the height decreases while the width increases.

An interesting special case is solutions with zero speed, $c = 0$. These solutions

$$U(\xi) = 3|b|\text{sech}^2\big(\tfrac{1}{2}\sqrt{|b|}\,\xi\big) - |b| \tag{2.24}$$

represent *static humps* on a negative background $b < 0$.

For solutions (2.23) with positive background, $b > 0$, the wave speed is positive $c > 0$. The wave peak is $3c - 2b > 0$, and the wave height is $h = 3(c - b) > 0$ while the width of the wave is proportional to $w = 2/\sqrt{c - b}$. See Fig. 2.3.

Physically, this type of solution describes either a *bright solitary wave on a positive background*, or a *bright solitary hole in a positive background*. As b increases, the height decreases while the width increases.

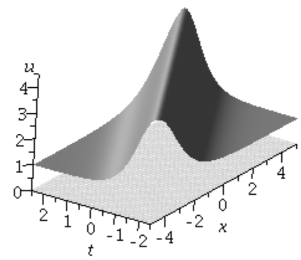


Figure 2.3: KdV solitary waves on a positive background for $c > 0$

In all cases, the height and the width satisfy the scaling relation

$$hw^2 = 12 \tag{2.25}$$

which is independent of both the speed and the background. Note this is the same relation that holds for the standard soliton. Moreover, the profile of the solution expressed in terms of the height and the background is simply

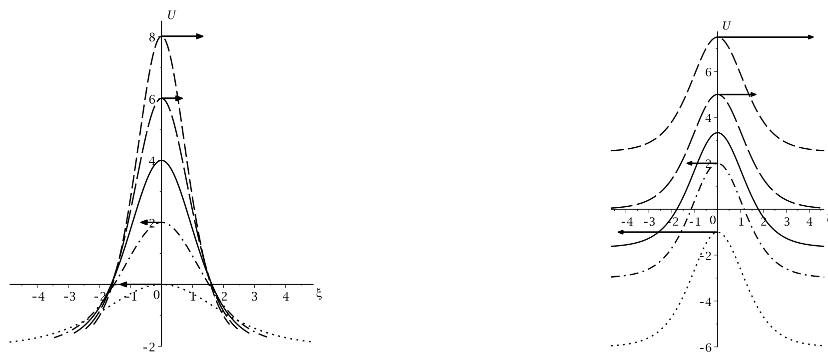
$$U(\xi) = h\,\mathrm{sech}^2\big(\tfrac{1}{6}\sqrt{3h}\,\xi\big) + b \tag{2.26}$$

where

$$c = \tfrac{1}{3}h + b \tag{2.27}$$

is the speed. This shows that the solution consists of linearly superimposing the standard soliton profile on an arbitrary background, b, and adjusting the speed by adding b to it. See Fig 2.4.

Finally, it is interesting to consider the conservation laws of mass, momentum, energy, and Galilean momentum for the solutions (2.23). For a zero background, $b = 0$, since the familiar KdV soliton solution (2.21) vanishes rapidly as $|x| \to \infty$, each of these four conservation laws lead to a corresponding conserved integral $\int_{-\infty}^{\infty} T\,dx = -\Phi\big|_{-\infty}^{\infty}$. When the background is non-zero,



(a) (b)

Figure 2.4: (a) KdV solitary waves on a same negative background with different heights, (b) KdV solitary waves with the same height on different backgrounds

$b \neq 0$, the conserved integrals are infinite. However, a finite conserved integral can be obtained by subtraction of the constant background term in the density T. This leads to the following finite expressions for the mass, momentum, energy, and Galilean momentum of the solutions (2.20):

$$M = 12\sqrt{c-b} \tag{2.28}$$

$$P = 12c\sqrt{c-b} \tag{2.29}$$

$$E = \tfrac{6}{5}\sqrt{c-b}((c-b)^2 + 5c^2) \tag{2.30}$$

$$G = 0 \tag{2.31}$$

For $b = 0$, these conserved integrals reduce to the well-known mass, momentum, energy, and Galilean momentum of the KdV soliton, which obey the free-particle relations $P = cM$ and $E \propto P^2/M$. In the case $b \neq 0$, only the free-particle relation $P = cM$ continues to hold. Interestingly, E can be expressed as a linear combination of P^2/M and M^3.

It is also worth remarking that the first integrals (2.7) and (2.8) evaluated for the solutions (2.20) are given by $J = \frac{1}{2}b(b-2c)$ and $E = \frac{1}{6}b^2(2b-3c)$. These expressions can be straightforwardly inverted to give c and b in terms of J and E.

Chapter 3

Non-periodic travelling wave solutions of mKdV equation

By scaling t, x, and u in the mKdV equation (1.2), we can put $\beta = 1$ and $\alpha = \pm 1$. Consequently, the mKdV equation has two different scaled forms:

$$u_t + u^2 u_x + u_{xxx} = 0 \tag{3.1}$$

called the focussing case; and

$$u_t - u^2 u_x + u_{xxx} = 0 \tag{3.2}$$

called the defocussing case. The types and features of travelling wave solutions will turn out to be very different in these two cases.

It will be useful to recall the well-known conservation laws $D_t T + D_x \Phi = 0$ for mass, momentum, energy, and Galilean energy:

$$T = u, \quad \Phi = \pm\tfrac{1}{3}u^3 + u_{xx} \tag{3.3}$$

$$T = \tfrac{1}{2}u^2, \quad \Phi = \pm\tfrac{1}{4}u^4 + uu_{xx} - \tfrac{1}{2}u_x^2 \tag{3.4}$$

$$T = -\tfrac{1}{2}u_x^2 \pm \tfrac{1}{12}u^4, \quad \Phi = \tfrac{1}{2}(u_{xx} \pm \tfrac{1}{3}u^3)^2 + u_t u_x \tag{3.5}$$

$$T = \tfrac{1}{4}t(6u_x^2 \mp u^4) + \tfrac{1}{2}xu^2, \quad \Phi = t(\mp u^3 u_{xx} - \tfrac{3}{2}u_{xx}^2 - 3u_t u_x - \tfrac{1}{6}u^6) + x(uu_{xx} - \tfrac{1}{2}u_x^2 \pm \tfrac{1}{4}u^4) - uu_x \tag{3.6}$$

Here T denotes a conserved density, Φ denotes a conserved flux, and D is directional derivative.

Consider travelling wave solutions (1.4) of the scaled mKdV equation (3.1)–(3.2). These solutions satisfy the third-order ODE

$$U''' + (\pm U^2 - c)U' = 0 \tag{3.7}$$

First integrals of this ODE are given by

$$U'' \pm \tfrac{1}{3}U^3 - cU = J = \text{const} \tag{3.8}$$

$$UU'' - \tfrac{1}{2}U'^2 \pm \tfrac{1}{4}U^4 - \tfrac{1}{2}cU^2 = E = \text{const} \tag{3.9}$$

They can be derived directly from the conservation laws for mass (3.3), momentum (3.4), and energy (3.5), by the method described in Ref. [10, 11]. This yields $J, E, cJ + \frac{1}{2}E^2$, of which only the first two are functionally independent.

Hence the travelling wave ODE reduces to a first-order separable differential equation

$$\tfrac{1}{2}U'^2 \pm \tfrac{1}{12}U^4 - \tfrac{1}{2}cU^2 - JU - E = 0 \tag{3.10}$$

which can be expressed as a nonlinear oscillator equation (1.5) where the potential is

$$V(U) = \pm\tfrac{1}{12}U^4 - \tfrac{1}{2}cU^2 - JU \tag{3.11}$$

and where E is the oscillator total energy. A useful observation is that this potential (3.11) is invariant under the reflection

$$(U, J, c) \to (-U, -J, c) \tag{3.12}$$

In the focussing case, since $V(U)$ is an upward quartic, it defines a potential well which has no rims, with $E_{\text{max}} = \infty$. Depending on the type of critical points of $V(U)$, there are three different possibilities for the shape of this potential well: it can have a single local minimum point and no other critical points, or a local minimum and an inflection point, or two local minimum points and a local maximum point. See Fig. 3.1

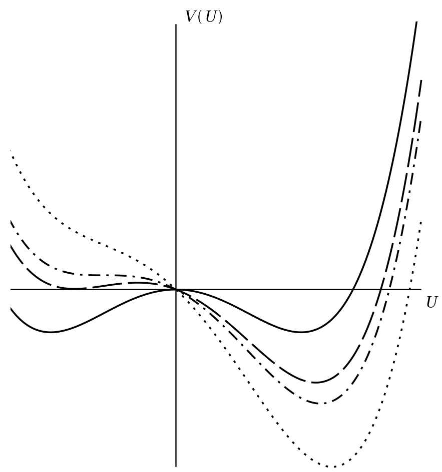


Figure 3.1: Focussing-mKdV potential for $J > 0$

Solitary waves on zero/non-zero backgrounds will arise only for $E = V_{\text{max}}$ when $V(U)$ has a local maximum. Since $V(U)$ never has two local maximums, there are no kink waves.

In contrast, in the defocussing case, since $V(U)$ is a downward quartic, a potential well exists only when $V(U)$ has three critical points which consist of two local maximums and a local minimum. If the maximums have different heights then the smaller maximum will be one of the rims of the potential well. Instead if the maximums have the same height, then each maximum will be a rim of the potential well. See Fig. 3.2

Here the travelling waves arising for $E = V_{\text{max}}$ will be solitary waves on zero/non-zero backgrounds in the former situation, while in the latter situation, they will be kink waves.

Consequently, in both the focussing and defocussing cases, the solutions of interest to us occur precisely when $V(U)$ has three distinct critical points. We will also look at the limit when two of the critical points coalesce.

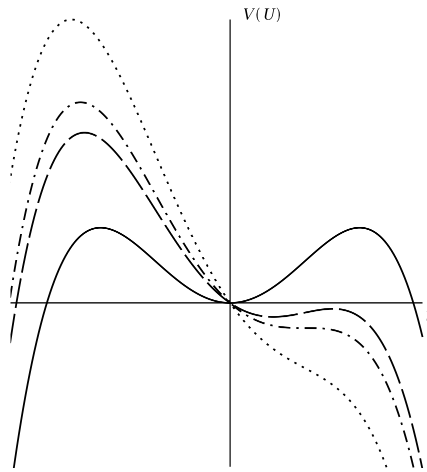


Figure 3.2: Defocussing-mKdV potential for $J > 0$

The critical points of $V(U)$ are determined by

$$0 = V'(U) = \pm\tfrac{1}{3}U^3 - cU - J \tag{3.13}$$

which is a cubic equation having either one real root when the discriminant $\pm\frac{4}{3}c^3 - 3J^2$ is negative, or three (distinct) real roots when the discriminant $\pm\frac{4}{3}c^3 - 3J^2$ is positive. Note that a positive discriminant arises iff $|J| < \frac{2}{3}\sqrt{\pm c^3}$. Hence, the focussing case $(+)$ will have three critical points iff

$$(+) \qquad |J| < \tfrac{2}{3}\sqrt{c^3}, \quad c \geq 0; \tag{3.14}$$

and the defocussing case $(-)$ will have three distinct critical points iff

$$(-) \qquad |J| < \tfrac{2}{3}\sqrt{-c^3}, \quad c \leq 0; \tag{3.15}$$

A necessary condition is therefore $\pm\text{sgn}(c) \geq 0$ in the two different cases of focussing and defocussing. This is in contrast to the KdV case, where the wave speed c could have any sign.

Note that, in both the focussing and defocussing cases, the critical points (3.13) will be invariant under the reflection (3.12).

To continue, we will consider each of the following four cases: focussing, with the interior of the potential well containing a local maximum; defocussing, with one rim of the potential well being a local maximum; defocussing, with both rims of the potential well being local maximums; focussing, with the interior of the potential well containing an inflection. This last case is interesting because the travelling waves arising for $E = V_{\text{inflect}}$ will be seen to have the form of heavy-tail waves on a non-zero background.

3.1 Focussing-mKdV solitary waves on a background

In the focussing case (3.14), the critical points (3.13) of $V(U)$ are given by

$$V_{\text{min},1} = c^2\big(\cos(\theta_1(\tfrac{3}{2}J/\sqrt{c^3})) - \tfrac{3}{2}J/\sqrt{c^3}\big)\cos(\tfrac{3}{2}\theta_1(J/\sqrt{c^3})), \quad U_1 = -2\sqrt{c}\,\cos(\theta_1(\tfrac{3}{2}J/\sqrt{c^3})) \tag{3.16a}$$

$$V_{\text{min},2} = -c^2\big(\cos(\theta_2(\tfrac{3}{2}J/\sqrt{c^3})) + \tfrac{3}{2}J/\sqrt{c^3}\big)\cos(\tfrac{3}{2}\theta_2(J/\sqrt{c^3})), \quad U_2 = 2\sqrt{c}\,\cos(\theta_2(\tfrac{3}{2}J/\sqrt{c^3})) \tag{3.16b}$$

$$V_{\text{max}} = c^2\big(\cos(\theta_3(\tfrac{3}{2}J/\sqrt{c^3})) - \tfrac{3}{2}J/\sqrt{c^3}\big)\cos(\tfrac{3}{2}\theta_3(J/\sqrt{c^3})), \quad U_3 = -2\sqrt{c}\,\cos(\theta_3(\tfrac{3}{2}J/\sqrt{c^3})) \tag{3.16c}$$

where

$$\theta_j(z) = \tfrac{1}{3}(\arccos(z) + (j-2)\pi), \quad -1 \le z \le 1, \quad j = 1,2,3 \tag{3.17}$$

These angles have the range $\frac{j-2}{3}\pi \le \theta_j \le \frac{j-1}{3}\pi$ for $-\frac{2}{3}\sqrt{c^3} \le J \le \frac{2}{3}\sqrt{c^3}$, which gives $-2 \le U_1 \le -1$, $1 \le U_2 \le 2$, and $1 \ge U_3 \ge -1$. Thus, $U_1 < U_3 < U_2$ holds for $|J| < \frac{2}{3}\sqrt{c^3}$. Note that under $J \to -J$ the critical points transform as $(U_1, U_2, U_3) \to (-U_2, -U_1, -U_3)$ and $(V_{\text{min},1}, V_{\text{min},2}) \to (V_{\text{min},2}, V_{\text{min},1})$, due to the reflection property (3.12).

The resulting potential well is defined by

$$V(U) = \tfrac{1}{12}U^4 - \tfrac{1}{2}cU^2 - JU \leq E_{\text{max}}, \quad E_{\text{max}} = \infty \tag{3.18}$$

with the domain $-\infty < U < \infty$.

To obtain the solitary wave solutions $U(\xi)$, we take $E = V_{\text{max}}$ in the nonlinear oscillator equation (1.5). The corresponding energy equation

$$0 = V(U) - E = \tfrac{1}{12}U^4 - \tfrac{1}{2}cU^2 - JU - E, \quad E = V_{\text{max}} \tag{3.19}$$

is a quartic that has two simple roots and a double root, corresponding to the factorization

$$V_{\text{max}} - V(U) = (U - U_3)^2(U - U_-)(U_+ - U) \tag{3.20}$$

where

$$U_\pm = -U_3 \pm \sqrt{6c - 2(U_3)^2} \tag{3.21}$$

Note that the shape of $V(U)$ implies $U_- < U_3 < U_+$. These roots define two turning points and an asymptotic turning point, and therefore we have two different solitary wave solutions $U(\xi)$ which correspond to the adjacent pairs of turning points (U_3, U_+) and (U_-, U_3). The qualitative features of the solutions are determined by the signs of these turning points.

From the quadrature (1.6), the first solution $U(\xi)$ is given by

$$\int_U^{U_+} \frac{dU}{(U - U_3)\sqrt{(U - U_-)(U_+ - U)}} = \tfrac{1}{\sqrt{6}}|\xi|, \quad U_3 \leq U \leq U_+ \tag{3.22}$$

while the second solution $U(\xi)$ is given by

$$\int_{U_-}^{U} \frac{dU}{(U_3 - U)\sqrt{(U - U_-)(U_+ - U)}} = \tfrac{1}{\sqrt{6}}|\xi|, \quad U_- \leq U \leq U_3 \tag{3.23}$$

It is straightforward to evaluate these integrals explicitly, which yields

$$U(\xi) = \frac{(U_+ - U_-)^2 - (4U_3)^2}{8U_3 \pm 2(U_+ - U_-)\cosh\left(\frac{1}{\sqrt{24}}\sqrt{(U_+ - U_-)^2 - (4U_3)^2}\,\xi\right)} + U_3 \tag{3.24}$$

Here the $+/-$ cases respectively correspond to the first and second solutions. Note that the solutions (3.24) can be expressed purely in terms of the parameters (J, c) through the expressions (3.21) and (3.16c) for the three turning points.

This family of solutions has the following features. The wave peak is $U_\pm$ and the background (asymptote) is $b = U_3$, and so for the first solution the wave height is $h = U_+ - U_3 > 0$ while for the second solution the wave height is $h = U_- - U_3 < 0$. The width of the wave is proportional to $w = 2/\sqrt{c - (U_3)^2}$. Consequently, the wave speed must obey the relation

$$c > b^2 \geq 0 \tag{3.25}$$

The well-known mKdV soliton is given by the solution that has a zero background: $b = 0$. This condition has the explicit form $\theta_3(\frac{3}{2}J/\sqrt{c^3})) = \frac{1}{2}\pi$, where, from expression (3.17),

$$\theta_3(\tfrac{3}{2}J/\sqrt{c^3})) = \tfrac{1}{3}(\arccos(\tfrac{3}{2}J/\sqrt{c^3}) + \pi) \tag{3.26}$$

This yields $\arccos(\frac{3}{2}J/\sqrt{c^3}) = \frac{1}{2}\pi$, and hence $J = 0$. Thus, when the background is zero, the turning points are given by $U_3 = 0$ and $U_\pm = \pm\sqrt{3c}$ from expressions (3.21). Consequently, the two solutions (3.24) become

$$U(\xi) = \pm\sqrt{6c}\,\mathrm{sech}\big(\sqrt{c}\,\xi\big), \quad c > 0 \tag{3.27}$$

which is the standard soliton, with an up/down orientation in the $+/-$ case.

The solutions (3.24) with a non-zero background are much less widely known and their properties have not been studied to-date. It will be useful to employ a physical parameterization given by replacing J in terms of b through inverting the relation $b = U_3 = -2\sqrt{c}\cos(\theta_3(\frac{3}{2}J/\sqrt{c^3}))$. This relation yields $\theta_3(\frac{3}{2}J/\sqrt{c^3}) = \arccos(-\frac{1}{2}b/\sqrt{c})$, and hence from expression (3.26), we obtain

$$J = \tfrac{1}{3}b^3 - cb, \quad |J| \leq \tfrac{2}{3}\sqrt{c^3} \tag{3.28}$$

Then we find that solutions (3.24) have the simple form

$$U(\xi) = \frac{6(c-b^2)}{2b \pm \sqrt{6c-2b^2}\,\cosh(\sqrt{c-b^2}\,\xi)} + b, \quad c > b^2 \geq 0 \tag{3.29}$$

giving the focussing mKdV solitary wave with a background b and a speed c. The background b can be positive, negative, or zero, with the speed c being larger than $c_{\text{min}} = b^2$. Alternatively, the speed c can be taken to be any positive value, with the size of the background b being restricted by $|b|_{\text{max}} = \sqrt{c}$.

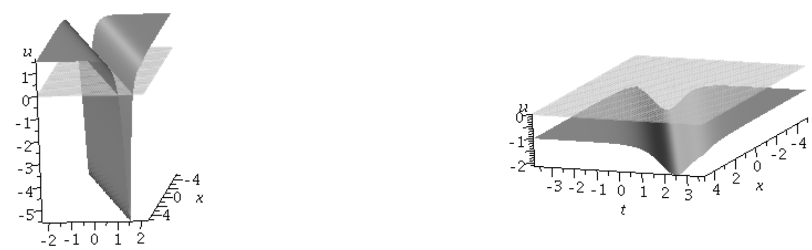
The wave peak is $-b \pm \sqrt{6c-2b^2}$, and the wave height is $h = -2b \pm \sqrt{6c-2b^2}$, while the width of the wave is proportional to $w = 2/\sqrt{c-b^2}$. In contrast to the soliton solution, the height and the width for non-zero backgrounds do not satisfy any scaling relation.

In the plus-sign case, the wave height obeys $h > 0$ for all allowed values of b. Physically, this type of solution describes either a *bright solitary wave on a positive/negative background*, or a *bright solitary hole in a positive/negative background*. See Fig. 3.3.

(a) (b)

Figure 3.3: Focussing-mKdV solitary waves for $(+)$-sign in equation (3.29)

In the minus-sign case, the wave height obeys $h < 0$ for all allowed values of b. See Fig. 3.4.



(a) (b)

Figure 3.4: Focussing-mKdV solitary wave for $(-)$-sign in equation (3.29)

Physically, this type of solution describes either a *dark solitary wave on a positive/negative background*, or a *dark solitary hole in a positive/negative background*.

For backgrounds with $\pm b > 0$, as the size of the background $|b|$ increases, the height of the wave decreases while the width increases. In the limit $|b| \to \sqrt{c}$, the wave flattens to become $U \to b$. The situation for opposite-sign backgrounds, $\pm b < 0$, is very different and will be discussed in section 3.4. See Fig. 3.5.

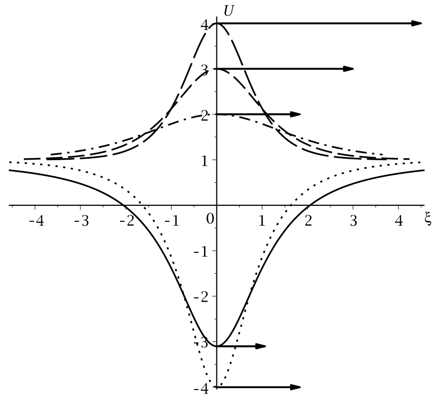


Figure 3.5: Focussing-mKdV solitary waves on positive and negative backgrounds

Finally, it is worth remarking that the first integrals (3.8) and (3.9) evaluated for the solutions (3.24) are given by $J = \frac{1}{3}b(b^2 - 3c)$ and $E = \frac{1}{4}b^2(b^2 - 2c)$. These expressions can be straightforwardly inverted to give c and b in terms of J and E.

3.2 Defocussing-mKdV solitary waves on a background

In the defocussing case (3.15), the critical points (3.13) of $V(U)$ are given by

$$V_{\text{max},1} = c^2\big(\cos(\theta_1(\tfrac{3}{2}J/\sqrt{|c|^3})) - \tfrac{3}{2}J/\sqrt{|c|^3}\big)\cos(\tfrac{3}{2}\theta_1(J/\sqrt{|c|^3})), \quad U_1 = 2\sqrt{|c|}\,\cos(\theta_1(\tfrac{3}{2}J/\sqrt{|c|^3})) \tag{3.30a}$$

$$V_{\text{max},2} = c^2\big(\cos(\theta_2(\tfrac{3}{2}J/\sqrt{|c|^3})) + \tfrac{3}{2}J/\sqrt{|c|^3}\big)\cos(\tfrac{3}{2}\theta_2(J/\sqrt{|c|^3})), \quad U_2 = -2\sqrt{|c|}\,\cos(\theta_2(\tfrac{3}{2}J/\sqrt{|c|^3})) \tag{3.30b}$$

$$V_{\text{min}} = c^2\big(\cos(\theta_3(\tfrac{3}{2}J/\sqrt{|c|^3})) - \tfrac{3}{2}J/\sqrt{|c|^3}\big)\cos(\tfrac{3}{2}\theta_3(J/\sqrt{|c|^3})), \quad U_3 = 2\sqrt{|c|}\,\cos(\theta_3(\tfrac{3}{2}J/\sqrt{|c|^3})) \tag{3.30c}$$

where

$$\theta_j(z) = \tfrac{1}{3}(\arccos(z) + (j-2)\pi), \quad -1 \le z \le 1, \quad j = 1, 2, 3 \tag{3.31}$$

These angles have the range $\frac{j-2}{3}\pi \leq \theta_j \leq \frac{j-1}{3}\pi$ for $-\frac{2}{3}\sqrt{|c|^3} \leq J \leq \frac{2}{3}\sqrt{|c|^3}$, which gives $2 \geq U_1 \geq 1$, $-1 \geq U_2 \geq -2$, and $-1 \leq U_3 \leq 1$. Thus, $U_2 < U_3 < U_1$ holds for $|J| < \frac{2}{3}\sqrt{|c|^3}$.

The resulting potential well is defined by

$$V(U) = -\tfrac{1}{12}U^4 + \tfrac{1}{2}|c|U^2 - JU \leq E_{\text{max}}, \quad E_{\text{max}} = \begin{cases} V_{\text{max},1}, & \text{if } \operatorname{sgn}(J) = 1 \\ V_{\text{max},2}, & \text{if } \operatorname{sgn}(J) = -1 \end{cases} \tag{3.32}$$

Under $J \to -J$, the critical points transform as $(U_1, U_2, U_3) \to (-U_2, -U_1, -U_3)$ and $(V_{\text{max},1}, V_{\text{max},2}) \to (V_{\text{max},2}, V_{\text{max},1})$, due to the reflection property (3.12). Hence we will only consider the case $\operatorname{sgn}(J) = 1$ hereafter.

To obtain the solitary wave solutions $U(\xi)$, we take $E = V_{\text{max},1}$ in the nonlinear oscillator equation (1.5). The corresponding energy equation is given by

$$0 = V(U) - E = -\tfrac{1}{12}U^4 + \tfrac{1}{2}|c|U^2 - JU - E, \quad E = V_{\text{max},1} \tag{3.33}$$

This quartic has two simple roots and a double root, corresponding to the factorization

$$V_{\text{max},1} - V(U) = (U - U_1)^2(U - U_-)(U - U_+) \tag{3.34}$$

where

$$U_\pm = -U_1 \pm \sqrt{6|c| - 2(U_1)^2} \tag{3.35}$$

Since $J > 0$, the shape of $V(U)$ implies that $U_- < U_+ < U_1$ where $U = U_1$ is the right side rim which represents an asymptotic turning point, while $U = U_+$ is a turning point which is the left side rim. In particular the domain of the potential well is $U_+ \leq U \leq U_1$. The signs of these points will determine the qualitative features of the solitary wave solutions.

From the quadrature (1.6), the solution $U(\xi)$ is given by

$$\int_{U_+}^{U} \frac{dU}{(U-U_1)\sqrt{(U-U_-)(U-U_+)}} = \tfrac{1}{\sqrt{6}}|\xi|, \quad U_+ \le U \le U_1 \tag{3.36}$$

This integral is straightforward to evaluate, which yields

$$U(\xi) = \frac{(U_+ - U_-)^2 - (4U_1)^2}{8U_1 + 2(U_+ - U_-)\cosh\left(\frac{1}{\sqrt{24}}\sqrt{(4U_1)^2 - (U_+ - U_-)^2}\,\xi\right)} + U_1 \tag{3.37}$$

Note that the solution (3.37) can be expressed purely in terms of the parameters (J, c) through the expressions (3.35) and (3.30a) for the three roots.

This family of solutions has the following features. The wave peak is $U_+ (< U_1)$ and the background (asymptote) is $b = U_1 > 0$, and so the wave depth is $h = U_1 - U_+ > 0$. The width of the wave is proportional to $w = 2/\sqrt{(U_1)^2 - |c|}$.

It will be useful to employ a physical parameterization given by replacing J in terms of b through inverting the relation $b = U_1 = 2\sqrt{|c|}\cos(\theta_1(\frac{3}{2}J/\sqrt{|c|^3}))$. This relation yields $\theta_1(\frac{3}{2}J/\sqrt{|c|^3}) = \arccos(\frac{1}{2}b/\sqrt{|c|})$, where, from expression (3.31),

$$\theta_1(\tfrac{3}{2}J/\sqrt{|c|^3})) = \tfrac{1}{3}(\arccos(\tfrac{3}{2}J/\sqrt{|c|^3}) - \pi) \tag{3.38}$$

Hence we obtain

$$J = |c|b - \tfrac{1}{3}b^3, \quad 0 \le J \le \tfrac{2}{3}\sqrt{|c|^3} \tag{3.39}$$

Then we find that the solution (3.37) takes a simple form

$$U(\xi) = b - \frac{6(b^2 - |c|)}{2b + \sqrt{6|c| - 2b^2}\,\cosh(\sqrt{b^2 - |c|}\,\xi)} \tag{3.40}$$

giving the defocussing mKdV solitary wave with a background $b > 0$ and a speed $c < 0$, with $b^2 > |c|$ and $3|c| > b^2$.

The wave peak is $-b + \sqrt{6|c| - 2b^2}$, and the wave depth is $h = 2b - \sqrt{6|c| - 2b^2}$, while the width of the wave is proportional to $w = 2/\sqrt{b^2 - |c|}$. Clearly, if the background b is taken to be any positive value, then the speed is restricted to lie in the interval

$$b^2 > |c| > \tfrac{1}{3}b^2 \tag{3.41}$$

whereby $c_{\text{min}} = -b^2$ and $c_{\text{max}} = -\frac{1}{3}b^2$ are lower and upper limits on the speed. Alternatively, the speed c can be taken to be any negative value, and then the background is restricted by

$$\sqrt{3|c|} > b > \sqrt{|c|} \tag{3.42}$$

with $b_{\text{min}} = \sqrt{|c|}$ and $b_{\text{max}} = \sqrt{3|c|}$ being the lower and upper limits.

Physically, the solution (3.40) describes either a *dark solitary wave on a positive background*, or a *dark solitary hole in a positive background*.

Under reflection $(U, J, c) \to (-U, -J, c)$, we have $b \to -b$, and hence the solution (3.40) becomes

$$U(\xi) = b + \frac{6(b^2 - |c|)}{\sqrt{6|c| - 2b^2}\,\cosh(\sqrt{b^2 - |c|}\,\xi) - 2b} \tag{3.43}$$

giving a defocussing mKdV solitary wave with a background $b < 0$ and a speed $c < 0$, with $b^2 > |c|$ and $3|c| > b^2$. The wave peak is $|b| - \sqrt{6|c| - 2b^2}$, and the wave height is $h = 2|b| - \sqrt{6|c| - 2b^2}$, while the width of the wave is proportional to $w = 2/\sqrt{b^2 - |c|}$. If the background b is taken to be any negative value, then the speed is restricted to lie in the interval (3.41). Alternatively, if the speed c is taken to be any negative value, then the background is restricted by $-\sqrt{3|c|} < b < -\sqrt{|c|}$.See Fig. 3.6

(a) (b)

Figure 3.6: Defocussing-mKdV solitary wave (a) positive background and (b) negative background

Physically, this solution (3.43) describes either a *bright solitary wave on a negative background*, or a *bright solitary hole in a negative background*.

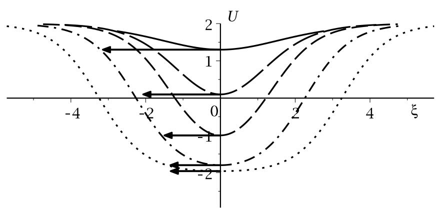
Both solutions (3.40) and (3.43) have not been studied previously.

For any fixed negative speed, as the size of the background $|b|$ decreases, the depth/height of the wave decreases while the width increases. In the limit $|b| \to \sqrt{|c|}$, the wave flattens to become $U \to b$. But in the opposite limit $|b| \to \sqrt{3|c|}$, the wave widens and has the approximate form

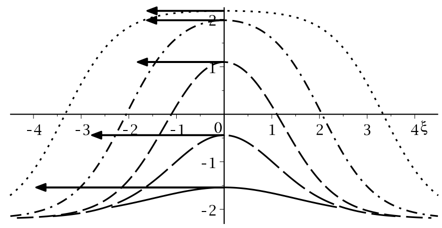
$$U \simeq \pm\sqrt{3|c|}\left(1 - \frac{2}{\sqrt{1-|b|/\sqrt{3}}\,\cosh(\xi\sqrt{2|c|})+1} + O(1-|b|/\sqrt{3})\right) \qquad (3.44)$$

where the $+/-$ case corresponds to $\text{sgn}(b) = \pm 1$. See Fig. 3.7

Finally, it is worth remarking that the first integrals (2.7) and (2.8) evaluated for the solutions (3.37) are given by $J = -\frac{1}{3}b(b^2+3c)$ and $E = -\frac{1}{4}b^2(b^2+2c)$. These expressions can be straightforwardly inverted to give c and b in terms of J and E.



(a)



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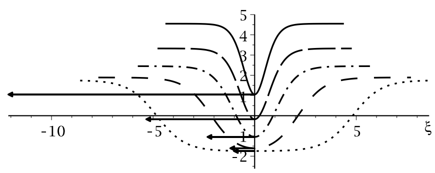


(c)

Figure 3.7: Defocussing-mKdV solitary waves with (a) positive background and negative heights (b) negative background and positive heights (c) on different backgrounds with the same heights

3.3 Defocussing-mKdV kink waves

When $J = 0$ in the defocussing case (3.15), the three critical points (3.30) become

$$V_{\text{max},1} = V_{\text{max},2} = \tfrac{3}{4}c^2, \quad U_1 = \sqrt{3|c|}, \quad U_2 = -\sqrt{3|c|} \tag{3.45a}$$

$$V_{\text{min}} = 0, \quad U_3 = 0 \tag{3.45b}$$

The potential well (3.32) now becomes

$$V(U) = -\tfrac{1}{12}U^4 + \tfrac{1}{2}|c|U^2 \leq E_{\text{max}} = V_{\text{max},1} = V_{\text{max},2} \tag{3.46}$$

with the domain given by $U_2 \leq U \leq U_1$, which corresponds to the factorization $\frac{3}{4}c^2 - V(U) = (U - U_1)^2(U - U_2)^2 = (U^2 - 3|c|)^2$. In particular, the previous roots (3.35) of the energy equation coalesce into $U_\pm = -U_1 = U_2$. Thus, both rims of the potential well are given by the pair of symmetric maximum points (3.45a), where the domain is $U_2 \leq U \leq U_1$.

The previous quadrature (3.36) for the solution $U(\xi)$ needs to be altered by changing the end-point, because U_+ is now a local maximum point. A convenient choice is the minimum point $U_3 = 0$. Thus, we have

$$\int_0^U \frac{dU}{(U_1 - U)(U - U_2)} = \int_0^U \frac{dU}{3|c| - U^2} = \tfrac{1}{\sqrt{6}}\xi, \quad U_2 \leq U \leq U_1 \tag{3.47}$$

which straightforwardly yields

$$U(\xi) = \sqrt{3|c|}\, \tanh\left(\tfrac{1}{\sqrt{2}}\sqrt{|c|}\,\xi\right), \quad c < 0 \tag{3.48}$$

This solution has the following features. As $\xi \to \pm\infty$, the asymptotes are $U \to \mp\sqrt{3|c|}$. Hence the net change in amplitude (height) is $h = 2\sqrt{3|c|}$. The width is proportional to $w = \sqrt{2}/\sqrt{|c|}$. Physically, the solution describes a *kink wave*.

3.4 Focussing-mKdV heavy-tail waves on a background

When $|J| = \frac{2}{3}\sqrt{c^3}$ in the limit of the focussing case (3.14), the three critical points (3.16) coalesce into two points, consisting of a local minimum and an inflection

$$V_{\text{min}} = -2c^2, \quad U_2 = 2\sqrt{c} \text{ for } \operatorname{sgn}(J) = 1, \quad U_1 = -2\sqrt{c} \text{ for } \operatorname{sgn}(J) = -1 \tag{3.49a}$$

$$V_{\text{inflect}} = \tfrac{1}{4}c^2, \quad U_3 = U_1 = -\sqrt{c} \text{ for } \operatorname{sgn}(J) = 1, \quad U_3 = U_2 = \sqrt{c} \text{ for } \operatorname{sgn}(J) = -1 \tag{3.49b}$$

The potential well (3.18) thereby becomes

$$V(U) = \tfrac{1}{12}U^4 - \tfrac{1}{2}cU^2 - \mathrm{sgn}(J)\tfrac{2}{3}\sqrt{c^3}\,U \leq E_{\text{max}}, \quad E_{\text{max}} = \infty \tag{3.50}$$

with the same domain as before. The previous turning points (3.21) given by the energy equation (3.19) with $E = V_{\text{inflect}}$ now become $U_\pm = \sqrt{c}(\mathrm{sgn}(J) \pm 2)$, corresponding to the factorization $\frac{1}{4}c^2 - V(U) = (U - U_+)^3(U - U_-)$. Note that $U_3 = U_-$ for $\mathrm{sgn}(J) = 1$ and $U_3 = U_+$ for $\mathrm{sgn}(J) = -1$.

Hence, in the respective cases $\mathrm{sgn}(J) = \pm 1$, the quadratures (3.22) and (3.23) for the solutions $U(\xi)$ take the form

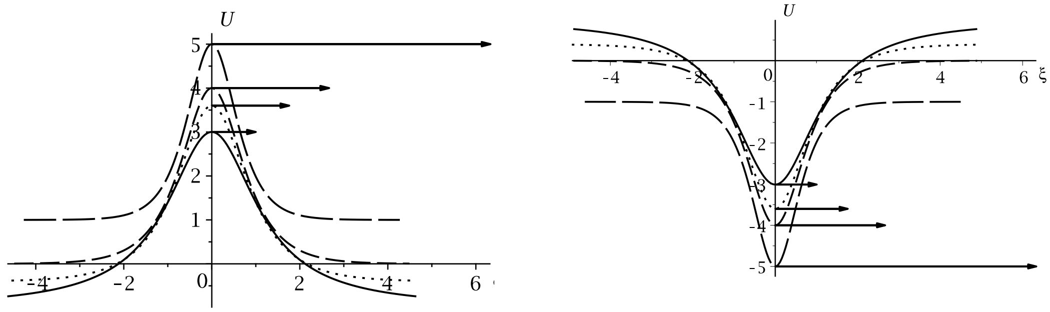
$$\mathrm{sgn}(J) \int_U^{\mathrm{sgn}(J)3\sqrt{c}} \frac{dU}{\sqrt{(\mathrm{sgn}(J)\,2\sqrt{c} - U)(U + \mathrm{sgn}(J)\sqrt{c})^3}} = \tfrac{1}{\sqrt{6}}|\xi|, \quad U_- \leq U \leq U_+ \tag{3.51}$$

A straightforward integration yields

$$U(\xi) = \pm\sqrt{c}\left(1 - \frac{12}{2c\xi^2 + 3}\right), \quad c > 0 \tag{3.52}$$

where the $+/-$ cases correspond to $\mathrm{sgn}(J) = \mp 1$ respectively.

This solution has the following features. The wave peak is $\mp 3\sqrt{c}$, and the background (asymptote) is $\pm\sqrt{c}$. Hence, the wave height/depth is $h = 4\sqrt{c}$. Physically, the solution describes a *dark/bright heavy-tail wave* on a positive/negative background, or a *dark/bright heavy-tail hole* in a positive/negative background. See fig. 3.8.



(a) (b)

Figure 3.8: Focussing-mKdV solitary waves with (a) positive heights and (b) negative heights

Chapter 4

Non-periodic travelling wave solutions of gKdV equation

In the gKdV equation (1.3), by scaling t and x we can put $\beta = 1$; and further by scaling u, we can put $\alpha = 1$ when p is odd and $\alpha = \pm 1$ when p is even. This yields the scaled form of the gKdV equation

$$u_t + \sigma u^p u_x + u_{xxx} = 0, \quad \sigma = \begin{cases} 1, & p \text{ is odd} \\ \pm 1, & p \text{ is even} \end{cases} \tag{4.1}$$

where p is an arbitrary positive integer; $\sigma = 1$ is called the focussing case, and $\sigma = -1$ is called the defocussing case, in analogy to the mKdV equation. (Note that: for $p = 0$ the equation would be linear; for p a negative integer the equation will have a singularity at $u = 0$.)

When $p = 1, 2$, the gKdV equation (4.1) reduces to the KdV equation (2.1) and the mKdV equation (3.1)–(3.2), respectively. Hereafter we will be interested in the cases $p \geq 3$.

It will be useful to recall the well-known mass, momentum, and energy conservation laws $D_t T + D_x \Phi = 0$ of the gKdV equation:

$$T = u, \quad \Phi = \tfrac{1}{p+1} u^{p+1} + u_{xx} \tag{4.2}$$

$$T = \tfrac{1}{2} u^2, \quad \Phi = \tfrac{1}{p+2} u^{p+2} - \tfrac{1}{2} u_x^2 + u u_{xx} \tag{4.3}$$

$$T = -\tfrac{1}{2} u_x^2 + \tfrac{1}{(p+1)(p+2)} u^{p+2}, \quad \Phi = \tfrac{1}{2}(u_{xx} + \tfrac{1}{p+1} u^{p+1})^2 + u_t u_x \tag{4.4}$$

Here T denotes a conserved density, Φ denotes a conserved flux, and D is directional derivative.

Consider travelling wave solutions (1.4) of the scaled gKdV equation (4.1). These solutions satisfy the third-order ODE

$$U''' + (\sigma U^p - c) U' = 0 \tag{4.5}$$

It has first integrals

$$U'' + \tfrac{\sigma}{p+1}U^{p+1} - cU = J = \text{const} \tag{4.6}$$

$$UU'' - \tfrac{1}{2}U'^2 + \tfrac{\sigma}{p+2}U^{p+2} - \tfrac{1}{2}cU^2 = E = \text{const} \tag{4.7}$$

These integrals can be derived directly from the conservation laws for mass (4.2), momentum (4.3), and energy (4.4), by the method described in Ref. [10, 11]. Specifically, each of these conservation laws applied to travelling waves (1.4) yields a first integral $\Psi = T - c\Phi = C = \text{const}$ for the travelling wave ODE (4.5). The first integral has the physical meaning that it describes the conserved flux in a reference frame moving with the travelling wave. This yields J, E, $cE + \frac{1}{2}J^2$, of which only the first two are functionally independent.

When the two first integrals (4.6) and (4.7) are combined, they yield a reduction of the travelling wave ODE to a first-order separable differential equation

$$\tfrac{1}{2}U'^2 + \tfrac{\sigma}{(p+2)(p+1)}U^{p+2} - \tfrac{1}{2}cU^2 - JU - E = 0 \tag{4.8}$$

This ODE (4.8) has the form of a nonlinear oscillator equation (1.5) where

$$V(U) = \tfrac{\sigma}{(p+2)(p+1)}U^{p+2} - \tfrac{1}{2}cU^2 - JU \tag{4.9}$$

is the potential and E is the oscillator energy.

When p is odd, then similarly to the KdV case $p = 1$, a potential well exists only if $V(U)$ has at least one local minimum and one local maximum. In this case the travelling waves of interest to us will be solitary waves on zero/non-zero backgrounds, which arise for $E = V_{\text{max}}$. As we will see later, no kink waves exist because $V(U)$ turns out to never have a pair of local maximums with equal heights.

When p is even, then similarly to the mKdV case $p = 2$, in the focussing case a potential well always exists because $V(U)$ is upward, whereas in the defocussing case a potential well exists only if $V(U)$ has at least two local maximums, because $V(U)$ is downward. Solitary waves on zero/non-zero backgrounds will arise whenever $V(U)$ has at least one local minimum and one local maximum, and heavy-tail waves on a non-zero background will arise in the limiting case when $V(U)$ has a local minimum and an inflection. Kink waves will arise whenever $V(U)$ has at least one pair of local maximums with equal heights.

The critical points of $V(U)$ are determined by

$$0 = V'(U) = \tfrac{\sigma}{p+1} U^{p+1} - cU - J \tag{4.10}$$

Now, we look at the separate cases of odd and even $p \geq 3$ to find the potential well and the resulting solutions of interest in each case.

4.1 Odd-power gKdV potential

For odd $p \geq 3$, the critical points of the potential (4.9) are the real roots of the polynomial (4.10) with even degree $p+1 \geq 4$. To determine the number and type of these critical points, we will write the polynomial in a scaled form in terms of

$$\tilde{U} = U/c^{1/p}, \quad \tilde{J} = J/c^{1+1/p}, \tag{4.11}$$

giving

$$0 = \tfrac{1}{2(q+1)} \tilde{U}^{2q+2} - \tilde{U} - \tilde{J} \tag{4.12}$$

where $p = 2q+1$, $q = 1, 2, \ldots$. This equation can be rearranged

$$\tfrac{1}{2(q+1)} \tilde{U}^{2q+2} = \tilde{U} + \tilde{J}. \tag{4.13}$$

where the intersection points of the two sides are precisely the scaled critical points of the potential. Note the left side is an non-negative even-power (convex) function whose slope is $\tilde{U}^{2q+1}$ while the right side is a linear function whose slope is 1. Therefore, graphically, the number of intersections is either two, one, or none depending on $\tilde{J}$. There will be one intersection when $\tilde{U}+\tilde{J}$ is tangent to $\frac{1}{2(q+1)}\tilde{U}^{2q+2}$, which occurs when their slopes are equal: $1=\tilde{U}^{2q+1}$. This intersection is given by $\tilde{U}=1$ and $\tilde{J}=-\frac{2q+1}{2(q+1)}$. Consequently, if $\tilde{J}<-\frac{2q+1}{2(q+1)}$, there will be no intersection, whereas if $\tilde{J}>-\frac{2q+1}{2(q+1)}$, there will be two intersections, which lie to the right and left of $\tilde{U}=1$.

Since p is odd, we have $c^{1+1/p}\geq 0$ and so $\operatorname{sgn}(\tilde{J})=\operatorname{sgn}(J)$. Hence for $J\leq -\frac{p}{p+1}c^{1+1/p}$, the potential $V(U)$ has at most one critical point, and in this case there is no potential well. Hereafter we consider the case

$$J>-\tfrac{p}{p+1}c^{1+1/p}, \tag{4.14}$$

whereby $V(U)$ has two critical points, $U=U_1$ and $U=U_2$, which consist of a local maximum and a local minimum because $V''(U)=U^p-c$ changes sign at $U=c^{1/p}$ which lies between the critical points. In particular,

$$U_1<c^{1/p}<U_2,\quad V''(U_1)<0,\quad V''(U_2)>0, \tag{4.15}$$

where the maximum point U_1 sits to the right of the minimum point U_2. This is exactly analogous to the KdV case.

Thus, when J satisfies the condition (4.14), there is a potential well defined by

$$V(U)=\tfrac{1}{(p+1)(p+2)}U^{p+2}-\tfrac{1}{2}cU^2-JU\leq E_{\max}\quad E_{\max}=V_{\max}=V(U_1) \tag{4.16}$$

Its domain is

$$U_1\leq U<\infty \tag{4.17}$$

Since this potential well has a single local maximum, it supports solitary waves on non-zero backgrounds but not kink waves.

4.1.1 Odd-power gKdV solitary waves on a zero/non-zero background

Similarly to the KdV case, solitary wave solutions $U(\xi)$ for an odd power $p \geq 3$ are obtained by taking $E = V_{\max} = V(U_1)$. Then the potential (4.9) has the factorization

$$V(U_1) - V(U) = (U - U_1)^2 (U_+ - U) W(U), \tag{4.18}$$

where

$$U_+ > U_2 \tag{4.19}$$

is the turning point, and where $W(U)$ is a polynomial that has even degree $p-1$ and that is positive on $U_1 \leq U \leq U_+$. Here U_1, U_2 are the critical points (4.15).

Hence, the quadrature (1.6) for $U(\xi)$ is given by

$$\int_U^{U_+} \frac{dU}{(U - U_1)\sqrt{(U_+ - U)W(U)}} = \sqrt{2}\,|\xi|, \quad U_1 \leq U \leq U_+ \tag{4.20}$$

When $p = 3$, the integral can be evaluated in terms of elliptic functions, but for $p \geq 5$, the integral has no explicit evaluation in general.

This quadrature (4.20) implicitly defines a family of the solitary wave solutions $U(\xi)$, parameterized by (J, c), with the following features. The wave peak is U_+ and the background (asymptote) is $b = U_1$, and so the wave height is $h = U_+ - U_1$, while $U(\xi)$ is an even function of ξ. The width of the wave is proportional to $w = 2/\sqrt{U_1^p - c}$, which is obtained from an asymptotic expansion of the quadrature as $U \to U_1$, combined with $V''(U_1) = 2(U_+ - U_1)W(U_1) = U_1^p - c$ from the factorization (4.18).

The well-known solitary wave on a zero background, $b = 0$, can be obtained in a straightforward way as follows. By putting $b = U_1 = 0$ into the critical point equation (4.10), we get

$J = 0$. Then the energy equation (4.18) simplifies to give $-V(U) = U^2(U_+ - U)W(U) = \frac{1}{2}cU^2 - \frac{1}{(p+1)(p+2)}U^{p+2}$ with $U_+ = (\frac{(p+2)(p+1)}{2}c)^{1/p}$ and $W(U) = \frac{1}{(p+1)(p+2)}U_+^{p-1}S_p(U/U_+)$ in terms of the positive function $S_n(z) = (z^n - 1)/(z-1) = \sum_{j=1}^{n}\binom{n}{j}(z-1)^{j-1}$ defined on $0 \leq z \leq 1$, for $n \in \mathbb{Z}^+$. Note here that the relations (4.15) and (4.19) satisfied by the critical points and the turning point together imply $c > 0$ due to $U_+ > U_2 > U_1 = 0$ and $-V(U) > 0$. The resulting quadrature (4.20) can be evaluated explicitly, yielding

$$U(\xi) = (\tfrac{(p+2)(p+1)}{2}c)^{1/p}\,\mathrm{sech}^{2/p}\big(\tfrac{p}{2}\sqrt{c}\,\xi\big), \quad p \text{ is odd} \tag{4.21}$$

This solitary wave has zero background, $b = 0$, height $h = (\frac{(p+2)(p+1)}{2}c)^{1/p}$, and width (proportional to) $w = 2/(p\sqrt{c})$, while its speed is positive, $c > 0$.

The solitary wave solutions (4.20) with a non-zero background are new and their properties have not been studied to-date. The background $b = U_1$ can be positive or negative, with the speed c being restricted by $c > b^p$. Alternatively, the speed can be taken to be positive, negative, or zero, with the background being restricted to be $b < c^{1/p}$.

Similarly to the KdV case, it is interesting that, unlike the zero-background solution, the solutions (4.20) with a negative background $b < 0$ can propagate in the opposite direction, namely $c < 0$. Physically, we can interpret this type of solution as being either a *bright solitary wave on a negative background* or a *bright solitary hole in a negative background*. As the size $|b|$ of the background increases, the height decreases while the width increases.

For solutions (4.20) with positive speed, $c > 0$, the non-zero background, $b < c^{1/p}$ can be positive or negative. Physically, this type of solution describes either a *bright solitary wave on a positive/negative background*, or a *bright solitary hole in a positive/negative background*. As b increases, the height decreases while the width increases.

An interesting special case is solutions with zero speed, $c = 0$. These solutions represent *static humps* on a negative background $b = -((p+1)J)^{1/(p+1)} < 0$, with $J > 0$.

4.2 Even-power gKdV potential

For even $p \geq 4$, the critical points of the potential (4.9) are the real roots of the polynomial (4.10) with odd degree $p+1 \geq 5$. We will write this polynomial in a scaled form in terms of

$$\tilde{U} = U/|c|^{1/p}, \quad \tilde{J} = J/|c|^{1+1/p}, \tag{4.22}$$

which gives

$$0 = \tfrac{\sigma}{2q+1}\tilde{U}^{2q+1} - \text{sgn}(c)\tilde{U} - \tilde{J} \tag{4.23}$$

where $p = 2q$, $q = 2, 3, \ldots$, and where $\sigma = \pm 1$ in the focussing and defocussing cases, respectively. This equation can be rearranged so that we have a linear function on one side and an odd-power (non convex) function on the other side:

$$\tfrac{\sigma}{2q+1}\tilde{U}^{2q+1} = \text{sgn}(c)\tilde{U} + \tilde{J}. \tag{4.24}$$

The intersection points of the two sides are precisely the scaled critical points of the potential. Note the left side has slope $\sigma\tilde{U}^{2q}$ while the right side has $\text{sgn}(c)$. Therefore, graphically, we have one, two, or three different intersections, depending on $\tilde{J}$ and $\text{sgn}(c)$. Two intersections occur when $\text{sgn}(c)\tilde{U} + \tilde{J}$ is tangent to $\tfrac{\sigma}{2q+1}\tilde{U}^{2q+1}$, which requires that the slopes of two functions are equal: $\sigma\tilde{U}^{2q} = \text{sgn}(c)$, and hence $\sigma = \text{sgn}(c)$. One of these two intersections is the tangential intersection point given by $\tilde{U} = \pm 1$, with $\tilde{J} = \mp\tfrac{2q\sigma}{2q+1}$. As a consequence, if $|\tilde{J}| < \tfrac{2q}{2q+1}$ and $\sigma = \text{sgn}(c)$, there will be three intersections, with one point lying between $\tilde{U} = -1$ and $\tilde{U} = 1$, while the other two points lie outside. In contrast, if $|\tilde{J}| > \tfrac{2q}{2q+1}$ or $\sigma \neq \text{sgn}(c)$, there will be only one intersection.

Hence when $|J| > \tfrac{p}{p+1}|c|^{1+1/p}$ or $\sigma \neq \text{sgn}(c)$, the potential $V(U)$ has a single critical point, which consists of a local minimum if $\sigma = 1$ or a local maximum if $\sigma = -1$. In the latter situation, no potential well exists, while in the former situation, there is a potential well but it does not support either solitary wave solutions or kink wave solutions.

Next we consider the conditions

$$|J| < \tfrac{p}{p+1}|c|^{1+1/p}, \quad \text{sgn}(c) = \sigma \tag{4.25}$$

whereby the potential $V(U)$ has three critical points $U = U_1, U_2, U_3$. A useful observation is that, like in the mKdV case, here the potential is invariant under the reflection (3.12),

In the focussing case, $\sigma = 1$, $V(U)$ is upward and defines a potential well

$$V(U) = \tfrac{1}{(p+1)(p+2)}U^{p+2} - \tfrac{1}{2}cU^2 - JU \leq E_{\text{max}}, \quad E_{\text{max}} = \infty \tag{4.26}$$

with the domain begin $-\infty < U < \infty$. This potential well has two local minimums and a local maximum

$$(+) \quad \begin{aligned} & U_1 < -c^{1/p}, \quad U_3 > c^{1/p}, \quad V''(U_1) > 0, \quad V''(U_3) > 0, \\ & |U_2| < c^{1/p}, \quad V''(U_2) < 0, \end{aligned} \tag{4.27}$$

where the maximum point U_2 sits between the two minimum points U_1 and U_3, and moreover, $\text{sgn}(U_2) = -\text{sgn}(J)$. The potential well supports solitary waves on zero/non-zero backgrounds but not kink waves since $V(U)$ never has two local maximums. This is exactly analogous to the focussing mKdV case. Note that, under the reflection (3.12), the critical points transform as $(U_1, U_2, U_3) \to (-U_3, -U_2, -U_1)$.

In defocussing case, $\sigma = -1$, $V(U)$ is downward and has two local maximums and a local minimum. In particular, when $J \neq 0$, we have

$$(-) \quad \begin{aligned} & \text{sgn}(J)\, U_1 > |c|^{1/p}, \quad \text{sgn}(J)\, U_3 < -|c|^{1/p}, \quad V''(U_1) < 0, \quad V''(U_3) < 0, \quad V(U_1) < V(U_3), \\ & |U_2| < |c|^{1/p}, \quad V''(U_2) > 0, \end{aligned} \tag{4.28}$$

where the minimum point U_2 sits between the two maximum points U_1 and U_3, and where the relative heights of the maximum points directly follows from the term $-JU$ in the expression (4.9) for the potential. The potential well is defined by

$$V(U) = -\tfrac{1}{(p+1)(p+2)}U^{p+2} - \tfrac{1}{2}cU^2 - JU \leq E_{\text{max}}, \quad E_{\text{max}} = V(U_1) \tag{4.29}$$

which has a rim on each side. One rim is the maximum at U_1, and the other rim is a turning point, which defines the domain. Solitary waves on non-zero backgrounds, but not kink waves, are supported by this potential well. Note that, under the reflection (3.12), the critical points transform as $(U_1, U_2, U_3) \to (-U_1, -U_2, -U_3)$.

In the special case $J = 0$, the potential is symmetric about $U = 0$, and we have

$$(-) \quad \begin{aligned} & U_1 = -U_3 = ((p+1)|c|)^{1/p}, \quad V''(U_1) < 0, \quad V''(U_3) < 0, \quad V(U_1) = V(U_3), \\ & U_2 = 0, \quad V''(U_2) > 0, \end{aligned} \tag{4.30}$$

The potential well is again defined by $E_{\text{max}} = V(U_1)$, with the domain being $-U_1 \leq U \leq U_1$. Kink waves, but not solitary waves, are supported by this potential well.

Both of these cases are exactly analogous to the defocussing mKdV case with $J \neq 0$ and $J = 0$, respectively.

Finally, we also consider the limiting case of the conditions (4.25) given by

$$|J| = \tfrac{p}{p+1}|c|^{1+1/p}, \quad \text{sgn}(c) = \sigma \tag{4.31}$$

whereby the potential $V(U)$ has only two critical points. This occurs when a local minimum and a local maximum coalesce into an inflection. In the focussing case, the potential well (4.26) will now have a local minimum and an inflection

$$(+) \qquad \begin{aligned} &\operatorname{sgn}(J)\, U_0 > c^{1/p}, \quad V''(U_0) > 0, \\ &U_2 = -\operatorname{sgn}(J)\, c^{1/p}, \quad V''(U_2) = 0, \end{aligned} \tag{4.32}$$

which arises from either $U_1 = U_2$ when $\operatorname{sgn}(J) = 1$, or $U_3 = U_2$ when $\operatorname{sgn}(J) = -1$. This potential well supports a heavy-tail wave on a non-zero background. In defocussing case, the potential well (4.29) will similarly have a local maximum and an inflection, but it will not support any solutions of interest to us.

4.2.1 Focussing even-power gKdV solitary waves on a zero/non-zero background

Solitary wave solutions $U(\xi)$ for an even power $p \geq 4$ are obtained by taking $E = V_{\max} = V(U_2)$, where U_2 is the local maximum point (4.27) in the potential well (4.26). The potential thereby has the factorization

$$V(U_2) - V(U) = (U - U_2)^2 (U_+ - U)(U - U_-) W(U), \tag{4.33}$$

where $U_+ > U_3$ and $U_- < U_1$ are turning points, with

$$U_- < U_2 < U_+, \tag{4.34}$$

and where $W(U)$ is a polynomial that has even degree $p-2$ and that is positive on $U_- \leq U \leq U_+$. We therefore have two turning points and an asymptotic turning point, which yield two different solitary wave solutions $U(\xi)$ corresponding to the adjacent pairs of turning points (U_2, U_+) and (U_-, U_2).

The quadrature (1.6) for $U(\xi)$ is given by

$$\int_{U}^{U_+} \frac{dU}{(U-U_2)\sqrt{(U_+-U)(U-U_-)W(U)}} = \sqrt{2}\,|\xi|, \quad U_2 \le U \le U_+ \tag{4.35}$$

for the first solution, and by

$$\int_{U_-}^{U} \frac{dU}{(U_2-U)\sqrt{(U_+-U)(U-U_-)W(U)}} = \sqrt{2}\,|\xi|, \quad U_- \le U \le U_2 \tag{4.36}$$

for the second solution. When $p = 4$, these integrals can be evaluated in terms of elliptic functions, but for $p \ge 6$, the integrals have no explicit evaluation in general.

These two quadratures (4.35) and (4.36) each implicitly define a family of the solitary wave solutions $U(\xi)$, parameterized by (J, c). Their background (asymptote) is $b = U_2$, and the first solution has the wave peak $U_+ > 0$ and the wave height $h = U_+ - U_2 > 0$, while the second solution has the wave peak $U_- < 0$ and the wave depth $h = U_2 - U_- > 0$. Both solutions $U(\xi)$ are even functions of ξ, and their width is proportional to $w = 2/\sqrt{|U_2|^p - c}$, which is obtained from an asymptotic expansion of the quadratures as $U \to U_2$, combined with $V''(U_2) = 2(U_+ - U_2)(U_2 - U_-)W(U_2) = U_2^p - c$ from the factorization (4.33). The wave speed here is $c > 0$, due to condition (4.25).

The well-known solitary wave on a zero background, $b = 0$, arises when we plug $b = U_2 = 0$ into the critical point equation (4.10), yielding $J = 0$ and $U_3 = -U_1 = ((p+1)c)^{1/p}$. Then the energy equation (4.33) simplifies to give $-V(U) = U^2(U_+ - U)(U - U_-)W(U) = \frac{1}{2}cU^2 - \frac{1}{(p+1)(p+2)}U^{p+2}$ with $U_\pm = \pm(\frac{(p+2)(p+1)}{2}c)^{1/p}$ and $W(U) = \frac{1}{(p+1)(p+2)}U_+^{p-2}S_{p/2}(U/U_+)$ in terms of the positive function $S_n(z) = \sum_{j=1}^{n} z^{2(n-j)}$ defined previously. The resulting quadratures (4.35) and (4.36) can be evaluated explicitly, yielding

$$U(\xi) = \pm(\tfrac{(p+2)(p+1)}{2}c)^{1/p}\,\mathrm{sech}^{2/p}\big(\tfrac{p}{2}\sqrt{c}\,\xi\big), \quad p \text{ is even} \tag{4.37}$$

These solitary waves have zero background, $b = 0$, height $h = \pm(\frac{(p+2)(p+1)}{2}c)^{1/p}$, width (proportional to) $w = 2/(p\sqrt{c})$, and positive speed, $c > 0$. The sign of h represents an up/down orientation.

The solutions (4.35)–(4.36) with a non-zero background $b = U_2 \neq 0$ are new and their properties have not been studied to-date.

The background $b = U_2$ can be positive or negative, with the speed c being larger than $c_{\text{min}} = |b|^p$. Alternatively, the speed can be taken to be any positive value, with the background being restricted by $|b|_{\text{max}} = c^{1/p}$. When the speed is $c = c_{\text{max}}$, and hence the size of the background is $|b| = |b|_{\text{max}}$, the width becomes $w = \infty$ and so has wave flattened.

Physically, the first solution (4.35) describes either a *bright solitary wave on a positive/negative background*, or a *bright solitary hole in a positive/negative background*. The second solution (4.36) describes either a *dark solitary wave on a positive/negative background*, or a *dark solitary hole in a positive/negative background*.

4.2.2 Focussing even-power gKdV heavy-tail waves on a non-zero background

In the limiting case (4.32) of the previous potential well (4.26), heavy-tail wave solutions $U(\xi)$ are obtained by taking $E = V_{\text{inflect}} = V(U_2)$, where

$$V_{\text{inflect}} = \tfrac{p}{2p+4}c^{1+p/2} \tag{4.38}$$

There are two separate cases, given by $\text{sgn}(J) = \pm 1$. In the case $\text{sgn}(J) = 1$, the turning points in the potential well are $U_- = U_2$ and $U_+ > U_0$, which correspond to the factorization $V_{\text{inflect}} - V(U) = (U - U_-)^3(U_+ - U)W(U)$. In the other case $\text{sgn}(J) = -1$, the turning points are $U_+ = U_2$ and $U_- < U_0$, corresponding to the factorization $V_{\text{inflect}} - V(U) = (U_+ - U)^3(U - U_-)W(U)$. Here, in both cases, $W(U)$ is a positive polynomial with even degree $p - 2$ on $U_- \leq U \leq U_+$.

The previous quadratures (4.35) and (4.36) now take the form

$$\pm \int_{U}^{U_\pm} \frac{dU}{\sqrt{c^{1/p} \pm U}^{3} \sqrt{\pm(U_\pm - U)W(U)}} = \sqrt{2}|\xi|, \quad U_- \leq U \leq U_+ \tag{4.39}$$

where the $+/-$ in the integral is given by $\operatorname{sgn}(J)$. When $p = 4$, this integral can be evaluated in terms of elliptic functions, but for $p \geq 6$, it has no explicit evaluation in general.

The quadrature (4.39) implicitly defines a family of the heavy-tail wave solutions $U(\xi)$, parameterized by (J, c), where $c > 0$. The background (asymptote) is $b = U_2 = \mp c^{1/p}$ and the wave peak is $U_\pm$, so the wave height is $h = U_\pm - U_2$. The tail of the wave has the form

$$U \simeq b - \tfrac{12}{pb^{p-1}}\xi^{-2} = \mp c^{1/p}\left(1 - \frac{12}{pc\xi^2}\right) \tag{4.40}$$

as $|\xi| \to \infty$, which is obtained from an asymptotic expansion of the quadrature as $U \to b$, combined with $V'''(b) = 6(U_+ - U_-)W(b) = pb^{p-1}$ from the factorization of $V_{\text{inflect}} - V(U)$. Thus, since the wave does not decay to b exponentially in ξ, it is not localized in the sense of a solitary wave.

These solutions are new. Physically, they describe a *dark/bright heavy-tail wave on a positive/negative background*, or a *dark/bright heavy-tail hole in a positive/negative background*.

4.2.3 Defocussing-gKdV solitary waves on a non-zero background

Solitary wave solutions $U(\xi)$ for an even power $p \geq 4$ are obtained by taking $E = V_{\text{max}} = V(U_1)$, where U_1 is the local maximum point (4.28) with least height in the potential well (4.29) such that $J \neq 0$. The potential has the factorization

$$V(U_1) - V(U) = (U - U_1)^2(U - U_+)(U - U_-)W(U), \tag{4.41}$$

where U_+ and U_- are roots of the energy equation, and where $W(U)$ is a polynomial that has even degree $p-2$. The shape of this potential depends on $\text{sgn}(J) = \pm 1$. For $\text{sgn}(J) = 1$, the roots obey

$$U_- < U_+ < U_1, \quad U_1 > U_2 > 0, \quad U_- < U_3 < 0 \tag{4.42}$$

where U_+ is a turning point which is the left side rim in the potential well, and U_1 is an asymptotic turning point which is the right side rim. In this case $W(U)$ is positive on $U_+ \leq U \leq U_1$. For $\text{sgn}(J) = -1$, the roots obey

$$U_1 < U_- < U_+, \quad U_1 < 0, \quad U_+ > 0 \tag{4.43}$$

where U_- is a turning point which is the right side rim in the potential well, and U_1 is an asymptotic turning point which is the left side rim. In this case $W(U)$ is positive on $U_1 \leq U \leq U_-$.

From the quadrature (1.6), the solution $U(\xi)$ in both cases is given by

$$\pm \int_{U_\pm}^{U} \frac{dU}{(U - U_1)\sqrt{(U - U_-)(U - U_+)W(U)}} = \sqrt{2}|\xi| \tag{4.44}$$

where the $+/-$ sign in the integral corresponds to $\text{sgn}(J) = \pm 1$, and where $U_+ \leq U \leq U_1$ and $U_1 \leq U \leq U_-$ are the respective integration domains. When $p = 4$, this integral can be evaluated in terms of elliptic functions, but for $p \geq 6$, it has no explicit evaluation in general.

The quadrature (4.44) implicitly defines a family of the solitary wave solutions $U(\xi)$ which are even functions of ξ, parameterized by (J, c). This family of solutions has the following features. The wave speed is $c < 0$, due to condition (4.25). The wave peak is $U_\pm$ and the background (asymptote) is $b = U_1$, and so the wave height $h = U_\pm - U_1$ is positive when $\text{sgn}(J) = 1$ and is negative when $\text{sgn}(J) = -1$. The width is proportional to $w = 2/\sqrt{|U_1|^p - |c|}$, which is obtained from an asymptotic expansion of the quadrature as $U \to U_1$, combined with $V''(U_1) = -2(U_1 - U_+)(U_1 - U_-)W(U_1) = -U_1^p - c$ from the factorization (4.41).

In these solutions $U(\xi)$, the speed can be taken to be any negative value, $c < 0$, with the background being restricted by $|b| > |c|^{1/p}$ so that the width is $w < \infty$. The background is also restricted to have a maximum $|b|_{\text{max}} = ((p+1)|c|)^{1/p}$, which arises from the limit $J \to 0$ given by the potential well (4.30). Hence, we have

$$|c|^{1/p} < |b| < ((p+1)|c|)^{1/p} \tag{4.45}$$

Alternatively, the background b can be chosen to have any positive or negative value, but not zero. Then the speed is restricted by

$$\tfrac{1}{p+1}|b|^p < |c| < |b|^p \tag{4.46}$$

Physically, these solutions describe either a *dark/bright solitary wave on a positive/negative background*, or a *dark/bright solitary hole in a positive/negative background*.

For any fixed negative speed, as the size of the background $|b|$ decreases, the height/depth of the wave decreases while the width increases. In the limit $|b| \to |c|^{1/p}$, the wave flattens to become $U \to b$. But in the opposite limit $b \to |b|_{\text{max}} = ((p+1)|c|)^{1/p}$, the wave widens and the peak flattens, similarly to the mKdV case.

4.2.4 Defocussing-gKdV kink waves

Kink wave solutions $U(\xi)$ arise from the potential well (4.29) with $J = 0$, where the two maximum points (4.30) have equal height $V_{\text{max}} = V(U_1) = V(-U_1)$. The potential has the factorization

$$V_{\text{max}} - V(U) = (U - U_1)^2 (U + U_1)^2 W(U), \tag{4.47}$$

where $W(U)$ is a polynomial that has even degree $p - 2$ and that is positive on $-U_1 \leq U \leq U_1$. In particular, the previous roots of the energy equation (4.41) coalesce into $U_\pm = -\text{sgn}(J)\, U_1$.

The previous quadrature (4.44) for $U(\xi)$ needs to be altered by changing the endpoint, because

$U_\pm$ is now a local maximum point. A convenient choice is the minimum point $U_2 = 0$ of the potential (4.30). Thus, we have

$$\int_0^U \frac{dU}{(U^2 - U_1^2)\sqrt{W(U)}} = \sqrt{2}\xi, \quad -U_1 \le U \le U_1 \tag{4.48}$$

When $p = 4, 6$, this integral can be evaluated in terms of elementary functions and elliptic functions, respectively, while for $p \geq 8$, the integral cannot be evaluated explicitly in general.

This quadrature (4.48) implicitly defines a family of the solitary wave solutions $U(\xi)$ which are even functions of ξ, with the following features. As $\xi \to \pm\infty$, the asymptotes are $U \to \mp((p+1)|c|)^{1/p}$. Hence the net change in amplitude (height) is $h = 2((p+1)|c|)^{1/p}$. The width is proportional to $w = 2/\sqrt{p|c|}$. Physically, the solution describes a *kink wave*.

Chapter 5

Conclusion

The results are obtained in this thesis:

- The energy analysis method is defined in five steps and applied on KdV, mKdV, and gKdV equations to find different types of travelling solutions. Using this method we can see that the integral constant J is equivalent to background.

- In case of the KdV equation, the maximum point coincide on the boundary of periodic orbits set and non-periodic orbits set. So, we can observe one solitary wave. Also, we showed that the solitary wave on a non-zero background can be obtained by superimposing the standard soliton profile on an arbitrary background with an adjustment of speed corresponding to that background.

- In case of the focussing mKdV equation, the maximum point coincide on the boundary of two sets of periodic orbits and we derived a pair of bright and dark solitary waves. The heavy-tail waves are special limit which decay slower than the solitary waves to the background. These types of waves arise the case that we have inflection point instead of maximum point.

- In case of the defocussing mKdV equation, although we have two maximum points, but the lower maximum point coincide on the boundary of periodic orbits set and non-periodic orbits set, like the KdV case, and the higher maximum point causes to have a fake turning point. Therefore, we could find a family of solitary waves on a non-zero background. In a special case that the maximum points have the same height, kink waves can be observed.

- In case of the gKdV equation, we split up into two odd power and even power cases. In both cases, we found the same behaviour of critical points as of the KdV and the mKdV equations, respectively.

For future work, it will be interesting to:

- study interactions of non-zero background KdV solitary waves as well as non-zero background mKdV waves and heavy-tailed mKdV waves, by using the integrability properties of these two equations;

- find all non-periodic travelling wave solutions for the Gardner equation $u_t + \alpha u u_x + \beta u^2 u_x + \gamma u_{xxx} = 0$, and study their interactions.

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