

Linear Forms in Logarithms and Fibonacci Numbers

Benjamin Earp-Lynch

Department of Mathematics and Statistics

Submitted in partial fulfillment
of the requirements for the degree of

Master of Science

Faculty of Mathematics and Science, Brock University
St. Catharines, Ontario

©2019

Abstract

The main work included in these pages is from a paper co-written by myself and my brother, Simon Earp-Lynch, under the supervision of Omar Kihel, pertaining to Diophantine triples of Fibonacci numbers. To go along with this will be introductory material not included in said paper which establishes the mathematical concepts therein and offers some historical perspective and motivation.

The initial aim of the paper was to explore the possibility of a generalization of the main result in [2] on $D(4)$ -Diophantine triples of Fibonacci numbers. The paper managed to extend the ideas in [2] to results for $D(9)$ -Diophantine triples and $D(64)$ -Diophantine triples. A generalization of Lemma 1 of [1] was also found, a lemma on Diophantine triples and Pellian equations which is key in establishing the main result in [2]. This paper includes this result and its proof, which involves a correction of the proof of Lemma 1 of [1]. This result may prove useful in the extension of the results in the paper, and potentially others as well.

I will begin by introducing Diophantine equations, leading to Diophantine triples, followed by a section on the necessary preliminaries on Fibonacci numbers, which concludes with the statements of our main results. Following this, I establish the primary machinery used in the proof of the main result, linear forms in logarithms. I then move to the generalization of the aforementioned Lemma 1 of [1], before finally commencing the proof of the main results.

Key words and phrases: Linear forms and logarithms; Fibonacci numbers; Diophantine triples; Pellian equations.

Acknowledgements

I would like to thank those who provided advice in the shaping of this paper: my supervisor Dr. Omar Kihel, as well as the members of my thesis committee, Dr. Beatrice Ombuki, Dr. Hichem Ben-el-Mechaiekh and Dr. Bill Ralph. Additional thanks to Jesse Larone.

Contents

1	Introduction	iv
1.1	Diophantine Equations	iv
1.2	Diophantine Triples	vi
1.3	Fibonacci Numbers	vi
2	Linear Forms in Logarithms	viii
2.1	Preliminaries	viii
2.2	Baker's Theorem and Baker-Davenport Reduction	x
2.3	Lemmas of MATEEV and LAURENT	xii
2.4	Useful Lemmas	xiv
3	Lemma on Pellian Equations	xvi
3.1	Case when $0 \leq c' < b$	xix
3.2	We may assume $c' < b$	xx
4	Proving Theorem 1.4	xxii
4.1	Setting up the $D(9)$ -Diophantine Triple	xxii
4.2	A Linear Form in Three Logarithms	xxiv
4.3	Linear Form in Two Logarithms	xxix
4.4	Refining Our Bounds	xxxii
4.5	Reduction of the Bounds	xxxiv
5	Proving Theorem 1.5	xxxv
5.1	Setting up the $D(64)$ -Diophantine triple	xxxv
5.2	A Linear form in Three Logarithms (2)	xxxvi
5.3	A Linear Form in Two Logarithms (2)	xl
5.4	Refining Our Bounds	xliii
5.5	Baker-Davenport Reduction	xlvi
6	Conclusion	xlvi
A	D(9) Baker-Davenport Reduction Calculations	xlix
B	D(64) Baker-Davenport Reduction Calculations	lii

1 Introduction

1.1 Diophantine Equations

A Pythagorean triple is a set of three positive integers $\{a, b, c\}$ such that

$$a^2 + b^2 = c^2.$$

Though named after the Greek mathematician Pythagoras (who is famously associated with, but may not have proven, what is known as Pythagoras's theorem regarding the length of the sides of a right angled triangle), there is historical record of the existence of Pythagorean triples being known in Sumer more than 1200 years before his birth (see [24]).

The equation $a^2 + b^2 = c^2$ where a, b and c are unknown integers is one of the earliest known examples of a Diophantine equation (a polynomial of two or more variables to which we seek integer (or possibly rational) solutions), named after another Greek mathematician, Diophantus. Diophantus produced a prodigious amount of work, on what are now known as Diophantine equations, in the form of volumes of problems. His work (that which survived the fire of Alexandria) had a profound influence on subsequent mathematicians, notably Pierre de Fermat. Fermat, a French mathematician, famously conjectured, in the margin of his copy of Diophantus's work, that there are no solutions to equations similar to Pythagoras's with exponent $n > 2$. That is, there do not exist positive integers $\{a, b, c\}$, $n > 2$, for which

$$a^n + b^n = c^n.$$

This came to be known as Fermat's last theorem. Though Fermat did prove this for the case of $n = 4$, it was not until 1994 (published in 1995), after centuries of fruitful failure, that Andrew Wiles and Richard Taylor proved it for all $n > 2$ in [14]. For more on the history of Fermat's Last Theorem, see [15].

Fermat's last theorem, though it is of interest as a historical problem and for the mathematics that it inspired, is far from the last word on Diophantine equations. Indeed, the study of Diophantine equations remains a very active branch of number theory. The following theorem is a result of Thue ([23]) on what are known as Thue equations, another kind of Diophantine equation. This theorem of Thue was later improved upon by Baker, who bounded the size of the solutions using linear forms in logarithms (see [9]).

Theorem 1.1. *Let*

$$a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0 \in \mathbb{Z}[X]$$

be an irreducible polynomial over \mathbb{Q} of degree $d \geq 3$. Then, for every nonzero integer m , the Diophantine equation

$$a_d X^d + a_{d-1} X^{d-1} Y + \cdots + a_1 X Y^{d-1} + a_0 Y^d = m$$

has only finitely many integer solutions $(X = p, Y = q)$.

Recall that a polynomial is irreducible over \mathbb{Q} provided it cannot be factored over \mathbb{Q} . In examining a particular case of the above theorem, we see that equations of the form $X^3 - dY^3 = 1$ with an arbitrary integer d have at most finitely many integer solutions. However, equations of the form $X^2 - dY^2 = 1$, where d is a positive square-free integer, are not classified in the above theorem, and have infinitely many integer solutions. Such equations are called Pellian equations, and their (positive) solutions are nicely classified in the following theorem of Nagell (see [12]), upon which later results rely.

Theorem 1.2. *If d is a square-free positive integer, the Diophantine equation*

$$X^2 - dY^2 = 1$$

has infinitely many solutions $X + Y\sqrt{d}$. All solutions with positive X and Y are obtained by the formula

$$X_n + Y_n \sqrt{d} = (X_1 + Y_1 \sqrt{d})^n$$

where $X_1 + Y_1 \sqrt{d}$ is the fundamental solution of $X^2 - dY^2 = 1$, n runs through all positive integers, and where

$$X_n = X_1^n + \sum_{k=1}^n \binom{n}{2k} X_1^{n-2k} Y_1^{2k} d^k,$$

$$Y_n = \sum_{k=1}^n \binom{n}{2k-1} X_1^{n-2k+1} Y_1^{2k-1} d^{k-1}.$$

Here the **fundamental solution**, $X_1 + Y_1 \sqrt{d}$, is the least positive solution of the Pellian equation.

1.2 Diophantine Triples

Another product of the work of Diophantus is the Diophantine n -tuple. In his work, Diophantus found a set of rational numbers,

$$\left\{ \frac{1}{16}, \frac{33}{16}, \frac{68}{16}, \frac{105}{16} \right\},$$

for which the product of any two plus one is a square of another rational number. Fermat considered similar sets of integers, finding the set $\{1, 3, 8, 120\}$, which is thought to be the first example of a $D(1)$ -quadruple (see [16] for more on Fermat and the problems of Diophantus).

Definition 1.3. We call a set of n positive integers, $\{a_1, \dots, a_n\}$ a **$D(l)$ -Diophantine n -tuple** ($l \in \mathbb{Z}$) if $a_i a_j + l$ is a perfect square for all $i \neq j$ in $\{1, \dots, n\}$.

Though rational Diophantine n -tuples exist, as seen above with the work of Diophantus, it is the case of integers that is of interest for the purposes of this paper. Much of the existing work on $D(l)$ -Diophantine n -tuples pertains to the case where $l = 1$ (see, for example, [9] and [10]). For instance, a result (formerly a long-standing conjecture) that was proven in 2019 by He, Togbé and Ziegler ([22]) states that there are no $D(1)$ -Diophantine quintuples (and hence no $D(1)$ -Diophantine n -tuples for $n \geq 5$). $D(l)$ -Diophantine n -tuples where $l = -1$ and $l = 4$ have also been studied (see [1], [2], [4], [5], and [9]) as have further cases, though all less extensively than the case when $l = 1$.

1.3 Fibonacci Numbers

Like simple Diophantine equations, there is historical record of the idea of Fibonacci numbers (which are expressed in a sequence known as the Fibonacci sequence) having been known to humans for thousands of years, notably in India (see [21]). The Italian Leonardo Bonacci, who, years after his death, came to be known simply as Fibonacci, introduced the Fibonacci sequence to Europe, having been influenced by Hindu-Arabic mathematics. Fibonacci's book, *Liber Abaci*, played a large role in inspiring Europeans to adopt the Hindu-Arabic numeral system that we use today. For more on the history of Fibonacci and Fibonacci numbers, see [17].

The Fibonacci sequence is defined as follows, where F_n denotes the n 'th Fibonacci number:

$$F_1 = 1, \quad F_2 = 1, \quad F_{n+1} = F_{n-1} + F_n.$$

The Lucas numbers, named after the 19th century French mathematician Édouard Lucas, are defined similarly, where L_n denotes the n 'th Lucas number:

$$L_1 = 1, \quad L_2 = 3, \quad L_{n+1} = L_{n-1} + L_n.$$

The Fibonacci numbers and Lucas numbers have many remarkable properties, indeed far too many to mention here. If we let

$$\alpha = \frac{1 + \sqrt{5}}{2}, \quad \bar{\alpha} = \frac{1 - \sqrt{5}}{2}$$

then Binet's formula gives us:

$$F_k = \frac{\alpha^k - \bar{\alpha}^k}{\sqrt{5}}, \quad \text{and} \quad L_k = \alpha^k + \bar{\alpha}^k \quad \text{for all } k \geq 1.$$

The above α is often referred to as the golden ratio. It is a value that has had great significance in mathematics throughout history, particularly in the connection between mathematics and art.

A property of the Fibonacci numbers that is of particular note is called Catalan's identity,

$$F_n^2 - F_{n+r}F_{n-r} = (-1)^{n-r}F_r^2,$$

which shows a connection between the Fibonacci sequence and Diophantine triples. For example, for every $n \geq 1$, we have the $D(1)$ -triple,

$$\{F_{2n}, F_{2n+2}, F_{2n+4}\},$$

which is examined in many places, including [11]. Likewise we have the $D(4)$ -triple,

$$\{F_{2n}, F_{2n+6}, 4F_{2n+4}\},$$

examined in [2], and [4], which is a special case of an application of Catalan's identity, and only holds since $F_1 = F_2 = 1$. More generally, Catalan's identity and the additional identity:

$$F_{2n} = F_n L_n$$

shows that for any non-negative integer n and positive integer m ,

$$\{F_{2n}, L_m^2 F_{2n+2m}, F_{2n+4m}\}$$

is a $D(F_{2m}^2)$ Diophantine Triple.

The next section introduces the results on linear forms in logarithms that are needed for the main proofs. Following this, a small mistake in the proof of Lemma 1 of [1] is addressed (see Section 3 in this paper). This solidifies the result in [2], where this lemma was used. Then, through a slight modification of Lemma 1 in [1], combined with application of the lemmas stated in Section 2, the following Theorems are proven:

Theorem 1.4. *If $\{F_{2n+8}, 9F_{2n+4}, F_k\}$ is a $D(9)$ Diophantine Triple, then if $n > 1$, we must have $k = 2n$. If $n = 1$, we have $F_1 = 1 = F_2$, so $k = 1$ and $k = 2$ are both solutions.*

Theorem 1.5. *If $\{16F_{2n+6}, F_{2n+12}, F_k\}$ is a $D(64)$ Diophantine Triple, then if $n > 1$, we must have $k = 2n$. If $n = 1$, we have $F_1 = 1 = F_2$, so $k = 1$ and $k = 2$ are both solutions.*

In other words, for the pair $\{L_m^2 F_{2n+2m}, F_{2n+4m}\}$, when $m = 2$ or 3 , the only Fibonacci number that extends the pair to a $D(F_{2m}^2)$ -Diophantine triple is F_{2n} (with one exception in each case, as stated above).

2 Linear Forms in Logarithms

To arrive at the main results of this paper (Theorems (1.4) and (1.5)), it is first necessary to introduce the main tool, linear forms in logarithms. The theorems involving linear forms in logarithms will be used to give a bound on our possible solutions. This bound can then be improved to a point that it is possible to determine all possible solutions with a computer.

The following sections will introduce and define some basic concepts in algebra and number theory, the understanding of which is required for the subsequent results on linear forms in logarithms.

2.1 Preliminaries

Definition 2.1. *An infinite generalized continued fraction is an expression of the form*

$$a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{\ddots}}}$$

where a_0, a_1, a_2, \dots and b_1, b_2, \dots are either rational, real or complex numbers or functions of such variables.

Definition 2.2. For $b_i = 1, i \in \mathbb{N}$, the expression in 2.1 is called an **infinite simple continued fraction**. Its abbreviated notation is

$$[a_0; a_1, a_2, \dots].$$

Example 2.3. The golden ratio $\alpha = \frac{1 + \sqrt{5}}{2}$ can be expressed as a continued fraction:

$$1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\ddots}}}$$

In abbreviated notation, we can express α by

$$[1; 1, 1, \dots].$$

It is possible to approximate irrational numbers by way of continued fractions. This is done using what are known as **convergents**.

Definition 2.4. An expression

$$[a_0; a_1, a_2, \dots, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}}$$

is called a **finite simple continued fraction** with $a_i \geq 1, i = 1, \dots, n$ and $a_n \geq 2$ integers that are called **partial quotients**. The rational numbers

$$\frac{p_0}{q_0} = [a_0], \frac{p_1}{q_1} = [a_0; a_1], \frac{p_2}{q_2} = [a_0; a_1, a_2], \dots, \frac{p_n}{q_n} = [a_0; a_1, a_2, \dots, a_n]$$

are called the **convergents** and n is its length.

For an irrational number ϕ with abbreviated infinite simple continued fraction represented by $[a_0, a_1, a_2, \dots]$, each convergent gives a rational approximation of ϕ . Note that for an irrational number, the size of both the numerator and the denominator of the rational n 'th convergent increases as n increases. The rational

approximations of ϕ also become better as n increases, meaning that the difference between ϕ and the n 'th convergent is less than that between ϕ and the $n - 1$ 'th convergent. The previously mentioned golden ratio, however, is the irrational which is worst approximated by its convergents. We see in the following lemma that the smaller the subsequent partial quotient a_{r+1} , the worse $\frac{p_r}{q_r}$ is as an approximation of an irrational number ϕ . Since for the golden ratio, every a_i is 1, it has the smallest possible partial quotients.

Lemma 2.5. *Let ϕ be an irrational number, $\frac{p_r}{q_r}$ its r th convergent and a_{r+1} its $r + 1$ st partial quotient. The following inequality holds:*

$$\frac{1}{q_r^2(a_{r+1} + 2)} < \left| \phi - \frac{p_r}{q_r} \right| \leq \frac{1}{q_r^2 a_{r+1}}.$$

The following theorem due to Legendre will also be used (for a proof, see [9]).

Theorem 2.6. *Let p, q be integers such that $q \geq 1$ and*

$$\left| \phi - \frac{p}{q} \right| < \frac{1}{2q^2}.$$

Then $\frac{p}{q}$ is a convergent of ϕ .

For more on continued fractions, see [6] and [9].

2.2 Baker's Theorem and Baker-Davenport Reduction

The methods we use involving linear forms in logarithms arose from the study of transcendental numbers.

Definition 2.7. *A real number σ is called **algebraic** over \mathbb{Q} if it is a root of a polynomial equation with coefficients in \mathbb{Q} . A real number is said to be **transcendental** if it is not algebraic.*

In 1900, at the International Congress of Mathematicians in Paris, David Hilbert began presenting what would come to be a list of twenty three unsolved problems. He suggested these problems would have important implications in mathematics in the coming century.

The seventh of these problems concerns the irrationality and transcendence of certain numbers, and asks specifically whether the number σ^β is transcendental for algebraic $\sigma \neq 0, 1$ and irrational algebraic β . This problem was solved in 1935 by

Gelfond and Schneider, independently. On their way to solving Hilbert's seventh problem, they proved that if $\sigma_1, \sigma_2 \neq 0$ are algebraic numbers such that $\log \sigma_1, \log \sigma_2$ are linearly independent over \mathbb{Q} , then

$$\beta_1 \log \sigma_1 + \beta_2 \log \sigma_2 \neq 0$$

for all algebraic numbers β_1, β_2 . Gelfond also found a lower bound for the absolute value of the above linear form. See [9] for more on Gelfond and Schneider and Hilbert's seventh problem.

Linear independence is a condition for most linear form in logarithm results.

Definition 2.8. *Let $\sigma_1, \sigma_2, \dots, \sigma_n$ be n (real or complex) numbers. We call $\sigma_1, \sigma_2, \dots, \sigma_n$ **linearly dependent** over the rationals (equivalently integers) if there are rational numbers (integers) r_1, r_2, \dots, r_n , not all zero, such that*

$$r_1 \sigma_1 + r_2 \sigma_2 + \dots + r_n \sigma_n = 0$$

*If $\sigma_1, \sigma_2, \dots, \sigma_n$ are not linearly dependent over the rationals (integers), they are **linearly independent** over the rationals (integers).*

This linear form of Gelfond and Schneider is the first example of what would become known as a linear form in logarithms. In that case, it was a linear form in two logarithms. The idea would be generalized to linear forms in n logarithms.

Definition 2.9. *A **linear form in logarithms** of algebraic numbers is an expression of the form*

$$\Lambda = \beta_0 + \beta_1 \log \sigma_1 + \beta_2 \log \sigma_2 + \dots + \beta_n \log \sigma_n$$

where $\sigma_i, i = 1, \dots, n$ and $\beta_i, i = 1, \dots, n$ are complex algebraic numbers and \log denotes any determination of the logarithm.

Alan Baker was awarded a Fields Medal in 1970, in large part due to his work in three papers ([18], [19], [20]), in which he generalized the Gelfond-Schneider theorem to arbitrarily many logarithms.

Theorem 2.10. *Baker 1966 If $\sigma_1, \sigma_2, \dots, \sigma_n \neq 0, 1$ are algebraic numbers such that $\log \sigma_1, \log \sigma_2, \dots, \log \sigma_n, 2\pi i$ are linearly independent over the rationals, then*

$$\beta_0 + \beta_1 \log \sigma_1 + \beta_2 \log \sigma_2 + \dots + \beta_n \log \sigma_n \neq 0$$

for any algebraic numbers $\beta_0, \beta_1, \dots, \beta_n$ that are not all zero.

Baker used this theorem to generalize the solution to Hilbert's problem, showing the transcendence of the number $e^{\beta_0} \sigma_1^{\beta_1} \dots \sigma_n^{\beta_n}$ for algebraic σ_i and β_i . His work inspired many subsequent results in transcendental number theory and Diophantine analysis. In this paper, several of the results on linear forms in logarithms which arose from Baker's are used, including a modified version of the original Baker-Davenport reduction method due to Dujella and Pethő, named after Baker and his supervisor, Harold Davenport (See [3], Lemma 5a, and Lemma 2.3 of [2]).

Lemma 2.11. *Assume that κ and μ are real numbers and M is a positive integer. Let P/Q be a convergent of the continued fraction expansion of κ such that $Q > 6M$ and let*

$$\eta = \|\mu Q\| - M \cdot \|\kappa Q\|,$$

where $\|\cdot\|$ denotes the distance from the nearest integer. If $\eta > 0$, then there is no solution of the inequality

$$0 < j\kappa - k + \mu < AB^{-j}$$

in integers j and k with

$$\frac{\log(AQ/\eta)}{\log B} \leq j \leq M.$$

Here $\|x\|$ denotes the distance from x to the nearest integer.

This lemma will play a key role in lowering the bound on the number of possible solutions in the proofs of the main results.

2.3 Lemmas of Mateev and Laurent

The first two results listed here are identical to Lemmas 2.1 and 2.2 of [2]. See also [7] and [8]. It is first necessary to clarify some terminology.

Definition 2.12. *The **minimal polynomial** p of an algebraic number σ over \mathbb{Q} is the uniquely determined irreducible monic polynomial of minimal degree with rational coefficients satisfying*

$$p(\alpha) = 0.$$

*Elements that are algebraic over \mathbb{Q} and have the same minimal polynomial are called **conjugates** over \mathbb{Q} .*

For any non-zero algebraic number γ of degree d (the degree of its minimal polynomial) over \mathbb{Q} whose minimal polynomial over \mathbb{Z} is $a \prod_{j=1}^d (X - \gamma^{(j)})$, where the $\gamma^{(j)}$

are the conjugates of γ , we denote by

$$h(\gamma) = \frac{1}{d} \left(\log a + \sum_{j=1}^d \log \max(1, |\gamma^{(j)}|) \right)$$

its **absolute logarithmic height**. We will use the following result due to Mátéev [8] to establish a bound on a linear form in three logarithms.

Lemma 2.13. *Let Λ be a linear form in logarithms of multiplicatively independent totally real algebraic numbers $\alpha_1, \dots, \alpha_N$ with rational integer coefficients b_1, \dots, b_N ($b \neq 0$). Let $h(\alpha_j)$ denote the absolute logarithmic height of α_j for $1 \leq j \leq N$. Define the numbers D, A_j ($1 \leq j \leq N$) and E by $D := [\mathbb{Q}(\alpha_1, \dots, \alpha_N) : \mathbb{Q}]$, $A_j = \max\{Dh(\alpha_j), |\log \alpha_j|\}$, $E = \max\{1, \max\{|b_j|A_j/A_N; 1 \leq j \leq N\}\}$. Then*

$$\log |\Lambda| > -C(N)C_0W_0D^2\Omega,$$

where

$$\begin{aligned} C(N) &:= \frac{8}{(N-1)!} (N+2)(2N+3)(4e(N+1))^{N+1}, \\ C_0 &:= \log(e^{4.4N+7}N^{5.5}D^2 \log(eD)), \\ W_0 &:= \log(1.5eED \log(eD)), \quad \Omega = A_1 \dots A_N. \end{aligned}$$

Here $[\mathbb{Q}(\alpha_1, \dots, \alpha_N) : \mathbb{Q}]$ denotes the degree of the field extension $\mathbb{Q}(\alpha_1, \dots, \alpha_N)$ over \mathbb{Q} . We use the following result of Laurent [7] in order to bound a linear form in two logarithms.

Lemma 2.14. *Let $\gamma_1 > 1$ and $\gamma_2 > 1$ be two real multiplicatively independent algebraic numbers, $b_1, b_2 \in \mathbb{Z}$ not both 0 and*

$$\Lambda = b_2 \log \gamma_2 - b_1 \log \gamma_1.$$

Let $D := [\mathbb{Q}(\gamma_1, \gamma_2) : \mathbb{Q}]$. Let

$$h_i \geq \max \left\{ h(\gamma_i), \frac{|\log \gamma_i|}{D}, \frac{1}{D} \right\} \text{ For } i = 1, 2, \quad b' \geq \frac{|b_1|}{Dh_2} + \frac{|b_2|}{Dh_1}.$$

Then

$$\log |\Lambda| \geq -17.9 \cdot D^4 \left(\max \left\{ \log b' + 0.38, \frac{30}{D}, 1 \right\} \right)^2 h_1 h_2.$$

2.4 Useful Lemmas

The following two lemmas will be useful in applying 2.13.

Lemma 2.15. $F_{2n+4}F_{2n+8}$ is neither a square nor 5 times a square.

Proof. To see that $F_{2n+4}F_{2n+8}$ is not a square, simply note that $F_{2n+4}F_{2n+8} + 1 = F_{2n+6}^2$. Suppose that $F_{2n+4}F_{2n+8}$ is 5 times a square, so $F_{2n+4}F_{2n+8} = 5y^2$ for some integer y . Then we must have that $5y^2 + 1 = F_{2n+6}^2$. We examine the solutions of the Pellian equation

$$X^2 - 5Y^2 = 1, \quad (1)$$

with our interest lying in those solutions $X + Y\sqrt{5}$ for which $X = F_{2n+6}$ for some positive integer n .

This Pellian equation has fundamental solution $9 + 4\sqrt{5} = \alpha^6$, where here $\alpha = \frac{1 + \sqrt{5}}{2}$. Hence all solutions to equation (1) have the form $(9 + 4\sqrt{5})^j$ for some $j \in \mathbb{Z}^+$.

If we let $X_j + Y_j\sqrt{5} = (9 + 4\sqrt{5})^j$, then $X_j = \frac{(9 + 4\sqrt{5})^j + (9 + 4\sqrt{5})^{-j}}{2} = \frac{\alpha^{6j} + \alpha^{-6j}}{2}$.

Since $F_k = \frac{\alpha^k - (\frac{-1}{\alpha})^k}{\sqrt{5}}$, our aim becomes to solve the equation

$$\frac{\alpha^{6j} + \alpha^{-6j}}{2} = \frac{\alpha^{2n+6} - \alpha^{-2n-6}}{\sqrt{5}} \quad (2)$$

for $n, j \in \mathbb{Z}^+$.

We see here that for $n \geq 1$

$$\begin{aligned} \frac{\alpha^{2n+4} + \alpha^{-2n-4}}{2} &< \frac{\alpha^4}{2}\alpha^{2n} + \frac{1}{2} < 8\alpha^{2n} - 1 \\ &< \frac{\alpha^6\alpha^{2n} - \alpha^{-2n-6}}{\sqrt{5}} = \frac{\alpha^{2n+6} - \alpha^{-2n-6}}{\sqrt{5}}, \end{aligned}$$

hence we must have $j > \frac{n+2}{3}$. However,

$$\frac{\alpha^{2n+6} + \alpha^{-2n-6}}{2} > \frac{\alpha^{2n+6} + \alpha^{-2n-6}}{\sqrt{5}} > \frac{\alpha^{2n+6} - \alpha^{-2n-6}}{\sqrt{5}},$$

which means $j < \frac{n+3}{3}$.

The bounds $\frac{n+2}{3} < j < \frac{n+3}{3}$ mean that j cannot be an integer, which is a contradiction. Thus there is no solution to the Pellian equation (1) wherein X is a Fibonacci number with index $2n + 6, n \in \mathbb{Z}^+$, which means that $F_{2n+4}F_{2n+8}$ is neither a square nor 5 times a square. \square

Lemma 2.16. $F_{2n+6}F_{2n+12}$ is neither a square nor 5 times a square.

Proof. To see that it is not itself a square, observe that $F_{2n+6}F_{2n+12} + 4 = F_{2n+9}^2$, and the difference between any two nonzero, nonidentical squares is either odd or bigger than 4. Suppose that $F_{2n+6}F_{2n+12}$ is 5 times a square, so $F_{2n+6}F_{2n+12} = 5y^2$ for some integer y . We're interested in solutions to the Pellian equation

$$X^2 - 5Y^2 = 4 \quad (3)$$

with $X = F_{2n+9}$ for some $n \in \mathbb{Z}^+$. Since the fundamental solution of $X^2 - 5Y^2 = 1$ is $9 + 4\sqrt{5}$, a theorem of Nagell (see [12]) gives us that all classes of solutions to equation (3) have fundamental solution $u + v\sqrt{5}$ with

$$0 < v \leq \frac{4}{\sqrt{2(9+1)}}\sqrt{4} = \frac{4}{\sqrt{5}} < 2 \text{ and}$$

$$0 \leq |u| \leq \sqrt{\frac{1}{2}(9+1) \cdot 4} = 2\sqrt{5} < 5,$$

so all solutions to (3) are of the form

$$(\pm 3 + \sqrt{5})(9 + 4\sqrt{5})^j$$

for some integer $j \geq 1$. If we let $V_j + U_j\sqrt{5} = (9 + 4\sqrt{5})^j$, then this means that we have solutions $X_j + Y_j\sqrt{5}$ where $X_j = \pm 3V_j + 5U_j$ and $Y_j = V_j \pm 3U_j$. This means that we want to find solutions (j, n) to the equation

$$X_j = F_{2n+9} = \frac{\alpha^{2n+9} - \left(\frac{-1}{\alpha}\right)^{2n+9}}{\sqrt{5}} = \frac{\alpha^{2n+9} + \alpha^{-2n-9}}{\sqrt{5}}.$$

Since $(9 + 4\sqrt{5}) = \alpha^6$, we have $V_j = \frac{\alpha^{6j} + \alpha^{-6j}}{2}$, our aim is to solve the following

$$\pm 3\left(\frac{\alpha^{6j} + \alpha^{-6j}}{2}\right) + 5\left(\frac{\alpha^{6j} - \alpha^{-6j}}{2\sqrt{5}}\right) = \frac{\alpha^{2n+9} + \alpha^{-2n-9}}{\sqrt{5}}.$$

Noting that $-3\sqrt{5} + 5$ and $-3\sqrt{5} - 5$ are both less than 0 and will never yield a solution, we obtain the following after cancelling denominators:

$$(3\sqrt{5} + 5)\alpha^{6j} + (3\sqrt{5} - 5)\alpha^{6j} = 2\alpha^{2n+9} + 2\alpha^{-2n-9}. \quad (4)$$

We see from a brief observation that

$$(3\sqrt{5} + 5)\alpha^{2n+6} + (3\sqrt{5} - 5)\alpha^{-2n-6} > 2.76\alpha^{2n+9} + 2.76\alpha^{-2n-9} > 2\alpha^{2n+9} + 2\alpha^{-2n-9},$$

meaning that $j < \frac{n+3}{3}$, and

$$(3\sqrt{5}+5)\alpha^{2n+4}+(3\sqrt{5}-5)\alpha^{-2n-4} < 1.06\alpha^{2n+9}+18.95\alpha^{2n-9} < 2\alpha^{2n+9} < 2\alpha^{2n+9}+2\alpha^{-2n-9}$$

which gives us $\frac{n+2}{3} < j < \frac{n+3}{3}$, contradicting $j \in \mathbb{Z}$. Thus we see that $F_{2n+6}F_{2n+12}$ is neither a square nor 5 times a square. \square

One may note here that the previous two results become almost trivial given the veracity of a particular outstanding conjecture. It is known by Carmichael's Theorem that every Fibonacci number aside from $F_6 = 8$ and $F_{12} = 144$ contains at least one prime factor not appearing in any previous number in the sequence, a primitive prime factor (see [27]). It is conjectured that primitive prime factors will always appear to the first power (see [25], [26]). This of course would tell us that, aside from a few low-indexed exceptions, we have $F_n F_m$ is neither a square nor 5 times a square.

3 Lemma on Pellian Equations

In this section, we prove a lemma regarding Pellian equations that will be crucial in setting up the proof of the main result. The lemma is a version of Lemma 1 in [1], which was for $D(4)$ triples, and its proof is very similar. However, there was a small mistake in the proof found in [1] which is amended here. Due to the use of this lemma in [2], this section will serve to legitimize the results in that paper as well.

Let l be a positive integer and $\{a, b, c\}$ be a $D(l^2)$ Diophantine triple, i.e. there exist positive integers r, s, t such that

$$ab + l^2 = r^2, \quad ac + l^2 = s^2, \quad \text{and} \quad bc + l^2 = t^2.$$

Lemma 3.1. *Let $\{a, b, c\}$ is a $D(l^2)$ triple with $a < b < a\left(4 + \frac{4}{l^2}\right)$ and assume that if l is an odd prime then $l \mid ab$ and if l is not prime then $l^2 \mid a$ or $l^2 \mid b$. (If $l = 2$ these divisibility conditions are unnecessary). Then all solutions of the equation*

$$at^2 - bs^2 = l^2(a - b) \tag{5}$$

are of the form

$$t\sqrt{a} + s\sqrt{b} = (\pm l\sqrt{a} + l\sqrt{b})\left(\frac{r + \sqrt{ab}}{l}\right)^\nu,$$

where $\nu \in \mathbb{Z}^+$.

Proof. As previously mentioned, the proof is very similar to that of Lemma 1 in [1]; however, allowances are made for $l > 2$ and a mistake in the proof of that Lemma is fixed.

Define $s' = \frac{rs - at}{l}$, $t' = \frac{rt - bs}{l}$ and $c' = \frac{(s')^2 - l^2}{a}$. The triple $\{a, b, c'\}$ is also a $D(l^2)$ -Diophantine triple. In the case of $l = 2$ (the lemma found in [1]), $2 \mid (rs - at)$ and $2 \mid (rt - bs)$ no matter whether a and b are both odd or not. For $l > 2$ this may not be true, hence our divisibility requirement. Since if l is prime, $l \mid ab$ and if not, $l^2 \mid a$ or $l^2 \mid b$, we must have that $l \mid r$, and in addition that $l \mid s$ or $l \mid t$, thus s' and t' are always integers. Moreover, since

$$t\sqrt{a} + s\sqrt{b} = (t'\sqrt{a} + s'\sqrt{b})\left(\frac{r + \sqrt{ab}}{k}\right),$$

(t', s') belongs to the *same class* of solutions of (5) as (t, s) , thus we can replace $c = c'$ and follow the process again with the triple $\{a, b, c'\}$ while always remaining in the same class of solutions. This will be the key to our proof. The above information means that if we let

$$t_\nu^\pm \sqrt{a} + s_\nu^\pm \sqrt{b} = (\pm l \sqrt{a} + l \sqrt{b}) \left(\frac{r + \sqrt{ab}}{l}\right)^\nu,$$

then if after a certain number of times repeating the process of finding c' and replacing $c = c'$ in our triple, we have that $(t', s') = (\pm l, l)$, or equivalently, if $c' = a + b \pm 2r$, then we must have that $(t, s) = (t_\nu^\pm, s_\nu^\pm)$ for some positive integer ν . Thus at this point the proof will be complete.

We begin with a pair of useful facts:

Remark. s' is always positive.

To see this, observe the Pellian equation:

$$bs^2 - at^2 = l^2(b - a)$$

obtained by multiplying (5) by -1 . Multiplying by a , we get

$$abs^2 - a^2t^2 = (r^2 - l^2)s^2 - a^2t^2 = r^2s^2 - a^2t^2 - l^2s^2 = l^2a(b - a)$$

which means that

$$(rs - at)(rs + at) = l^2a(b - a) + l^2s^2.$$

Since r, s, a, t, l are all positive and $0 < a < b$, we must have that $rs - at = ls' > 0$.

Next we show that

Remark. $c' \geq 0$.

If l is prime and $l \mid a$, then $l \mid ab + l^2 = r^2$, which means $l \mid r$ and similarly, $l \mid s$. If l also divides b , then $l \mid t$, and so $l^2 \mid rs - at$, meaning $s' \geq l$, and so $c' \geq 0$. If $l \nmid b$, then $ab + l^2 = r^2$ implies $l^2 \mid a$, giving $l^2 \mid rs - at$, and again $c' \geq 0$. A similar process proves the remark in the case l prime and $l \mid b$.

If l is composite and $l^2 \mid a$, then $l^2 \mid ab + l^2 = r^2$ and $l^2 \mid ac + l^2 = s^2$. So $l^2 \mid rs - at$, which means that

$$l \leq \frac{rs - at}{l} = s',$$

and $c' = \frac{(s')^2 - l^2}{a} \geq 0$.

Similarly if $l^2 \mid b$, then $l^2 \mid ab + l^2 = r^2$ and $l^2 \mid bc + l^2 = t^2$. So $l^2 \mid rt - bs$, which means that

$$l \leq \left| \frac{rt - bs}{l} \right| = |t'|.$$

Therefore $c' = \frac{(t')^2 - l^2}{b} \geq 0$.

If $l = 2$, we show that $c' \geq 0$ regardless of whether it divides a or b . If $2 \mid ab$, then the proof is the same as above since we have l prime and $l \mid ab$. Suppose that both a and b are odd and suppose for a contradiction that $c' < 0$. This means that $s'^2 - 4 < 0$, which is equivalent to $s' < 2$. Since s' is always positive, this must mean that $s' = 1$, so we set $\frac{rs - at}{2} = 1$. Multiplying by 2 and adding at to each side, then squaring, we get $r^2s^2 = (2 + at)^2$, giving $a^2bc + 4ab + 4ac + 16 = 4 + 4at + a^2bc + 4a^2$. We subtract a^2bc , divide by 4 and rearrange to obtain

$$-3 = a(b + c - a - t).$$

Since we assumed that a and b are both $1 \pmod{2}$, this must mean that $c - t \equiv 1 \pmod{2}$. However because of the fact that $b \equiv 1 \pmod{2}$, we must have that $t^2 = bc + 4 \equiv c \pmod{2}$, and so $t \equiv c \pmod{2}$. Therefore $a(b + c - a - t) \equiv 0 \pmod{2}$ and we have obtained a contradiction. Hence $c' \geq 0$ when $l = 2$.

3.1 Case when $0 \leq c' < b$

If $c' = 0$, then $s' = t' = l$. So $c = a + b + c' + \frac{2}{l^2}(abc' + rs't') = a + b + 2r$, and the proof is complete.

Suppose that $0 < c' < b$. Define $r' = (s')' = \frac{rs' - at'}{l}$ and $b' = (c')' = \frac{(r')^2 - l^2}{a}$. Then $b' = (c')' = a + b + c' + \frac{2}{l^2}(abc' - rs't')$. We have that $\{a, (c')', c', b\}$ is a regular $D(l^2)$ -quadruple. Indeed

$$\begin{aligned} ab + l^2 &= r^2, & ac' + l^2 &= (s')^2, & bc' + l^2 &= (t')^2 \\ ab' + l^2 &= (r')^2, & bb' + l^2 &= (q')^2, & b'c' + l^2 &= u^2 \end{aligned}$$

where

$$u = \frac{s't' - rc'}{l} \quad \text{and} \quad q' = (t')' = \frac{rt' - bs'}{l} \quad \text{both} \in \mathbb{Z}.$$

Because $\{a, c', b\}$ is a $D(l^2)$ -triple, we have

$$ac' + l^2 = (s')^2, \quad ab + l^2 = r^2, \quad bc' + l^2 = q^2,$$

and

$$\begin{aligned} r &= \frac{(s')r' + aq'}{l}, & z &= \frac{(s')q' + cr'}{l} \\ r' &= \frac{(s')r - aq}{l}, & q' &= \frac{(s')q - c'r}{l} \end{aligned}$$

It can be seen that

$$\begin{aligned} ab + l^2 = r^2 &= \left(\frac{(s')r' + aq'}{l} \right)^2 = \frac{1}{l^2}((s')^2(ab' + l^2) + a^2(b'c' + l^2) + 2(s')ar'q') \\ &= \frac{1}{l^2}((s')^2ab' + l^2ac' + l^4 + a^2(b'c' + l^2) + 2(s')r'aq'), \end{aligned}$$

so

$$\begin{aligned} b &= \frac{1}{l^2}(s')^2b' + c' + \frac{1}{l^2}ab'c' + a + \frac{2}{l^2}(s')r'q' \\ &> \frac{2}{l^2}ab'c' + c' + a + \frac{2}{l^2}\sqrt{ac'}\sqrt{ab'}\sqrt{b'c'} \\ &= \frac{4}{l^2}ab'c'. \end{aligned}$$

Hence

$$b' < \frac{l^2 b}{4ac'} < \frac{l^2}{4} \left(4 + \frac{4}{l^2}\right) = \frac{l^2 + 1}{c'},$$

which means $b'c' < l^2 + 1$. But since $\{a, b', c'\}$ is a $D(l^2)$ -triple, $(c')' = b' = 0$, so $c' = a + b - 2r$.

3.2 We may assume $c' < b$

The proof proceeds by showing that we can assume $c' < b$. It is here that the mistake in [1] arose. The proof of Lemma 1 in that paper for $D(4)$ used a version of the inequalities (3.7) and (3.8) found in [10], modified for $D(4)$ triples in order to show that it could be assumed that $c' < r^2$

$$ac' = (s')^2 - 4 < \frac{s^2}{r^2} - 4 = \frac{ac + 4}{r^2} - 4 < \frac{ac}{r^2}.$$

In [10], the above inequality was shown to hold for $D(1)$ triples provided we assume that $b \leq a + c$, but in the case of $D(4)$ triples this inequality will never hold. In particular,

$$s' < \frac{s}{r} \tag{6}$$

is never true. To see this, note that we have

$$\begin{aligned} 4s &= s(r^2 - ab) = r^2s - art + art + abs \\ &= r(rs - at) + a(rt - bs) \\ &= \frac{4(rs' + at')}{2}. \end{aligned}$$

So $s = \frac{rs' + at'}{2}$, which means inequality (6) is

$$rs' < \frac{rs' + at'}{2},$$

which is equivalent to

$$rs' < at'.$$

Since s' is always positive the inequality will always fail when $t' \leq 0$, so we assume that $t' > 0$. Squaring both sides, we obtain the equivalent inequality

$$r^2(s')^2 < a^2(t')^2. \tag{7}$$

Substituting $r^2 = ab + 4$, $(s')^2 = ac' + 4$, $(t')^2 = bc' + 4$ into (7), we get

$$r^2(s')^2 = a^2bc' + 4ab + 4ac' + 16 < a^2(t')^2 = a^2bc' + 4a^2$$

so our original inequality is equivalent to

$$ab + ac' + 4 < a^2$$

which is never true since $a < b$.

Now let $\{a, b, c\}$ be a $D(l^2)$ -Diophantine triple with r, s, t, c', s', t' defined as above. If $c < b$ then we are done, since $c = \bar{c}'$ for the triple $\left\{a, b, \bar{c}' = a + b + c + \frac{2}{l^2}(abc + rst)\right\}$ which falls under the previous case (since $0 \leq \bar{c}' < b$).

Suppose that $c \geq b > a > 0$. We want to show that we can assume $c' < b$. To do this, we suppose that $c' \geq c$, and show that a contradiction arises. This means that if $c \geq b$ then we must have $c' < c$, meaning that we can keep on replacing $c' = c$ and repeating with the new triple and eventually we will obtain $c' < b$.

If $c' \geq c$, then

$$c' \geq a + b + c' + \frac{2}{l^2}(abc' + rs't')$$

$$\text{meaning } a + b + \frac{2}{l^2}(abc' + rs't') \leq 0$$

$$\text{which implies } t' < 0.$$

$t' < 0$ means we must have $rt - bs < 0$ and so

$$r^2t^2 < b^2s^2$$

$$\text{which means that } (ab + l^2)(bc + l^2) < b^2(ac + l^2),$$

$$\text{giving } ab + bc + l^2 < b^2$$

$$\text{which then gives } c < b - a,$$

contradicting our assumption that $c \geq b$.

Therefore any $D(l^2)$ Diophantine triple $\{a, b, c\}$ with $a < b < 4\left(1 + \frac{4}{l^2}\right)$ can, through repetition of this process of taking $c' = c$, be reduced to a triple for which $0 \leq c' < b$, and we have established that the Pellian equation (5) stemming from such triples has solutions in the class of $(\pm l, l)$. \square

4 Proving Theorem 1.4

The next two sections follow a process which resembles that contained in [2], where a similar result on $D(4)$ triples of the form $\{F_k, 4F_{2n+4}, F_{2n+6}\}$ was proven.

4.1 Setting up the $D(9)$ -Diophantine Triple

Let us begin by setting up a Pellian equation. Given that $\{F_{2n+8}, 9F_{2n+4}, F_k\}$ is a $D(9)$ Diophantine triple, we must have that

$$F_{2n+8}F_k + 9 = X^2, \text{ and } 9F_{2n+4}F_k + 9 = Y^2.$$

We subtract $9F_{2n+4}$ times the first equation from F_{2n+8} times the second to obtain

$$F_{2n+8}Y^2 - 9F_{2n+4}X^2 = 9(F_{2n+8} - 9F_{2n+4}). \quad (8)$$

Since $9 \mid 9F_{2n+4}$, $F_{2n+8} < 9F_{2n+4} < \frac{40}{9}F_{2n+8}$, and $9F_{2n+4}F_{2n+8} + 9 = (3F_{2n+6})^2$, Lemma 3.1 tells us that all solutions $Y\sqrt{F_{2n+8}} + X\sqrt{9F_{2n+4}}$ of this equation are given by

$$Y\sqrt{F_{2n+8}} + 3X\sqrt{F_{2n+4}} = (\pm 3\sqrt{F_{2n+8}} + 9\sqrt{F_{2n+4}})(F_{2n+6} + \sqrt{F_{2n+4}F_{2n+8}})^j.$$

Now we define the sequences $\{V_j\}_{j=1}^\infty$ and $\{U_j\}_{j=1}^\infty$ by

$$V_j + U_j\sqrt{F_{2n+4}F_{2n+8}} := (F_{2n+6} + \sqrt{F_{2n+4}F_{2n+8}})^j.$$

This gives us

$$\begin{aligned} X &= X_j = 3V_j \pm F_{2n+8}U_j, \text{ and} \\ Y &= Y_j = \pm V_j + 9F_{2n+4}U_j. \end{aligned}$$

Substituting these expressions into our earlier equations, we obtain

$$\begin{aligned} F_{2n+8}F_k + 9 &= X^2 = (3V_j \pm F_{2n+8}U_j)^2, \text{ and} \\ 9F_{2n+4}F_k + 9 &= Y^2 = (\pm 3V_j + 9F_{2n+4}U_j)^2. \end{aligned}$$

This gives us alternative expressions for F_k

$$F_k = \frac{9V_j^2 - 9}{F_{2n+8}} + F_{2n+8}U_j^2 \pm 6U_jV_j \text{ and } F_k = \frac{V_j^2 - 1}{F_{2n+4}} + 9U_j^2F_{2n+4} \pm 6U_jV_j,$$

which together give us

$$F_k = \pm 6U_j V_j + U_j^2 (F_{2n+8} + 9F_{2n+4}). \quad (9)$$

We call this sequence

$$C_j^\pm := \pm 6U_j V_j + U_j^2 (F_{2n+8} + 9F_{2n+4}), \quad (10)$$

and aim to solve $F_k = C_j^\pm$ for positive integer j and k . We remark here that

$$\begin{aligned} C_1^- &= -6U_1 V_1 + U_1^2 (F_{2n+8} + 9F_{2n+4}) = -6F_{2n+6} + F_{2n+8} + 9F_{2n+4} \\ &= F_{2n+5} - 4F_{2n+6} + 9F_{2n+4} = 2F_{2n+4} - 3F_{2n+3} = F_{2n} \end{aligned}$$

and that

$$\begin{aligned} C_1^+ &= 6F_{2n+6} + F_{2n+8} + 9F_{2n+4} = 12F_{2n+6} + F_{2n} \\ &= F_{2n+10} + F_{2n+7} + 2F_{2n+5} + 6F_{2n+4} + F_{2n} \\ &= F_{2n+11} + F_{2n+5} + F_{2n+3} + 2F_{2n} \end{aligned}$$

and therefore $F_{2n+11} < C_1^+ < F_{2n+12}$, thus we assume $j \geq 2$. In addition, because $X_1^+ > X_1^- > 0$ and

$$\begin{aligned} X_{j+1}^\pm &= 3F_{2n+6}(3V_j \pm F_{2n+8}U_j) + F_{2n+8}(\pm V_j + 9F_{2n+4}U_j) \\ &= (9F_{2n+6} \pm F_{2n+8})V_j + 3F_{2n+8}(3F_{2n+4} \pm F_{2n+6})U_j \\ &> 3V_j \pm F_{2n+8}U_j = X_j^\pm, \end{aligned}$$

and since we're looking for solutions such that $F_k = \frac{(X_j)^2 - 9}{F_{2n+8}}$ we may assume that $k > 2n$ when $j \geq 2$.

Define $\beta_n := F_{2n+6} + \sqrt{F_{2n+6}^2 - 1}$. Then we have

$$V_j = \frac{\beta_n^j + \beta_n^{-j}}{2}, \text{ and } U_j = \frac{\beta_n^j - \beta_n^{-j}}{2\sqrt{F_{2n+4}F_{2n+8}}}.$$

So we can rewrite C_j^\pm as

$$C_j^\pm = \pm 6 \frac{\beta_n^{2j} - \beta_n^{-2j}}{4\sqrt{F_{2n+6}^2 - 1}} + (F_{2n+8} + 9F_{2n+4}) \cdot \frac{\beta_n^{2j} + \beta_n^{-2j} - 2}{4(F_{2n+6}^2 - 1)}$$

$$\begin{aligned}
&= \frac{\pm 6\beta_n^{2j}}{4\sqrt{F_{2n+6}^2 - 1}} + \frac{(F_{2n+8} + 9F_{2n+4})\beta_n^{2j}}{4(F_{2n+6}^2 - 1)} - \frac{\pm 6\beta_n^{-2j}}{4\sqrt{F_{2n+6}^2 - 1}} \\
&\quad + \frac{(F_{2n+8} + 9F_{2n+4})\beta_n^{-2j}}{4(F_{2n+6}^2 - 1)} - \frac{(F_{2n+8} + 9F_{2n+4})}{2(F_{2n+6}^2 - 1)}.
\end{aligned}$$

If we define the sequence γ_n^\pm by

$$\gamma_n^\pm := \frac{\pm 6}{4\sqrt{F_{2n+6}^2 - 1}} + \frac{(F_{2n+8} + 9F_{2n+4})}{4(F_{2n+4}F_{2n+8})},$$

then we have

$$C_j^\pm = \beta_n^{2j}\gamma_n^\pm - \frac{(F_{2n+8} + 9F_{2n+4})}{2(F_{2n+6}^2 - 1)} + \beta_n^{-2j}\gamma_n^\mp,$$

and our problem can be expressed as finding solutions $j \geq 2$ and $k > 2n$ to the equation

$$\beta_n^{2j}\gamma_n^\pm - \frac{(F_{2n+8} + 9F_{2n+4})}{2(F_{2n+6}^2 - 1)} + \beta_n^{-2j}\gamma_n^\mp = \frac{\alpha^k - \bar{\alpha}^k}{\sqrt{5}}. \quad (11)$$

4.2 A Linear Form in Three Logarithms

We begin by finding bounds for γ_n^\pm .

Lemma 4.1. *The following bounds apply to γ_n^\pm ,*

$$\begin{aligned}
0.011\alpha^{-2n-4} &< \gamma_n^- < 0.013\alpha^{-2n-4} \\
2.574\alpha^{-2n-4} &< \gamma_n^+ < 2.585\alpha^{-2n-4}.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
\sqrt{\gamma_n^\pm} &= \frac{3}{2\sqrt{F_{2n+8}}} \pm \frac{1}{2\sqrt{F_{2n+4}}} \\
&= \frac{3}{2\sqrt{(\alpha^{2n+8} - \alpha^{-2n-8})/\sqrt{5}}} \pm \frac{1}{2\sqrt{(\alpha^{2n+4} - \alpha^{-2n-4})/\sqrt{5}}} \\
&= \frac{5^{1/4}\alpha^{-n-2}}{2} \left(\frac{3}{\alpha^2\sqrt{(1 - 1/\alpha^{4n+16})}} \pm \frac{1}{\sqrt{(1 - 1/\alpha^{2n+8})}} \right)
\end{aligned}$$

As a result of the Taylor series of $(1 - x)^{1/2}$, we have for $0 < x < 1$

$$1 + \frac{1}{2}x < \frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \cdots < 1 + \frac{x}{2} \left(\frac{1}{1-x} \right)$$

so

$$\frac{3}{\alpha^2} \left(1 + \frac{1}{2} \alpha^{-4n-16}\right) < \frac{3}{\alpha^2 \sqrt{(1 - 1/\alpha^{4n+16})}} < \frac{3}{\alpha^2} \left(1 + \frac{\alpha^{-4n-16}}{2(1 - \alpha^{-4n-16})}\right),$$

hence we must have

$$1.1458980 < \frac{3}{\alpha^2} < \frac{3}{\alpha^2 \sqrt{(1 - 1/\alpha^{4n+16})}} < 1.14599721$$

and similarly

$$1 < \frac{1}{\sqrt{(1 - 1/\alpha^{2n+8})}} < 1.0041.$$

We then obtain bounds for $\sqrt{\gamma_n^\pm}$

$$0.141798 < \frac{2}{5^{1/4} \alpha^{-n-2}} \sqrt{\gamma_n^-} < 0.1499721$$

and

$$2.145898 < \frac{2}{5^{1/4} \alpha^{-n-2}} \sqrt{\gamma_n^+} < 2.15001,$$

and finally the following bounds on γ_n^\pm :

$$\begin{aligned} 0.011 \alpha^{-2n-4} &< \gamma_n^- < 0.013 \alpha^{-2n-4} \\ 2.574 \alpha^{-2n-4} &< \gamma_n^+ < 2.585 \alpha^{-2n-4}. \end{aligned}$$

□

We now define a linear form in three logarithms, Λ by

$$\Lambda := 2j \log \beta_n - k \log \alpha + \log(\sqrt{5} \gamma_n^\pm)$$

and proceed with the following lemma.

Lemma 4.2. $0 < \Lambda < 1162 \beta_n^{-2j}$ for $j \geq 2$.

Proof. We can turn (11) into

$$\beta_n^{2j} \gamma_n^\pm - \frac{\alpha^k}{\sqrt{5}} = \frac{(F_{2n+8} + 9F_{2n+4})}{2(F_{2n+6}^2 - 1)} - \frac{\bar{\alpha}^k}{\sqrt{5}} - \beta_n^{-2j} \gamma_n^\mp$$

$\Lambda = \log \sqrt{5} \gamma_n^\pm \beta_n^{2j} \alpha^{-k} > 0$ if and only if $\sqrt{5} \gamma_n^\pm \beta_n^{2j} \alpha^{-k} > 1$. Therefore in order to show that $\Lambda > 0$ we will assume, for a contradiction, that $\beta_n^{2j} \gamma_n^+ \leq \frac{\alpha^k}{\sqrt{5}}$. This would mean that

$$\frac{\sqrt{5}}{\alpha^k} \leq \frac{\beta_n^{-2j}}{\gamma_n^\pm} \leq \frac{\beta_n^{-2j}}{\gamma_n^-},$$

and because $\frac{1}{F_{2n+4}} < \frac{9}{F_{2n+8}}$, and by (11) and our assumption, this gives us the following inequality

$$\begin{aligned} \frac{1}{F_{2n+4}} &< \frac{1}{2F_{2n+4}} + \frac{9}{2F_{2n+8}} = \frac{F_{2n+8} + 9F_{2n+4}}{2(F_{2n+6}^2 - 1)} \\ &\leq \beta_n^{-2j} \gamma_n^\pm + \frac{\bar{\alpha}^k}{\sqrt{5}} < \beta_n^{-2j} \left(\gamma_n^+ + \frac{1}{5\gamma_n^-} \right), \end{aligned}$$

which we apply below along with the bounds for γ_n^\pm obtained in 4.1.

$$\begin{aligned} F_{2n+4}^j F_{2n+8}^j &= (F_{2n+6}^2 - 1)^j < \left(2F_{2n+6}^2 - 1 + 2F_{2n+6} \sqrt{F_{2n+6}^2 - 1} \right)^j = \beta_n^{2j} \\ &< F_{2n+4} \left(\gamma_n^+ + \frac{1}{5\gamma_n^-} \right) < F_{2n+4} (2.585\alpha^{-2n-4} + 18.2\alpha^{2n+4}) \end{aligned}$$

And $F_{2n+4}^{j-1} F_{2n+8}^j < 2.585\alpha^{-2n-4} + 18.182\alpha^{2n+4}$ is only true if $j < 2$, which is a contradiction. Hence $\beta_n^{2j} \gamma_n^+ > \frac{\alpha^k}{\sqrt{5}}$ and so $\Lambda > 0$. Now we have

$$\begin{aligned} \left| \alpha^k 5^{-1/2} \beta_n^{-2j} (\gamma_n^\pm)^{-1} - 1 \right| &< \frac{1}{\beta_n^{2j} \gamma_n^\pm} \left(\frac{F_{2n+8} + 9F_{2n+4}}{2(F_{2n+6}^2 - 1)} + \frac{1}{\sqrt{5}\alpha^k} \right) \\ &< \frac{1}{\beta_n^{2j} \gamma_n^\pm} \left(\frac{2}{F_{2n+4}} + \frac{1}{\sqrt{5}\alpha^{2n+1}} \right) \\ &< \frac{581}{\beta_n^{2j}} < \frac{1}{2}, \end{aligned}$$

so by the fact that

$$|e^\Lambda - 1| < \frac{1}{2} \text{ implies that } |\Lambda| < 2|e^\Lambda - 1| \tag{12}$$

we must have that $\Lambda < 1162\beta_n^{-2j}$. □

Using these two lemmas we prove a useful proposition

Proposition 4.3. *If equation (9) has a positive integer solution (j, k) with $j > 1$ then*

$$j < 1.15 \cdot 10^{12} (2n + 7) \log(78j(2n + 7))$$

In order to prove this proposition, we will apply Lemma 2.13 to the linear form in three logarithms Λ as defined above.

$$\Lambda := 2j \log \beta_n - k \log \alpha + \log(\sqrt{5}\gamma_n^\pm)$$

We take

$$\begin{aligned} N = 3, \quad D = 4, \quad b_1 = 2j, \quad b_2 = -k, \quad b_3 = 1, \\ \alpha_1 = \beta_n, \quad \alpha_2 = \alpha, \quad \alpha_3 = \sqrt{5}\gamma_n^\pm. \end{aligned}$$

The lemma calls for $\alpha_1, \alpha_2, \alpha_3$ to be multiplicatively independent (i.e. that their logarithms be linearly independent). We have that $\alpha_2 \in \mathbb{Q}(\sqrt{5})$ and $\alpha_1, \alpha_3^2 \in \mathbb{Q}(\sqrt{F_{2n+4}F_{2n+8}})$, and by Lemma 2.15, $F_{2n+4}F_{2n+8}$ is neither a square nor 5 times a square. Therefore, writing $F_{2n+4}F_{2n+8} = du^2$ for $u, d \in \mathbb{Z}$, $5 \neq d$ square-free, since no non-zero power of α_2 can be in $\mathbb{Q}(\sqrt{d})$, if $\alpha_1, \alpha_2, \alpha_3$ are multiplicatively dependent we must have that α_1 and α_3^2 are multiplicatively dependent. Since α_1 is a unit in $\mathbb{Q}(\sqrt{d})$, we must have that $\alpha_3^2 = 5(\gamma_n^\pm)^2$ is a unit, but the norm of $5(\gamma_n^\pm)^2$ is

$$25(\gamma_n^+\gamma_n^-)^2 = 25\left(\frac{9F_{2n+4} - F_{2n+8}}{4F_{2n+4}F_{2n+8}}\right)^4 < 1,$$

so this is never an integer for any n , therefore α_3^2 is not a unit for any n .

The absolute logarithmic heights for α_1 and α_2 are

$$h(\alpha_1) = h(\beta_n) = \frac{1}{2} \log \beta_n, \quad h(\alpha_2) = h(\alpha) = \frac{1}{2} \log \alpha.$$

For α_3 , since

$$(x - \gamma_n^+)(x - \gamma_n^-) = x^2 - 2\left(\frac{F_{2n+8} + 9F_{2n+4}}{4(F_{2n+4}F_{2n+8})}\right)x + \left(\frac{9F_{2n+4} - F_{2n+8}}{4F_{2n+4}F_{2n+8}}\right)^2,$$

clearing denominators we get the minimal polynomial

$$16F_{2n+4}^2F_{2n+8}^2x^2 - 8(F_{2n+8}^2F_{2n+4} + 9F_{2n+4}^2F_{2n+8})x + (9F_{2n+4} - F_{2n+8})^2,$$

and since $|\gamma_n^\pm| \leq |\gamma_n^+| < 2.585\alpha^{-6} < 1$, and $F_\lambda < \alpha^\lambda/\sqrt{5}$ for positive even λ we have

$$h(\gamma_n^\pm) = \frac{1}{2} \log(16F_{2n+4}^2F_{2n+8}^2) = \log(4F_{2n+4}F_{2n+8}) < (4n + 12) \log \alpha + \log 4/5.$$

Hence we can take

$$\begin{aligned} h(\alpha_3) &= h(\sqrt{5}\gamma_n^\pm) \leq h(\sqrt{5}) + h(\gamma_n^\pm) < \frac{1}{2} \log 5 + (4n + 12) \log \alpha + \log 4/5 \\ &< 2 \log \alpha + (4n + 12) \log \alpha = (4n + 14) \log \alpha \end{aligned}$$

and finally since we need $A_i \geq D \cdot h(\alpha_i)$, we take

$$A_1 = 2 \log \beta_n, \quad A_2 = 2 \log \alpha, \quad A_3 = 8(2n + 7) \log \alpha.$$

Next, since $\alpha^{\lambda-2} \leq F_\lambda \leq \alpha^{\lambda-1}$, we see that

$$\beta_n < 2F_{2n+8} < 2\alpha^{2n+5} < \alpha^{2n+7}$$

and in addition,

$$\begin{aligned} \alpha^{k-1} &< 2\alpha^{k-2} < 2F_k \leq 12U_jV_j + 2U_j^2(F_{2n+8} + 9F_{2n+4}) \\ &< 20U_jV_j + F_{2n+4}F_{2n+8}U_j^2 < (V_j + U_j\sqrt{F_{2n+4}F_{2n+8}})^2 - V_j^2 \\ &< (V_j + U_j\sqrt{F_{2n+4}F_{2n+8}})^2 = (F_{2n+6} + \sqrt{F_{2n+4}F_{2n+8}})^{2j} \\ &< (2F_{2n+6})^{2j} < (2\alpha^{2n+5})^{2j} < \alpha^{2j(2n+7)}. \end{aligned}$$

Due to the results above, we can take

$$\begin{aligned} E &= \max \{1, \max \{|b_j|A_j/A_N; 1 \leq j \leq N\}\} \\ &\leq \max \left\{ 2j, k, 1, \frac{2j \log \beta_n}{\log \alpha}, 4j \log \beta_n, \frac{k \log \alpha}{\log \beta_n}, 2k \log \alpha, 4(2n + 7), \frac{4(2n + 7) \log \alpha}{\log \beta_n} \right\} \\ &= \max \left\{ k, \frac{2j \log \beta_n}{\log \alpha}, 4(2n + 7) \right\} \leq 2j(2n + 7) \end{aligned}$$

and

$$\begin{aligned} C(3) &= \frac{8}{2}(5)(9)(16e)^4 < 6.45 \times 10^8 \\ C_0 &= \log e^{20.2} 3^{5.5} (16) \log(4e) < 30 \\ W_0 &= \log(1.5eE \cdot 4 \log 4e) < \log(78j(2n + 7)) \\ \Omega &= (2 \log \beta_n)(2 \log \alpha)(8(2n + 7) \log \alpha). \end{aligned}$$

Proof of Proposition 4.3. Applying Lemma 2.13 and combining this with the previous lemma, we see,

$$2j \log \beta_n - \log 1162 < -\log |\Lambda| < 1.434 \cdot 10^{11}(2n + 7)(\log \beta_n)(\log(78j(2n + 7))),$$

therefore

$$j < 1.15 \cdot 10^{12}(2n + 7) \log(78j(2n + 7)),$$

as desired. □

4.3 Linear Form in Two Logarithms

Using $j = 1, k = 2n$ in Λ we define the linear form in three logarithms, Λ_0 , by

$$\Lambda_0 := 2 \log \beta_n - 2n \log \alpha + \log(\sqrt{5}\gamma_n^\pm).$$

We will use Λ_0 to help form a linear form in two logarithms later, first we find the following upper bound on Λ_0

Lemma 4.4. $|\Lambda_0| < 9473\beta_n^{-2}$.

Proof. Assume for now that $n \geq 2$. After substituting the one known solution, $j = 1, k = 2n$, into our equation (11) from earlier, it becomes

$$\beta_n^2 \gamma_n^\pm - \frac{\alpha^{2n}}{\sqrt{5}} = \frac{(F_{2n+8} + 9F_{2n+4})}{2(F_{2n+6}^2 - 1)} - \frac{\bar{\alpha}^{2n}}{\sqrt{5}} - \beta_n^{-2} \gamma_n^\mp.$$

If $\beta_n^2 \gamma_n^\pm \leq \frac{\alpha^{2n}}{\sqrt{5}}$, then $\frac{\alpha^{-2n}}{\sqrt{5}} \leq \frac{1}{5\beta_n^2 \gamma_n^\pm}$ and

$$\begin{aligned} \left| \alpha^{2n} 5^{-1/2} \beta_n^{-2} (\gamma_n^\pm)^{-1} - 1 \right| &< \frac{\beta_n^{-2} \gamma_n^\mp + \frac{\alpha^{-2n}}{\sqrt{5}}}{\beta_n^2 \gamma_n^\pm} \\ &< \frac{\gamma_n^\mp + \frac{1}{5\gamma_n^\pm}}{\beta_n^4 \gamma_n^\pm} \\ &< \frac{235.3 + 1656.2\alpha^{4n+8}}{\beta_n^4} \\ &< \beta_n^{-2} (235.3 / (55 + 12\sqrt{21})^2 + 1656.2) \\ &< 1657\beta_n^{-2} < \frac{1}{2}. \end{aligned}$$

If $\beta_n^2 \gamma_n^\pm > \frac{\alpha^{2n}}{\sqrt{5}}$, then

$$\begin{aligned} \left| \alpha^{2n} 5^{-1/2} \beta_n^{-2} (\gamma_n^\pm)^{-1} - 1 \right| &< \frac{1/(2F_{2n+4}) + 1/F_{2n+4}}{\beta_n^2 \gamma_n^\pm} \\ &< \frac{3}{2F_{2n+4} \beta_n^2 \gamma_n^\pm} < 306\beta_n^{-2} < \frac{1}{2}. \end{aligned}$$

For $n = 1$, $|\Lambda_0| = 2 \log(21 + \sqrt{440}) - 2 \log \alpha + \log(\sqrt{5}\gamma_1^+) < 9473\beta_n^{-2}$. In every case we have (by inequality (12)) $|\Lambda_0| < 9473\beta_n^{-2}$. \square

We now form the aforementioned linear form in two logarithms, which will help to give us a hard bound on j . Define Λ_1 by

$$\Lambda_1 := K \log \alpha - (j - 1) \log(5/4),$$

where $K = (2j - 1)(2n + 6) - k - 6$. We have the following bound on Λ_1

Lemma 4.5. $|\Lambda_1| < (9j + 17042)\alpha^{-4n-12}$.

Proof. Note first that

$$\begin{aligned} \beta_n &= F_{2n+6} + \sqrt{F_{2n+6}^2 - 1} = \frac{(F_{2n+6} + \sqrt{F_{2n+6}^2 - 1})^2}{F_{2n+6} + \sqrt{F_{2n+6}^2 - 1}} \\ &= \frac{2F_{2n+6}^2 + 2F_{2n+6}\sqrt{F_{2n+6}^2 - 1} - 1}{F_{2n+6} + \sqrt{F_{2n+6}^2 - 1}} = 2F_{2n+6} - \frac{1}{F_{2n+6} + \sqrt{F_{2n+6}^2 - 1}} \\ &= 2F_{2n+6} \left(1 - \frac{1}{2F_{2n+6}(F_{2n+6} + \sqrt{F_{2n+6}^2 - 1})} \right) \end{aligned}$$

and $2F_{2n+6} = \frac{2}{\sqrt{5}}(\alpha^{2n+6} - \bar{\alpha}^{2n+6}) = \frac{2}{\sqrt{5}}\alpha^{2n+6}\left(1 - \frac{1}{\alpha^{4n+12}}\right)$. So if we define δ_n by

$$\delta_n = \left(1 - \frac{1}{2F_{2n+6}(F_{2n+6} + \sqrt{F_{2n+6}^2 - 1})} \right) \left(1 - \frac{1}{\alpha^{4n+12}} \right),$$

then $\log \beta_n = \log\left(\frac{2}{\sqrt{5}}\right) + (2n + 6) \log \alpha + \log \delta_n$ and we can see that

$$\begin{aligned} \Lambda - \Lambda_0 &= \left(2j \log \beta_n - k \log \alpha + \log(\sqrt{5}\gamma_n^\pm) \right) - \left(2 \log \beta_n - 2n \log \alpha + \log(\sqrt{5}\gamma_n^\pm) \right) \\ &= (2j - 2) \log \beta_n - (k - 2n) \log \alpha \\ &= (2j - 2) \log(2/\sqrt{5}) + K \log \alpha + (2j - 2) \log \delta_n, \end{aligned}$$

where $K = (2j - 1)(2n + 6) - k - 6$. Hence we have $\Lambda_1 = \Lambda - \Lambda_0 - (2j - 2) \log \delta_n$. We can bound $|\log \delta_n|$ by taking

$$\begin{aligned} |\log \delta_n| &\leq \left| \log \left(1 - \frac{1}{2F_{2n+6}(F_{2n+6} + \sqrt{F_{2n+6}^2 - 1})} \right) \right| + \left| \log \left(1 - \frac{1}{\alpha^{4n+12}} \right) \right| \\ &< \frac{1}{F_{2n+6}(F_{2n+6} + \sqrt{F_{2n+6}^2 - 1})} + \frac{1}{\alpha^{4n+12}} < \frac{1}{2\alpha^{4n+8}} + \frac{1}{\alpha^{4n+12}} \\ &< \frac{9}{2\alpha^{4n+12}}. \end{aligned}$$

Here we have used the triangle inequality and (12). This then gives us

$$|\Lambda_1| \leq |\Lambda| + |\Lambda_0| + |2j - 2| |\log \delta_n| < \frac{9473}{\beta_n^2} + \frac{9(j-1)}{\alpha^{4n+12}},$$

and since

$$\begin{aligned} \beta_n &= F_{2n+6} + \sqrt{F_{2n+6}^2 - 1} > 2\alpha^{2n+4}, \text{ we must have} \\ \beta_n^2 &> \frac{4}{7}\alpha^{4n+12}. \end{aligned}$$

Therefore

$$|\Lambda_1| < (9j + 17042)\alpha^{-4n-12}.$$

□

Lemma 4.6. *If equation (9) has a positive integer solution (j, k) with $j > 1$, then*

$$j < 3.55 \times 10^{19}, \text{ and } n < 246806.$$

Proof. We apply Lemma 2.14. Let

$$D = 2, \quad \gamma_1 = \frac{5}{4}, \quad \gamma_2 = \alpha, \quad b_1 = (j-1), \quad b_2 = K.$$

In addition, take $h_1 = \log 5, h_2 = \frac{1}{2}$. By the previous lemma, we have

$$\begin{aligned} K &< \frac{(j-1) \log(5/4) + (9j + 17042)\alpha^{-4n-12}}{\log \alpha} \\ &< 0.4638(j-1) + 0.0085j + 16.0466 < 0.48j + 15.59, \end{aligned}$$

and because

$$\frac{|b_1|}{Dh_2} + \frac{|b_2|}{Dh_1} = (j-1) + \frac{K}{2 \log 5} < 1.15j + 3.85,$$

Lemma 2.14 gives us a lower bound on Λ_1 ,

$$\log |\Lambda_1| > -17.9 \cdot 8 \cdot \log 5 \cdot \left(\max \{ \log(1.15j + 3.85) + 0.38, 15 \} \right)^2.$$

From the previous lemma, we see that

$$\log |\Lambda_1| < -(4n + 12) \log \alpha + \log(9j + 17042).$$

A combination of these two bounds yields

$$n < 119.75 \left(\max \{ \log(1.15j + 3.85) + 0.38, 15 \} \right)^2 + 0.52 \log(9j + 17042).$$

If

$$\log(1.15j + 3.85) + 0.38 \leq 15$$

then $j < 1943527$ and $n < 26952$. Otherwise,

$$n < 119.75 \left(\log(1.15j + 3.85) + 0.38 \right)^2 + 0.52 \log(9j + 17042)$$

and we substitute this bound into Proposition 4.3 to obtain

$$\begin{aligned} j < 1.15 \cdot 10^{12} (2(119.75 \left(\log(1.15j + 3.85) + 0.38 \right)^2 + 0.52 \log(9j + 17042)) + 7) \\ \times \log(78j(2(119.75 \left(\log(1.15j + 3.85) + 0.38 \right)^2 + 0.52 \log(9j + 17042)) + 7)), \end{aligned}$$

which means that $j < 3.55 \times 10^{19}$ and so $n < 246806$. □

4.4 Refining Our Bounds

We improve our bounds on n and j in this section before applying Baker-Davenport reduction on those bounds in the next section. Lemma 4.5 gives us

$$|K \log \alpha - (j - 1) \log(5/4)| < (9j + 17042) \alpha^{-4n-12}.$$

Thus we can divide by $j - 1$ to get

$$\left| \frac{\log(5/4)}{\log \alpha} - \frac{K}{j - 1} \right| < \frac{9j + 17042}{(j - 1) \alpha^{4n+12} \log \alpha}.$$

Assume that

$$\frac{9j + 17042}{(j - 1) \alpha^{4n+12} \log \alpha} < \frac{1}{2(j - 1)^2}. \tag{13}$$

By above,

$$\left| \frac{\log(5/4)}{\log \alpha} - \frac{K}{j - 1} \right| < \frac{1}{2(j - 1)^2}.$$

By Theorem (2.6), we must have that $\frac{K}{(j-1)}$ is a convergent in the simple continued fraction expansion of $\log(5/4)/\log \alpha$. Since the denominator of the 46th convergent

$$\frac{25158053660121411107}{54253653513327093513}$$

is greater than the upper bound of 3.55×10^{19} we established for j , we can use the denominator of the 45th convergent

$$\frac{4460457560349832575}{9619031832089360168}$$

which is bigger than 9.6×10^{18} to obtain the following lower bound,

$$\left| \frac{\log(5/4)}{\log \alpha} - \frac{K}{j-1} \right| \geq \left| \frac{\log(5/4)}{\log \alpha} - \frac{4460457560349832575}{9619031832089360168} \right| > 1.9 \times 10^{39}.$$

Combining these bounds

$$1.9 \times 10^{39} < \frac{9j + 17042}{(j-1)\alpha^{4n+12} \log \alpha} < 17060\alpha^{-4n-12}(\log \alpha)^{-1}$$

gives us $n < 49$. We also know from Lemma (2.5) that if $\frac{p_r}{q_r}$ is the r th convergent of $\frac{\log(5/4)}{\log \alpha}$, then

$$\left| \frac{\log(5/4)}{\log \alpha} - \frac{p_r}{q_r} \right| > \frac{1}{(a_{r+1} + 2)q_r^2},$$

where a_{r+1} is the $(r+1)$ st partial quotient of $\frac{\log(5/4)}{\log \alpha}$. Therefore, for $2 \leq r \leq 45$,

$$\min \left\{ \frac{1}{(a_{r+1} + 2)(j-1)^2} \right\} < \frac{9j + 17042}{(j-1)\alpha^{4n+12} \log \alpha}.$$

Since we have that $\max \{a_{r+1} : 2 \leq r \leq 45\} = a_{36} = 49$, we have

$$\alpha^{4n+12} < 51(j-1)(9j + 17042)(\log \alpha)^{-1}.$$

If (13) does not hold, i.e., if

$$\frac{9j + 17042}{(j-1)\alpha^{4n+12} \log \alpha} \geq \frac{1}{2(j-1)^2},$$

then we have

$$\alpha^{4n+12} \leq 2(j-1)(9j+17042)(\log \alpha)^{-1}.$$

In either case,

$$\alpha^{4n+12} < 51(j-1)(9j+17042)(\log \alpha)^{-1} < 444352j^2.$$

This leads to the following proposition.

Proposition 4.7. *If equation (9) has a positive integer solution (j, k) with $j > 1$, then*

$$n < 1.04 \log j + 3.76$$

When we combine this with the bound for j in Proposition 4.3, we get

$$j < 1.15 \cdot 10^{12}(2.08 \log j + 14.52) \log(78j(2.08 \log j + 14.52)),$$

which implies

Lemma 4.8. *If equation (9) has a positive integer solution (j, k) with $j > 1$, then $j < 4.63 \times 10^{15}$ and $n < 42$.*

4.5 Reduction of the Bounds

We now use the reduction method of Baker and Davenport described in Lemma 4.3 to bring the bounds for j and n down to something more computationally manageable. We then use a procedure written in MapleTM (see Appendix A) to check all remaining possibilities.

We know that

$$0 < 2j \log \beta_n - k \log \alpha + \log(\sqrt{5} \gamma_n^\pm) < 1162 \beta_n^{-2j}.$$

In order to apply Baker-Davenport reduction, we consider

$$\kappa = \frac{2 \log \beta_n}{\log \alpha}, \quad \mu = \frac{\log(\sqrt{5} \cdot \gamma_n^\pm)}{\log \alpha}, \quad A = \frac{1162}{\log \alpha}, \quad B = \beta_n^2, \quad M = 4.63 \times 10^{15}.$$

We then used a set of procedures written in Maple to undertake the computations. In all cases, we obtained $j \leq 6$ and therefore $1 \leq n \leq 6$. So we have the following result.

Lemma 4.9. *If equation (9) has a positive integer solution (j, k) with $j > 1$, then $j \leq 6$ and $n \leq 6$.*

Applying this result to equation (10) in order to prove Theorem 1.1, we see that no combination of n and j with $1 \leq n \leq 6$, $2 \leq j \leq 6$ yields a Fibonacci number. We have already established that $F_{11} < C_1^+ < F_{12}$, which means our only solution is $C_1^- = F_{2n}$. When $n = 1$, $F_{2n} = F_2 = 1 = F_1$, giving us one extra solution in that case.

5 Proving Theorem 1.5

5.1 Setting up the $D(64)$ -Diophantine triple

We now wish to show the same thing for the $D(64)$ -diophantine triple $\{F_k, F_{2n+12}, 16F_{2n+6}\}$. We wish to show that for this triple the only value k can take is $2n$. Note that in this case, we have $a = 16F_{2n+6}$ and $b = F_{2n+12}$ in Lemma 3.1, and $64 \mid a$ if we assume that $3 \mid n$, which gives us that all solutions of the equation $4Y\sqrt{F_{2n+6}} + X\sqrt{F_{2n+12}}$ are given by

$$4Y\sqrt{F_{2n+6}} + X\sqrt{F_{2n+12}} = (\pm 32\sqrt{F_{2n+6}} + 8\sqrt{F_{2n+12}}) \left(\frac{F_{2n+9}\sqrt{F_{2n+6}F_{2n+12}}}{2} \right)^j,$$

where $j \geq 0$.

Again we define sequences $(U_j)_{j \geq 0}$ and $(V_j)_{j \geq 0}$, this time by

$$V_j + U_j\sqrt{F_{2n+6}F_{2n+12}} := (F_{2n+9} + \sqrt{F_{2n+6}F_{2n+12}})^j.$$

This gives us

$$y = y_j = \pm 8V_j + 2U_jF_{2n+12} \text{ and } x = x_j = 8V_j \pm 32F_{2n+6}U_j,$$

so we get

$$F_k = \pm 32U_jV_j + 4U_j^2(F_{2n+12} + 16F_{2n+6}). \quad (14)$$

If we let $C_j^{(\pm)} := \pm 32U_jV_j + 4U_j^2(F_{2n+12} + 16F_{2n+6})$ for $j \geq 1$ then our goal now is to solve

$$F_k = C_j^{(\pm)} \quad (15)$$

for some positive j and k .

As in the previous section, the equation has the solution $C_j^{(\pm)} = F_{2n}$, so to prove our result we must prove that there are no other solutions. We can note immediately that

$$F_{2n+14} < C_1^{(+)} < F_{2n+15},$$

so we suppose for a contradiction that $j \geq 2$. In this case we have

$$\beta_n := \frac{F_{2n+9} + \sqrt{F_{2n+6}F_{2n+12}}}{2},$$

so we can express

$$V_j := \frac{\beta_n^j + \beta_n^j}{2} \text{ and } U_j := \frac{\beta_n^j - \beta_n^j}{2\sqrt{F_{2n+9}^2 - 4}}.$$

We find that $C_j^{(\pm)} = \beta_n^{2j} \gamma_n^{(\pm)} - 2 \left(\frac{F_{2n+12} + 16F_{2n+6}}{F_{2n+9}^2 - 4} \right) + \beta_n^{-2j} \gamma_n^{(\mp)}$ where

$$\gamma_n^{(\pm)} := \pm \frac{8}{\sqrt{F_{2n+9}^2 - 4}} + \frac{F_{2n+12} + 16F_{2n+6}}{F_{2n+9}^2 - 4}.$$

So we must solve:

$$C_j^{(\pm)} = \beta_n^{2j} \gamma_n^{(\pm)} - 2 \left(\frac{F_{2n+12} + 16F_{2n+6}}{F_{2n+9}^2 - 4} \right) + \beta_n^{-2j} \gamma_n^{(\mp)} = \frac{\alpha^k - \bar{\alpha}^k}{\sqrt{5}}.$$

5.2 A Linear form in Three Logarithms (2)

Lemma 5.1. *The following bounds apply to γ_n^\pm ,*

$$\begin{aligned} 0.00694321\alpha^{-2n-4} &< \gamma_n^- < 0.00733800\alpha^{-2n-4} \\ 8.45276900\alpha^{-2n-4} &< \gamma_n^+ < 8.46635831\alpha^{-2n-4}. \end{aligned}$$

Using the same identity from earlier, we get

$$\sqrt{\gamma_n^{(\pm)}} = \left(\frac{4}{\sqrt{F_{2n+12}}} \pm \frac{1}{\sqrt{F_{2n+6}}} \right) = 5^{1/4} \alpha^{-n-3} \left(\frac{4}{\alpha^3 \sqrt{1 - \frac{1}{\alpha^{4n+24}}}} \pm \frac{1}{\sqrt{1 - \frac{1}{\alpha^{4n+12}}}} \right),$$

giving us

$$1 + \frac{1}{2} \left(\frac{1}{\alpha^{4n+24}} \right) < \frac{1}{\sqrt{1 - \frac{1}{\alpha^{4n+24}}}} < 1 + \frac{\frac{1}{\alpha^{4n+24}}}{2(1 - \frac{1}{\alpha^{4n+24}})} < \frac{4}{\alpha^3} \cdot \frac{1}{\sqrt{1 - \frac{1}{\alpha^{4n+24}}}},$$

so

$$0.9442719 < \frac{4}{\alpha^3} \frac{1}{\sqrt{1 - \frac{1}{\alpha^{4n+24}}}} < 0.9442765.$$

Thus:

$$1 < \frac{1}{\sqrt{1 - \frac{1}{\alpha^{4n+24}}}} < 1 + \frac{\frac{1}{\alpha^{4n+12}}}{2(1 - \frac{1}{\alpha^{4n+12}})} < 1.00155765.$$

Rearranging

$$1.9442719 < \frac{\sqrt{\gamma_n^+}}{5^{1/4}\alpha^{-n-3}} < 1.94583415$$

and

$$-0.0572858 < \frac{\sqrt{\gamma_n^-}}{5^{1/4}\alpha^{-n-3}} < -0.0557234$$

gives us the result. \square

We define a linear form in logarithms, $\Lambda := 2j \log(\beta_n) - k \log(\alpha) + \log(\sqrt{5}(\gamma_n^{\pm}))$.

Lemma 5.2. $0 < \Lambda < 3868\beta_n^{-2j}$ for $j \geq 2$.

Proof. We have

$$\beta_n^{2j} \gamma_n^{(\pm)} - \frac{\alpha^k}{\sqrt{5}} = \frac{2(F_{2n+12} + 16F_{2n+6})}{F_{2n+9}^2 - 4} - \beta_n^{-2j} \gamma_n^{(\mp)} - \frac{\bar{\alpha}^k}{\sqrt{5}}.$$

Suppose for a contradiction that $\beta_n^{2j} \gamma_n^{(\pm)} \leq \frac{\alpha^k}{\sqrt{5}}$, thus

$$\frac{\sqrt{5}}{\alpha^k} \leq \frac{\beta_n^{-2j}}{\gamma_n^{(\pm)}} \leq \frac{\beta_n^{-2j}}{\gamma_n^{(-)}},$$

so

$$\begin{aligned} \frac{1}{F_{2n+6}} &< \frac{2}{F_{2n+6}} + \frac{32}{F_{2n+12}} = \frac{2(F_{2n+12} + 16F_{2n+6})}{F_{2n+9}^2 - 4} \\ &< \beta_n^{-2j} \gamma_n^{(\mp)} + \frac{\bar{\alpha}^k}{\sqrt{5}} < \beta_n^{-2j} \gamma_n^{(+)} + \frac{1}{\sqrt{5}\alpha^k} \\ &\leq \beta_n^{-2j} \gamma_n^{(+)} + \frac{1}{\sqrt{5}} \left(\frac{\beta_n^{-2j}}{\sqrt{5}\gamma_n^{(-)}} \right) = \beta_n^{-2j} \left(\gamma_n^{(+)} + \frac{1}{5\gamma_n^{(-)}} \right). \end{aligned}$$

We have

$$F_{2n+6}^j F_{2n+12}^j < \beta_n^{2j} < F_{2n+6} \left(\gamma_n^{(+)} + \frac{1}{5\gamma_n^{(-)}} \right) < F_{2n+6} (8.47\alpha^{-2n-6} + 28.81\alpha^{2n+6}),$$

which gives

$$F_{2n+6}^{j-1} F_{2n+12}^j < 8.47\alpha^{-2n-6} + 28.81\alpha^{2n+6},$$

which is a contradiction. Therefore $\beta_n^{2j}\gamma_n^{(\pm)} - \frac{\alpha^k}{\sqrt{5}} > 0$, so $\Lambda > 0$.

Furthermore,

$$\begin{aligned} |\alpha^k 5^{-1/1} \beta_n^{-2j} (\gamma_n^{(\pm)})^{-1} - 1| &= \frac{1}{\beta_n^{2j} \gamma_n^{(\pm)}} \left| \frac{\alpha^k}{\sqrt{5}} - \beta_n^{2j} \gamma_n^{(\pm)} \right| = \frac{1}{\beta_n^{2j} \gamma_n^{(\pm)}} \left(\beta_n^{2j} \gamma_n^{(\pm)} - \frac{\alpha^k}{\sqrt{5}} \right) \\ &= \frac{1}{\beta_n^{2j} \gamma_n^{(\pm)}} \left[\frac{2(F_{2n+12} + 16F_{2n+6})}{F_{2n+9}^2 - 4} - \beta_n^{-2j} \gamma_n^{(\mp)} - \frac{\alpha^k}{\sqrt{5}} \right] \\ &< \frac{1}{\beta_n^{2j} \gamma_n^{(\pm)}} \left[\frac{2(F_{2n+12} + 16F_{2n+6})}{F_{2n+9}^2 - 4} + \frac{1}{\alpha^k \sqrt{5}} \right] \\ &\leq \frac{1}{\beta_n^{2j} \gamma_n^{(\pm)}} \left[\frac{2(F_{2n+12} + 16F_{2n+6})}{F_{2n+9}^2 - 4} + \frac{1}{\alpha^{2n+1} \sqrt{5}} \right]. \end{aligned}$$

We assume k is odd (for k even the inequality is clear), and $k \geq 2n$ from earlier. We then have that the above:

$$\begin{aligned} &= \frac{1}{\beta_n^{2j} \gamma_n^{(\pm)}} \left[\frac{2}{F_{2n+6}} + \frac{32}{F_{2n+12}} + \frac{1}{\alpha^{2n+1} \sqrt{5}} \right] \\ &< \frac{144.026}{\beta_n^{2j}} \alpha^{2n+6} \left[\frac{2\sqrt{5}}{\alpha^{2n+6} - \frac{1}{\alpha^{2n+6}}} + \frac{32\sqrt{5}}{\alpha^{2n+12} - \frac{1}{\alpha^{2n+12}}} + \frac{1}{\alpha^{2n+1} \sqrt{5}} \right] \\ &= \frac{144.026}{\beta_n^{2j}} \left[\frac{2\sqrt{5}}{1 - \frac{1}{\alpha^{4n+12}}} + \frac{32\sqrt{5}}{\alpha^6 - \frac{1}{\alpha^{4n+18}}} + \frac{\alpha^5}{\sqrt{5}} \right] \\ &\leq \frac{144.026}{\beta_n^{2j}} \left[\frac{2\sqrt{5}}{1 - \frac{1}{\alpha^{16}}} + \frac{32\sqrt{5}}{\alpha^6 - \frac{1}{\alpha^{22}}} + \frac{\alpha^5}{\sqrt{5}} \right] < \frac{1934}{\beta_n^{2j}} < \frac{1}{2}. \end{aligned}$$

So we have $|\Lambda| < \frac{3868}{\beta_n^{2j}}$ □

Proposition 5.3. *If (14) has a positive integer solution (j, k) with $j > 1$, then*

$$j < 1.16 \cdot 10^{10} (2n + 9) \log(156j(n + 5)).$$

Proof. We take

$$\begin{aligned} N = 3, \quad D = 4, \quad b_1 = 2j, \quad b_2 = -k, \quad b_3 = 1, \\ \alpha_1 = \beta_n, \quad \alpha_2 = \alpha, \quad \alpha_3 = \sqrt{5}\gamma_n^{(\pm)}. \end{aligned}$$

We have from Lemma 2.16 that $\alpha_1, \alpha_2, \alpha_3$ are multiplicatively independent.

We have

$$h(\alpha_1) = h(\beta_n) = \frac{1}{2} \log \beta_n, \text{ and } H(\alpha_2) = h(\alpha) = \frac{1}{2} \log \alpha.$$

We now calculate $h(\alpha_3)$.

$$\begin{aligned} (X - \gamma_n^{(+)}) (X - \gamma_n^{(-)}) &= X^2 - (\gamma_n^{(+)} + \gamma_n^{(-)})X + \left(\frac{F_{2n+12} - 16F_{2n+6}}{F_{2n+6}F_{2n+12}} \right)^2 \\ &= F_{2n+6}^2 F_{2n+12}^2 X^2 - 2(F_{2n+12}^2 F_{2n+6} + 16F_{2n+12} F_{2n+6}^2)X + (F_{2n+12} - 16F_{2n+6})^2, \end{aligned}$$

so

$$\begin{aligned} h(\gamma_n^{(\pm)}) &= \frac{1}{2} \left[\log (F_{2n+6}^2 F_{2n+12}^2) + \log (1) + \log (1) \right] = \log (F_{2n+6} F_{2n+12}) \\ &< \log \left(\frac{\alpha^{2n+6} \alpha^{2n+12}}{5} \right) = (4n + 18) \log (\alpha) + \log \left(\frac{1}{5} \right). \end{aligned}$$

Thus

$$\begin{aligned} h(\alpha_3) &= h(\sqrt{5}\gamma_n^{(\pm)}) \leq h(\sqrt{5}) + h(\gamma_n^{(\pm)}) \\ &< \frac{1}{2} \log (5) + (4n + 18) \log (\alpha) + \log \left(\frac{1}{5} \right) \\ &= (4n + 18) \log (\alpha). \end{aligned}$$

We have

$$\begin{aligned} A_1 &= \max \{2 \log (\beta_n), |\log (\beta_n)|\} = 2 \log (\beta_n), \\ A_2 &= \max \{2 \log \alpha, |\log \alpha|\} = 2 \log \alpha, \\ A_3 &= \max \{4(4n + 18) \log (\alpha), |\log (\sqrt{5}\gamma_n^{(\pm)})|\} = 8(2n + 9) \log (\alpha). \end{aligned}$$

As $\alpha^{l-2} \leq F_l \leq \alpha^{l-1}$, we have that $\beta_n < 2F_{2n+9} < 2\alpha^{2n+8} < \alpha^{2(n+5)}$.

Moreover,

$$\begin{aligned}
\alpha^{k-1} &< 2\alpha^{k-2} < 2F_k \leq 64U_jV_j + 8U_j^2(F_{2n+12} + 16F_{2n+6}) \\
&< (V_j + U_j\sqrt{F_{2n+6}F_{2n+12}})^2 = \left(\frac{F_{2n+9} + \sqrt{F_{2n+9}^2 - 4}}{2}\right)^{2j} \\
&< F_{2n+9}^{2j} < (\alpha^{2n+8})^{2j} < \alpha^{4j(n+4)}.
\end{aligned}$$

Therefore we consider

$$\begin{aligned}
E &\leq 4j(n+5) \\
C(3) &< 6.45 \times 10^8 \\
C_0 &< 30 \\
W_0 &= \log(1.5eE4 \log(4e)) < \log(156j(n+5)) \\
\Omega &= A_1A_2A_3 \leq 32(2n+9) \log(\alpha) \log(\beta_n).
\end{aligned}$$

This gives, by Lemma (2.13),

$$\begin{aligned}
\log |\Lambda| &> -C(N)C_0W_0D^2\Omega \\
&\geq -6.45 \times 10^8 \cdot 30 \log(156j(n+5))(32(2n+9)(\log \alpha)^2(\log \beta_n)).
\end{aligned}$$

So

$$\begin{aligned}
\log \Lambda &< \log \frac{3868}{\beta_n^{2j}} = \log(3868) - 2j \log(\beta_n) \\
j &< 1.16 \times 10^{10}(2n+9) \log(156j(n+5)).
\end{aligned}$$

□

5.3 A Linear Form in Two Logarithms (2)

Lemma 5.4. *Define $\Lambda_0 = 2 \log \beta_n - 2n \log \alpha + \log(\sqrt{5}\gamma_n^{(\pm)})$. We have that*

$$|\Lambda_0| < \frac{270830}{\beta_n^2}.$$

Proof. Assume that $n \geq 2$. We have

$$\beta_n^2 \gamma_n^{(\pm)} - \frac{\alpha^{2n}}{\sqrt{5}} = \frac{2(F_{2n+12} + 16F_{2n+6})}{F_{2n+9} - 4} - \beta_n^{-2} \gamma_n^{(\mp)} - \frac{\alpha^{-2n}}{\sqrt{5}}.$$

If $\frac{1}{\beta_n^2 \gamma_n^{(\pm)}} \geq \alpha^{-2n} \sqrt{5}$, then $\frac{1}{5\beta_n^2 \gamma_n^{(\pm)}} \geq \frac{\alpha^{-2n}}{\sqrt{5}}$, and we have

$$\begin{aligned} |\alpha^{2n} 5^{-\frac{1}{2}} \beta_n^{-2} (\gamma_n^{(\pm)})^{-1} - 1| &= \frac{1}{\beta_n^2 \gamma_n^{(\pm)}} \left| \frac{\alpha^{2n}}{\sqrt{5}} - \beta_n^2 \gamma_n^{(\pm)} \right| \\ &< \frac{\beta_n^{-2} \gamma_n^{(\mp)} + \frac{\alpha^{-2n}}{\sqrt{5}}}{\beta_n^2 \gamma_n^{(\pm)}} < \frac{\gamma_n^{(\mp)} + \frac{1}{5\gamma_n^{(\pm)}}}{\beta_n^4 \gamma_n^{(\pm)}} \\ &< \frac{8.46635831 + \frac{\alpha^{4n+12}}{5(0.00694321)}}{0.00694321 \beta_n^{4n}} \\ &< \frac{1219.4 + 4148.7 \alpha^{4n+12}}{\beta_n^4} \\ &\leq \frac{\frac{1219.4}{\beta_n^2} + \frac{4148.7 \alpha^{4n+12}}{\beta_n^2}}{\beta_n^2} < 4149 \beta_n^{-2}. \end{aligned}$$

If $\beta_n^2 \gamma_n^{(\pm)} > \frac{\alpha^{2n}}{\sqrt{5}}$, then

$$\begin{aligned} |\alpha^{2n} 5^{-\frac{1}{2}} \beta_n^{-2} (\gamma_n^{(\pm)})^{-1} - 1| &= \frac{1}{\beta_n^2 \gamma_n^{(\pm)}} (\beta_n^2 \gamma_n^{(\pm)} - \alpha^{2n} 5^{-\frac{1}{2}}) \\ &< \frac{1}{\beta_n^2 \gamma_n^{(\pm)}} \frac{2(F_{2n+12} + 16F_{2n+6})}{F_{2n+9} - 4} = \frac{\left(\frac{2}{F_{2n+6}} + \frac{32}{F_{2n+12}} \right)}{\beta_n^2 \gamma_n^{(\pm)}} \\ &< \frac{\left(\frac{2}{F_{2n+6}} + \frac{32}{F_{2n+12}} \right)}{\beta_n^2 (0.00694321) \alpha^{-2n-6}} = \frac{\frac{2\alpha^{2n+6}}{F_{2n+6}} + \frac{32\alpha^{2n+6}}{F_{2n+12}}}{\beta_n^2 (0.00694321)} \\ &< \frac{2\alpha^2 + \frac{32}{\alpha^4}}{\beta_n^2 (0.00694321)} < \frac{9.91}{\beta_n^2 (0.00694321)} < \frac{1428}{\beta_n^2}. \end{aligned}$$

So $\Lambda_0 \leq 2 \log \beta_1 - 2 \log \alpha + \log(\sqrt{5}\gamma_1^{(+)}) < \frac{270830}{\beta_n^2}$. In all cases we have

$$|\Lambda_0| < \frac{270830}{\beta_n^2}.$$

□

Lemma 5.5. *Let*

$$\Lambda_1 := K \log \alpha - (j-1) \log \left(\frac{5}{4} \right), \text{ where } K := (2j-1)(2n+9) - k - 9.$$

Then we have

$$|\Lambda_1| < \frac{10(j+5493)}{\alpha^{4n+14}}.$$

Proof. We know

$$\begin{aligned} \beta_n &= F_{2n+9} + \sqrt{F_{2n+9}^2 - 4} = 2F_{2n+9} - \frac{4}{F_{2n+9} + \sqrt{F_{2n+9}^2 - 4}} \\ &= 2F_{2n+9} \left(1 - \frac{4}{2F_{2n+9}(F_{2n+9} + \sqrt{F_{2n+9}^2 - 4})} \right) \end{aligned}$$

and

$$2F_{2n+9} = \frac{2}{\sqrt{5}}(\alpha^{2n+9} - \bar{\alpha}^{2n+9}) = \frac{2}{\sqrt{5}}\alpha^{2n+9} \left(1 + \frac{1}{\alpha^{4n+18}} \right).$$

Define

$$\delta_n := \left(1 - \frac{4}{2F_{2n+9}(F_{2n+9} + \sqrt{F_{2n+9}^2 - 4})} \right) \left(1 + \frac{1}{\alpha^{4n+18}} \right).$$

Then $\log(\beta_n) = \log \left(\frac{2}{\sqrt{5}} \right) + (2n+9) \log \alpha + \log \delta_n$.

$$\begin{aligned} \Lambda - \Lambda_0 &= (2j-2) \log(\beta_n) - (k-2n) \log(\alpha) \\ &= (2j-2) \log \left(\frac{2}{\sqrt{5}} \right) + (2j-2)(2n+9) \log(\alpha) \\ &\quad + (2j-2) \log(\delta_n) - (k-2n) \log(\alpha) \\ &= (2j-2) \log(\delta_n) + K \log(\alpha) - (j-1) \log \left(\frac{5}{4} \right). \end{aligned}$$

So $\Lambda_1 = \Lambda - \Lambda_0 - (2j - 2) \log(\delta_n)$. By the triangle inequality we have

$$|\log(\delta_n)| \leq \left| \log \left(1 - \frac{4}{2F_{2n+9}(F_{2n+9} + \sqrt{F_{2n+9}^2 - 4})} \right) \right| + \left| \log \left(1 + \frac{1}{\alpha^{4n+18}} \right) \right|,$$

so

$$\begin{aligned} |\log(\delta_n)| &< \frac{4}{F_{2n+9}(F_{2n+9} + \sqrt{F_{2n+9}^2 - 4})} + \frac{1}{\alpha^{4n+18}} \\ &< \frac{4}{\alpha^{4n+14}} + \frac{1}{\alpha^{4n+18}} < \frac{5}{\alpha^{4n+14}}, \end{aligned}$$

and

$$|\Lambda_1| \leq |\Lambda| + |\Lambda_0| + |2j - 2| |\log \delta_n| < \frac{3868}{\beta_n^2} + \frac{270830}{\beta_n^2} + \frac{10(j-1)}{\alpha^{4n+14}}.$$

Clearly

$$\beta_n = F_{2n+9} \left(1 + \sqrt{1 - \frac{4}{F_{2n+9}^2}} \right) \geq F_{2n+9} \left(1 + \frac{89^2 - 4}{89^2} \right) > \frac{\alpha^{2n+9}}{\sqrt{5}} \left(1 + \sqrt{\frac{7917}{89^2}} \right).$$

Thus

$$\beta_n^2 > \alpha^{4n+18} \left(1 + \frac{\sqrt{7917}}{89} \right)^2 > \frac{3}{4} \alpha^{4n+18} > 5\alpha^{4n+14},$$

$$\text{and we have } |\Lambda_1| < \frac{274698}{\beta_n^2} + \frac{10(j-1)}{\alpha^{4n+14}} < \frac{10(j+5493)}{\alpha^{4n+14}}. \quad \square$$

5.4 Refining Our Bounds

Lemma 5.6. *If (14) has a positive integer solution (j, k) with $j > 1$ then*

$$j < 6.25 \times 10^{21} \text{ and } n < 37024766.$$

Proof. We apply Lemma 2.13. We have

$$D = 2, \quad \gamma_1 = \frac{5}{4}, \quad \gamma_2 = \alpha, \quad b_1 = (j-1), \quad b_2 = K.$$

We take $h_1 = \log(5)$, and $h_2 = \frac{1}{2}$. We have $K \log(\alpha) - (j-1) \log\left(\frac{5}{4}\right) < \frac{10(j+5493)}{\alpha^{4n+14}}$, so

$$K < \frac{(j-1) \log\left(\frac{5}{4}\right) + \frac{10(j+5493)}{\alpha^{4n+14}}}{\log(\alpha)} < 19.3 + 0.47j.$$

Thus we take

$$\frac{|b_1|}{Dh_2} + \frac{|b_2|}{Dh_1} = (j-1) + \frac{K}{2\log(5)} \leq 4.996 + 1.15j.$$

We let $b' = 5 + 1.15j$. Hence

$$\log |\Lambda_1| > (-17.9) \cdot 16 \cdot (\max\{\log(5 + 1.15j) + 0.38, 15\})^2 \cdot \log(5) \cdot \frac{1}{2}.$$

The previous Lemma gives: $\log |\Lambda_1| < -(4n + 14) \log(\alpha) + \log(10(j + 5493))$, so we have

$$\begin{aligned} (-17.9) \cdot 16 \cdot (\max\{\log(5 + 1.15j) + 0.38, 15\})^2 \cdot \log(5) \cdot \frac{1}{2} &< \log |\Lambda_1| \\ &< -(4n + 14) \log(\alpha) + \log(10(j + 5493)). \end{aligned}$$

Rearranging this we get

$$n < 120(\max\{\log(5 + 1.15j) + 0.38, 15\})^2 + 0.52 \cdot \log(10(j + 5493)).$$

If $\log(1.15j + 5) + 0.38 < 15$, then $j < 1943957$ and $n < 27009$. Otherwise

$$n < 120(\log(5 + 1.15j) + 0.38)^2 + 0.52 \log(10(j + 5493)).$$

Combining this with $j < 1.16 \times 10^{12}(2n + 9) \log(156j(n + 5))$ gives $j < 6.25 \times 10^{21}$ and $n < 37024766$. \square

Proposition 5.7. *If (14) has a positive integer solution (j, k) , with $j > 1$, then*

$$n < 1.04 \cdot \log j + 4.6.$$

Proof. We have that $|K \log(\alpha) - (j-1) \log\left(\frac{5}{4}\right)| < 10(j + 5493)\alpha^{-(4n+14)}$. Hence

$$\left| \frac{\log(5/4)}{\log(\alpha)} - \frac{K}{(j-1)} \right| < \frac{10(j + 5493)}{(j-1)\alpha^{4n+14} \log(\alpha)}.$$

First, assume

$$\frac{10(j + 5493)}{(j-1)\alpha^{4n+14} \log(\alpha)} < \frac{1}{2(j-1)^2}. \quad (16)$$

Then

$$\left| \frac{\log(5/4)}{\log(\alpha)} - \frac{k}{(j-1)} \right| < \frac{1}{2(j-1)^2}.$$

The denominator of the 49th convergent of $\frac{\log(5/4)}{\log(\alpha)}$ is greater than 6.25×10^{21} , our upper bound of j . The 48th convergent gives the lower bound:

$$\left| \frac{\log(5/4)}{\log(\alpha)} - \frac{K}{j-1} \right| > 4 \times 10^{-44}.$$

So

$$4 \times 10^{-44} < \frac{10(j+5493)}{(j-1)\alpha^{4n+14}\log(\alpha)} < 5513 \cdot \alpha^{-4n-14}(\log(\alpha))^{-1}.$$

This gives $\alpha^{4n} < \frac{4 \cdot 10^{44} \cdot (5513)}{\alpha^{14}\log(\alpha)}$, so $n < \frac{\log \frac{4 \times 10^{44} \cdot (5513)}{\alpha^{14}\log(\alpha)}}{4 \cdot \log(\alpha)}$ gives us $n < 55$.

Since we know that $\frac{p_r}{q_r}$ is the r th convergent, $\left| \frac{\log(5/4)}{\log(\alpha)} - \frac{p_r}{q_r} \right| > \frac{1}{(a_{r+1}+2)q_r^2}$, and since $\max\{a_{r+1} : 2 \leq r \leq 47\} = a_{36} = 49$,

$$\frac{1}{51(j-1)^2} < \frac{10(j+5493)}{(j-1)\alpha^{4n+14}\log(\alpha)} \implies \alpha^{4n+14} < \frac{510(j-1)(j+5493)}{\log(\alpha)}.$$

If (16) does not hold, we have

$$\frac{1}{2(j-1)^2} \leq \frac{10(j+5493)}{(j-1)\alpha^{4n+14}\log(\alpha)} \implies \alpha^{4n+14} \leq \frac{20(j-1)(j+5493)}{\log(\alpha)}.$$

In both cases

$$\alpha^{4n+14} < \frac{510(j-1)(j+5493)}{\log(\alpha)} < 1060j(j+5493) < 5823640j^2,$$

so $n < 1.04 \cdot \log(j) + 4.6$. □

Combining the above proposition with 5.3 yields:

Lemma 5.8. *If (14) has a positive integer solution (j, k) with $j > 1$, then*

$$j < 4.88 \times 10^{15} \text{ and } n < 43.$$

5.5 Baker-Davenport Reduction

Using the same method as before, we have obtained our bounds on j and n . All that remains to prove Theorem 1.2 is to apply Baker-Davenport reduction to these bounds.

We have

$$0 < 2j \log \beta_n - k \log \alpha + \log(\sqrt{5} \cdot \gamma_n^{(\pm)}) < 3868\beta_n^{-2j}.$$

To apply Baker-Davenport reduction, consider

$$\kappa = \frac{2 \log \beta_n}{\log \alpha}, \quad \mu = \frac{\log(\sqrt{5} \cdot \gamma_n^{(\pm)})}{\log \alpha}, \quad A = \frac{3868}{\log \alpha}, \quad M = 4.88 \times 10^{15}$$

We again use procedures written in MapleTM (see Appendix B) to find $j \leq 5$, which in turn gives us $1 \leq n \leq 5$.

Lemma 5.9. *If (14) has a positive integer solution (j, k) with $j > 1$, then*

$$j \leq 5 \text{ and } n \leq 5.$$

We can very quickly check all possible j between 2 and 5 and n from 1 to 5 in (15) to see that no solution exists. We already saw that $F_{14} < C_1^{(+)} < F_{15}$, so the only possible solution to (15) is $C_1^{(-)} = F_{2n}$. When $n = 1$, we have the solution $F_{2n} = F_2 = 1 = F_1$, so there is an additional solution in this case again.

6 Conclusion

In both cases the bounds were refined, through various linear forms in logarithms lemmas, to the extent that all that was needed was to run a simple program to check the remaining cases.

It is not as simple to extend this result to the larger families of Diophantine triples, $\{F_{2n}, L_m^2 F_{2n+2m}, F_{2n+4m}\}$. The divisibility condition in Lemma (3.1), that is that if l is prime then $l \mid a$ or $l \mid b$, otherwise $l^2 \mid a$ or $l^2 \mid b$, holds for the cases mentioned here, where $m = 2$, and conditionally for $m = 3$, but it does not hold in general. In the case of $m = 4$ for instance, the divisibility condition in Lemma (3.1) holds only for certain n . It would suffice to show, however, that this condition holds for the particular family of Diophantine triples, $\{F_{2n}, L_m^2 F_{2n+2m}, F_{2n+4m}\}$.

One additional complication lies in the application of the lemma of Mateev. It would be necessary to find a way to give a more general version of Lemmas (2.15) and (2.16), which is not possible with the methods used to prove said lemmas. The Pellian equations used in the proof may have multiple classes of solutions.

References

- [1] L. Bačić and A. Filipin, *The extensibility of $D(4)$ -pairs*, Math. Commun. **18** (2013), 447-456.
- [2] S.E. Rihane, M.O. Hernane, A. Togbé, *On the $D(4)$ -Diophantine triples of Fibonacci numbers*, Fibonacci Quart. **56.1** (2018)
- [3] A. Dujella and A. Pethő, *A generalization of a theorem of Baker and Davenport*, Quart. J. Math. Oxford Ser. (2) **49** (1998), 291-306.
- [4] A. Filipin, B. He, A. Togbé, *On the $D(4)$ -triple $\{F_{2k}, F_{2k+6}, F_{2k+4}\}$* , Fibonacci Quart. **48.3** (2010), 219-227.
- [5] B. He, F. Luca, A. Togbé, *Diophantine triples of Fibonacci numbers*, Acta Arithmetica **175** (2016), 57-70.
- [6] A. Ya. Khinchin, *Continued Fractions*, University of Chicago Press, Chicago 1964.
- [7] M. Laurent, *Linear forms in two logarithms and interpolation determinants II*, Acta Arith. **133** (2008), 325-348.
- [8] E. M. MATEEV, *An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers II*, Izv. Math. **64** (2000), 1217-1269.
- [9] S. Bujačić and A. Filipin *Linear Forms in Logarithms*, Chapter 1 in *Diophantine Analysis, Course Notes from a Summer School*, Birkhäuser
- [10] B.W. Jones, *A second variation on a problem of Diophantus and Davenport*, Fibonacci Quart. **16** (1978), 155-165.
- [11] A. Dujella, *Diophantine quadruples and Fibonacci numbers*, 2004.
- [12] T. Nagell, *Introduction to Number Theory*, Wiley, 1951.
- [13] Maple 2017. Maplesoft, a division of Waterloo Maple Inc., Waterloo, Ontario.
- [14] A. Wiles, *Modular elliptic curves and Fermat's Last Theorem*; Annals of Math. 142(1995), 443-551
- [15] S. Singh, *Fermat's Enigma*, Penguin Canada, 1997.

- [16] T. L. Heath, *Diophantus of Alexandria. A Study in the History of Greek Algebra, With a supplement containing an account of Fermat's theorems and problems connected with Diophantine analysis and some solutions of Diophantine problems by Euler*, Cambridge University Press, 1910.
- [17] J. Gies, F. Gies, *Leonardo of Pisa and the New Mathematics of the Middle Ages*, New Classics Library, 1983.
- [18] A. Baker, Linear forms in the logarithms of algebraic numbers, I. *Mathematika J. Pure Appl. Math.* **13** 204-216 (1966)
- [19] A. Baker, Linear forms in the logarithms of algebraic numbers, II. *Mathematika J. Pure Appl. Math.* **14** 102-107 (1967)
- [20] A. Baker, Linear forms in the logarithms of algebraic numbers, III. *Mathematika J. Pure Appl. Math.* **14** 220-228 (1967)
- [21] P. Singh, The so-called fibonacci numbers in ancient and medieval India, *Historia Mathematica*, **12** 229-244 (1985)
- [22] B. He, A. Togbé, V. Ziegler, There is no Diophantine quintuple, *Trans. Amer. Math. Soc.* **371** (2019), 6665-6709
- [23] A. Thue, Über Näherungswerte und Kettenbrüche, *J. Reine Angew. Math. (Crelle)*, **115**(3), 221-233 (1895)
- [24] O. Neugebauer, *The Exact Sciences in Antiquity*, Dover Publications, 1969
- [25] D. Wall, *Fibonacci series modulo m*, *Amer. Math. Monthly*, **67** (1960), 525-532
- [26] V. Andreji, *On Fibonacci Powers*, *Publikacije Elektrotehničkog Fakulteta, Serija Matematika*, textbf17 (2006), 38-44
- [27] R.D. Carmichael, *On the Numerical Factors of the Arithmetic Forms $\alpha^n \pm \beta^n$* , *Ann. of Math.* **15** (1913), 30-70

A D(9) Baker-Davenport Reduction Calculations

```

> with (combinat) :
> F := n → fibonacci(n) :
> gammaplus := n →  $\frac{3}{2 \cdot \sqrt{F(2 \cdot n + 4) \cdot F(2 \cdot n + 8)}} + \frac{(F(2 \cdot n + 8) + 9 \cdot F(2 \cdot n + 4))}{4 \cdot F(2 \cdot n + 4) \cdot F(2 \cdot n + 8)}$  :
> gammaminus := n →  $\frac{-3}{2 \cdot \sqrt{F(2 \cdot n + 4) \cdot F(2 \cdot n + 8)}} + \frac{(F(2 \cdot n + 8) + 9 \cdot F(2 \cdot n + 4))}{4 \cdot F(2 \cdot n + 4) \cdot F(2 \cdot n + 8)}$  :
> betan := n → F(2 · n + 6) + sqrt((F(2 · n + 6))2 - 1) :
> a :=  $\frac{(1 + \sqrt{5})}{2}$  :
> kappan := n →  $\frac{(2 \cdot \log(\text{betan}(n)))}{\log(a)}$  :
> muminus := n →  $\left( \frac{\log(\sqrt{5} \cdot \text{gammaminus}(n))}{\log(a)} \right)$  :
> muplus := n →  $\left( \frac{\log(\sqrt{5} \cdot \text{gammaplus}(n))}{\log(a)} \right)$  :
> A :=  $\frac{1162}{\log(a)}$  :
> Bn := n → (betan(n))2 :
> M := 4.63 · 1015 :
> with (numtheory) :
> with (NumberTheory) :
> cfkapn := n → ContinuedFraction(kappan(n)) :
> fc := proc(n, t)
  local m, i, Q;
  Q := 0;
  i := 0;
  m := cfkapn(n);
  while Q ≤ t do
    i := i + 1;
    Q := Denominator(m, i);
  end do;
  return (Q, i)
end proc;
> fc(1, 6 · M)

```

44045204789851415, 29 (1)

```

> etan := proc(n)
  local m, i, t, x, y, j;
  j := 6 · M;
  t := -1;
  y := 0;
  x := -1;
  while (evalf(t - y) ≤ 0) or (evalf(x - y) ≤ 0) do

```

```

t := abs(frac(muplus(n):fc(n,j)[1]) - round(frac(muplus(n):fc(n,j)[1]));
x := abs(frac(muminus(n):fc(n,j)[1]) - round(frac(muminus(n):fc(n,j)[1]));
y := M * (abs(frac(kappan(n):fc(n,j)[1]) - round(frac(kappan(n):fc(n,j)[1]))));
j := fc(n,j)[1]
end do:
return(evalf(t - y), evalf(x - y), j)
end proc:
> etan(42)
0.0086482577176300636077265821671585594635487205336798334,
3.130595754269031981857470813366567538574662885 10-10,
129498349586713017249929679
(2)
> bound := proc(n)
local x, y;
x :=  $\frac{\log\left(\frac{A \cdot \text{etan}(n)[3]}{\text{etan}(n)[1]}\right)}{\log(Bn(n))}$ ;
y :=  $\frac{\log\left(\frac{A \cdot \text{etan}(n)[3]}{\text{etan}(n)[2]}\right)}{\log(Bn(n))}$ ;
return(floor(evalf(x)), floor(evalf(y)))
end proc:
> bound(40)
0, 1
(3)
> listn := proc(n)
local i;
for i from 1 to n do
print(bound(i))
end do:
end proc:
> listn(42)
6, 6
5, 5
4, 4
3, 3
3, 3
2, 2
2, 2
2, 2
2, 2
1, 1
1, 1
1, 1

```


B D(64) Baker-Davenport Reduction Calculations

```

> with (combinat) :
> F := n → fibonacci(n) :
> gammaplus := n →  $\frac{8}{\sqrt{(F(2 \cdot n + 6) \cdot F(2 \cdot n + 12))} + \frac{(F(2 \cdot n + 12) + 16 \cdot F(2 \cdot n + 6))}{F(2 \cdot n + 6) \cdot F(2 \cdot n + 12)}}$  :
> gammaminus := n →  $\frac{-8}{\sqrt{(F(2 \cdot n + 6) \cdot F(2 \cdot n + 12))} + \frac{(F(2 \cdot n + 12) + 16 \cdot F(2 \cdot n + 6))}{F(2 \cdot n + 6) \cdot F(2 \cdot n + 12)}}$  :
> betan := n →  $\frac{(F(2 \cdot n + 9) + \sqrt{(F(2 \cdot n + 9))^2 - 4})}{2}$  :
> a :=  $\frac{(1 + \sqrt{5})}{2}$  :
> kappan := n →  $\frac{(2 \cdot \log(\text{betan}(n)))}{\log(a)}$  :
> muminus := n →  $\left( \frac{\log(\sqrt{5}) \cdot \text{gammaminus}(n)}{\log(a)} \right)$  :
> muplus := n →  $\left( \frac{\log(\sqrt{5}) \cdot \text{gammaplus}(n)}{\log(a)} \right)$  :
> A :=  $\frac{3868}{\log(a)}$  :
> Bn := n → (betan(n))2 :
> M := 4.88 · 1015 :
> with (numtheory) :
> with (NumberTheory) :
> cfkapn := n → ContinuedFraction(kappan(n)) :
> fc := proc(n, t)
  local m, i, Q;
  Q := 0;
  i := 0;
  m := cfkapn(n);
  while Q ≤ t do
    i := i + 1;
    Q := Denominator(m, i);
  end do;
  return(Q, i)
end proc;
> fc(1, 6 · M)

```

302835779297689803, 33

(1)

```

> etan := proc(n)
  local m, i, t, x, y, j;
  j := 6 · M;

```

```

t := -1;
y := 0;
x := -1;
while (evalf(t - y) ≤ 0) or (evalf(x - y) ≤ 0) do
t := abs(frac(muplus(n):fc(n,j)[1]) - round(frac(muplus(n):fc(n,j)[1])));
x := abs(frac(muminus(n):fc(n,j)[1]) - round(frac(muminus(n):fc(n,j)[1])));
y := M·(abs(frac(kappan(n):fc(n,j)[1]) - round(frac(kappan(n):fc(n,j)[1]))));
j := fc(n,j)[1]
end do;
return (evalf(t - y), evalf(x - y), j)
end proc:
> etan(43)
0.41018902218821349577881091509320177001879931769702987913,
3.93611804593920658506665318568192537527773274 10-12,
59518701483690965606765821
(2)

> bound := proc(n)
local x, y;
x :=  $\frac{\log\left(\frac{A \cdot \text{etan}(n)[3]}{\text{etan}(n)[1]}\right)}{\log(Bn(n))}$ ;
y :=  $\frac{\log\left(\frac{A \cdot \text{etan}(n)[3]}{\text{etan}(n)[2]}\right)}{\log(Bn(n))}$ ;
return (floor(evalf(x)), floor(evalf(y)))
end proc:
> bound(43)
0, 1
(3)

> listn := proc(n)
local i;
for i from 1 to n do
print(bound(i))
end do;
end proc:
> listn(43)
5, 5
4, 4
4, 4
3, 3
3, 2
2, 2
2, 2
2, 2
2, 2
2, 1

```

