Diophantine Triples and Linear Forms in Logarithms

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Abstract

This is the thesis for my master's degree in mathematics which I undertook with Dr. Omar Kihel. Over the last couple of years I have studied number theory with the aim being to develop a broader understanding of the theory of Diophantine equations and their (at times) elusive solutions. I begin my thesis by establishing some of the preliminary results while touching on their place within the history of number theory. This section finishes with an account of Alan Baker's work on linear forms in logarithms and some of its applications, after which the two theorems on Diophantine triples that this paper will aim to prove are stated. In the second section, I list a series of definitions and results of which the reader must be aware, but which I could not fit into the first section due to its historical slant. Following this, I prove a lemma on Pellian equations which generalizes the first lemma of [1]. This requires that a mistake from the proof of that lemma be fixed. Since this lemma was used in [2], this section serves to buttress that result as well. In the next two sections, I prove the two main theorems using results on linear forms in logarithms of algebraic numbers, extending the main result in [2] to $D(9)$ and $D(64)$ triples. The thesis ends with a few words on potential generalization and improvement of the main results, as well as other potential avenues of inquiry, and draws attention to some potential difficulties. The main results closely follow a paper co-written with my brother, Benjamin Earp-Lynch.

Key words and phrases: Linear forms in logarithms; Diophantine triples; Pellian equations; Fibonacci numbers.

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Contents

1 Introduction and Preliminaries

1.1 Mathematics BC

In the early twentieth century, a clay tablet was recovered from the desert in Iraq. It made its way to New York publisher George Plimpton in 1922, and was bequeathed to Columbia University in the 1930s. The tablet came to be known as Plimpton 322, and has been dated to around 1800 BC. Upon the tablet - carved in the customary Babylonian cuneiform script using their base 60 number system - is a list of positive integer (whole number) solutions (a,b,c) to the equation

$$a^2 + b^2 = c^2,$$

which alternatively can be thought of as a list of right-angle triangles with all integer sides. We call such solutions *Pythagorean triples.* This is due to the so-called *Pythagorean theorem*, named after the Greek mathematician Pythagoras (570-495 BC) - who legend has it recorded the first proof of the result, though no evidence of it has survived. The theorem states that the sum of the squares of the two shorter sides of a right triangle is equal to its hypotenuse. Due to the notation on the tablet, it seems that the Babylonians had some awareness of this result about 1200-1300 years before Pythagoras, but there is not much indication that they considered mathematics an intellectual discipline unto itself the way the ancient Greeks did. (See [14] and [33] for more on Plimption 322 and ancient Greek and Babylonian mathematics in general.)

Pythagoras himself was obsessed with integers and rational numbers (fractions of integers). He balked at the notion of irrational numbers - numbers that cannot be expressed as a ratio of two integers - but offend as they might his sense of mathematical aesthetic, it seems unlikely that he wasn't aware of them. Given a right triangle with shorter sides both equal to 1, by Pythagoras' eponymous theorem itself the length of the third side must square to 2, or in other words it must have length x such that

$$x^2 = 2.$$

The solutions to the above equation are written $\pm\sqrt{2}$, and it can be shown relatively easily that they cannot possibly be written as a ratio of integers. As the story goes, this was brought to his attention by a student, and that Pythagoras (or his followers) objected so vehemently that the whistleblower was drowned at sea and mention of the affair thenceforth forbidden. (See [15] for a compilation of classical accounts of Pythagoras, his followers and his philosophy.) His attempts to stifle news of irrational numbers, if indeed any attempts were actually made, would ultimately be futile. This is mostly due to the fact that the more one looks at expressions involving integers and rational numbers, the more irrational numbers inevitably rear their heads. They are (so to speak) essential trappings of the universe itself.

1.2 The Golden Ratio and Algebraic Numbers

The golden ratio held great aesthetic appeal to the ancient Greeks (and indeed to many others throughout the intervening centuries). Euclid, writing in his *Elements* (see [21] for an English translation), recorded a definition in around 300 BC.

A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the lesser.

This ratio is precisely

$$\left[1 : \varphi = \frac{1+\sqrt{5}}{2} = 1.618\ldots\right],$$

where φ is, despite the probable will of the Pythagoreans, irrational. It can, however, be shown to be the positive solution to the polynomial equation,

$$x^2 - x - 1 = 0.$$

The reader might note the similarity to the previous example of an irrational number, $\sqrt{2}$, which was a solution to the equation $x^2 - 2 = 0$. In both examples so far, the irrational number has been the solution to a polynomial equation with integer coefficients. We call such numbers (be they rational or irrational) *algebraic numbers*. The algebraic numbers encompass all numbers this paper has mentioned so far and many more besides. The solutions to the equation $x^2 + 1 = 0$ are given by $\pm i$, where i is what we call the *imaginary unit*. The imaginary unit gives rise to the *complex numbers*, numbers of the form $a + bi$ where a and b may be rational or irrational. Complex numbers were not studied until the 1500s [22].

Remark (Some Notation)**.** *Henceforth, where appropriate, the symbol* $\mathbb{Z}$ *shall signify the set of all integers and the symbol* $\mathbb{Q}$ *that of the rational numbers. Strictly irrational numbers will be said to belong to the set* $\mathbb{R}\setminus\mathbb{Q}$ *(R without Q) where* $\mathbb{R}$ *denotes the real numbers - which encompass all rational and irrational numbers. Lastly,* $\mathbb{C}$ *will represent the complex numbers.*

This leads to the seemingly reasonable thought: while it may be that not all numbers are rational, are all numbers algebraic? That is, can they be expressed as solutions to polynomial equations with integer coefficients? Alas, as elegant and appealing as that might seem, it is not the case. However, while they were strongly suspected to exist earlier, it wasn't until 1844 that French mathematician Joseph Liouville (1809-1882) proved the existence of numbers beyond the algebraic [9]. We call such numbers *transcendental*. Transcendental numbers will be the focus of section 1.6.

1.3 Diophantus and his Equations

During the third century AD, long after the Babylonians and even the Pythagoreans, there lived in Alexandria a prolific mathematician by the name of Diophantus. Known primarily for his work in number theory (indeed, his most famous and influential work was a multi-volume mathematical epic entitled *Arithmetica* - consisting of dozens of problems and their solutions), the echoes of Diophantus' impact upon mathematics have reverberated through the centuries that followed him.

One of Diophantus' chief concerns was finding integer and rational solutions to polynomial equations. So influential was his work in this regard that we have come to know polynomial equations in two or more variables to which integer (or rational) solutions are sought as *Diophantine equations*. As an example, take the familiar equation in three variable (a,b,c) studied two thousand years before Diophantus by the Babylonians, $a^2 + b^2 = c^2$, to which one may seek integer solutions, i.e. Pythagorean triples such as $(3, 4, 5), (5, 12, 13)$ and their multiples.

Another type of Diophantine equation which will reappear later in this work is the *Pellian equation*. It appears in the form

$$u^2 - Dv^2 = C,$$

where D and C are fixed integers. Particular cases of Pellian equations, such as $u^2 - 2v^2 = 1$, were studied by Diophantus himself. The equation is named after John Pell (1611-1685), an English mathematician. His involvement with the equation extended only as far as translating the work of others, but this was misunderstood by Leonhard Euler (1707-1783) and the name 'Pellian equation' stuck (see [19]).

The left side of the above equation can be factored as a difference of squares,

$$u^2 - Dv^2 = (u - v\sqrt{D})(u + v\sqrt{D}),$$

and solutions to the Pellian equation can be characterized into classes corresponding to a finite number of *fundamental solutions* (the smallest solution in its class). Solutions, we then say, are those (u, v) such that

$$u + v\sqrt{D} = (u^* + v^*\sqrt{D})(x + y\sqrt{D}),$$

where $u^* + v^*\sqrt{D}$ is a fundamental solution and $x + y\sqrt{D}$ runs through all solutions to $u^2 - Dv^2 = 1$. (If $x + y\sqrt{D}$ is the fundamental solution to $u^2 - Dv^2 = 1$, then $(x + y\sqrt{D})^j$ will run through all that equation's solutions as j runs through the positive integers.) Pellian equations and their solutions are thoroughly explored in chapter VI of *Introduction to Number Theory* by Nagell [12].

Though famous for his work on polynomial equations, Diophantus' work was not limited to them. For example, Diophantus noticed (see [19]) that there are sets of rational numbers such as

$$\left\{\frac{1}{16}, \frac{33}{16}, \frac{68}{16}, \frac{105}{16}\right\}$$

where if one multiplies any two of the members together and adds 1, the result is a rational square. There are similar sets whose members are integers, such as $\{1, 3, 8\}$.

$$1 \times 3 + 1 = 4 = 2^2$$
$$1 \times 8 + 1 = 9 = 3^2$$
$$3 \times 8 + 1 = 25 = 5^2.$$

This is an example of what we call a *Diophantine triple* with property $D(1)$. In general, we call a set with m elements (almost always assumed to be integers) $\{a_1, a_2, \ldots, a_m\}$ a *$D(l)$-Diophantine m-tuple* provided the product of any two of the set's elements plus l is a square, i.e. for $i \neq j$,

$$a_i a_j + l = t^2,$$

for some $t \in \mathbb{Z}$.

Diophantine m-tuples have been fairly well-studied in recent years (see for instance many papers by A. Dujella such as [11], [23], [24]). It is often of particular interest to narrow the focus of inquiry from all Diophantine triples (or quadruples or m-tuples) to just those in a particular family. A common way to do this is to exploit the properties of a certain sequence of numbers which will be described in the next section.

1.4 Filius Bonacci and Leporine Vigour

Leonardo Bigollo (of the Bonacci family) (1170-1250) was an Italian mathematician from Pisa better known by the name Fibonacci (short for filius Bonacci). He was an early adopter of the Hindu-Arabic numeral system, which he learned while accompanying his father who worked in Bugia (present-day Algeria). He engaged in quite sophisticated mathematics for the time period, including the construction of

Pythagorean triples and accurate approximations of certain irrational numbers, however he is best known for introducing a particular sequence of numbers to European mathematicians. In his 1202 work *Liber abaci*, when considering a question on the breeding habits of rabbits, Fibonacci answered with:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

The above sequence, where one obtains the next entry by adding together the two previous entries, has come to be known as the *Fibonacci sequence*, and a number which belongs to the sequence is referred to as a *Fibonacci number*. We write F_n to represent the nth element in the sequence, and define it recursively:

$$F_1 = 1, \quad F_2 = 1, \quad F_{n+1} = F_n + F_{n-1}.$$

For a biography of the man who came to be known as Fibonacci, see [20].

There is a close connection between the Fibonacci numbers and the aforementioned golden ratio, $\varphi = \frac{1+\sqrt{5}}{2}$. The quotient of one element in the sequence by the previous, $\frac{F_{n+1}}{F_n}$ approaches the golden ratio φ as n goes to infinity, so these quotients give increasingly precise approximations of φ. For example,

$$\begin{aligned} \varphi - \frac{F_6}{F_5} &= \varphi - \frac{8}{5} \approx 1.618033989 - 1.6 && < 0.02 \text{ and} \\ \varphi - \frac{F_{10}}{F_9} &= \varphi - \frac{55}{34} \approx 1.618033989 - 1.617647059 && < 0.0005. \end{aligned}$$

Indeed, any Fibonacci number can be represented using the following formula in φ called *Binet's formula*,

$$F_n = \frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}},$$

where $\overline{\varphi}$ is the other solution to the polynomial equation $x^2 - x - 1 = 0$,

$$\overline{\varphi} = \frac{-1}{\varphi} = \frac{1-\sqrt{5}}{2} \approx -0.618033989.$$

Fibonacci numbers also happen to have a close connection to Diophantine m-tuples. Recall that the example given in the last section of a D(1)-Diophantine triple of integers was $\{1, 3, 8\}$, whose elements are the first three even-indexed Fibonacci numbers (note that $F_1 = F_2 = 1$). Observe the following identity, called *Cassini's identity*:

$$F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1}.$$

This means that for any two consecutive even-indexed Fibonacci numbers, F_{2n} and F_{2n+2}, their product increased by 1: $F_{2n}F_{2n+2} + 1 = F_{2n+1}^2$ is a square. Cassini's identity happens to be a specific case of the more general *Catalan's identity*,

$$F_n^2 - F_{n-r}F_{n+r} = (-1)^{n-r}F_r^2,$$

which, in the case $r = 2$ gives $F_{2n}F_{2n+4} + 1 = F_{2n+2}^2$. From these two identities it can be seen that not only is the set $\{1, 3, 8\}$ a D(1)-Diophantine triple, but any set of three consecutive even-indexed Fibonacci numbers $\{F_{2n}, F_{2n+2}, F_{2n+4}\}$ will also form such a triple.

To extend this to similarly-constructed $D(l)$-Diophantine triples with $l > 1$ requires the introduction of the *Lucas numbers*, which are named after Édouard Lucas (1842-1891), a French mathematician who studied the Fibonacci sequence and similar sequences [9]. The Lucas numbers are a sequence similar to the Fibonacci numbers but starting with $1, 3$ instead of $1, 1$ or $1, 2$. The nth Lucas number is denoted L_n and the sequence is defined using the recurrence relation $L_{n+1} = L_n + L_{n-1}$. The sequence of Lucas numbers is then as follows:

$$1, 3, 4, 7, 11, 18, 29, 47, \ldots$$

The Lucas numbers obey a similar formula to that of Binet for Fibonacci numbers, $L_n = \varphi^n + \overline{\varphi}^n$. Since $\overline{\varphi}^n$ will shrink quickly as n grows, this formula means that L_n will always be the closest integer to the nth power of φ for $n > 1$.

There are many relations between the Fibonacci numbers and the Lucas numbers. In order to enlarge the earlier family of Diophantine triples, it need only be known that an even-indexed Fibonacci number F_{2n} can be factored into the product of a Lucas number L_n and a Fibonacci number F_n,

$$F_{2n} = L_nF_n.$$

This identity means that the triple $\{a = F_{2n}, b = L_r^2F_{2n+2r}, c = F_{2n+4r}\}$ is a $D(F_{2r}^2)$-Diophantine triple. To see this, combine Catalan's identity with the above factorization of even-indexed Fibonacci numbers:

$$\begin{aligned}
ab + F_{2r}^2 &= F_{2n} \cdot L_r^2F_{2n+2r} && + L_r^2 \cdot F_r^2 && = (L_rF_{2n+r})^2 \\
ac + F_{2r}^2 &= F_{2n} \cdot F_{2n+4r} && + F_{2r}^2 && = (F_{2n+2r})^2 \\
bc + F_{2r}^2 &= L_r^2F_{2n+2r} \cdot F_{2n+4r} && + L_r^2 \cdot F_r^2 && = (L_rF_{2n+3r})^2.
\end{aligned}$$

One of the main interests of this thesis is whether results on $D(1)$-Diophantine triples of the form $\{F_{2n}, F_{2n+2}, F_{2n+4}\}$ can be extended to these more general $D(F_{2r}^2)$-Diophantine triples. One of the first such results was due to a seventeenth century French lawyer responsible for many of the directions number theory has taken in the intervening centuries. This man will be the focus of the next section.

1.5 Fermat

Pierre de Fermat (1607-1665) was a French mathematician perhaps most famous for his formulation of what came to be known as *Fermat's Last Theorem.* The reader might recall the Diophantine equation $x^2 + y^2 = z^2$, integer solutions to which have been studied as far back as ancient Babylon, as well as by the Pythagoreans. One might readily wonder (as Fermat did) whether similar solutions could be obtained for the equation $x^3 + y^3 = z^3$ or indeed the more general equation

$$x^n + y^n = z^n.$$

In 1637, Fermat wrote in the margin of his copy of Diophantus' *Arithmetica* that there do not exist integers x, y, z all different from zero satisfying the above equation for any n larger than 2. He added that he had found a 'truly marvellous' proof of this result, but one which the margin was too narrow to contain. His copy of *Arithmetica* was published posthumously by his son, and generated a storm of mathematical intrigue (see [17]). No proof was found in the notes Fermat left, and it is widely believed nowadays that Fermat had not in fact found a valid proof of this result. In the intervening centuries dozens and dozens of attempts were made both to reconstruct Fermat's supposed proof and to prove it by any means necessary. Interesting mathematical advances were made in service of attempted proofs, but a proof seemed more and more distant as time passed. It wasn't until 1995, 358 years after Fermat's initial formulation of it, that a valid proof of Fermat's Last Theorem was finalized by Andrew Wiles with help from Richard Taylor [25]. It is perhaps fitting that Wiles, an English mathematician, was the one to finally prove the result. Fermat had a penchant for posing questions of English mathematicians the solutions for which he knew to be extremely difficult. This, combined with his refusal to share his proofs, led to no end of frustration amongst the English. Some 330 years after Fermat's death one could say that they finally got their own back, albeit far too late to gloat.

Another of Fermat's results that is of interest concerns Diophantine n-tuples. Fermat showed that the $D(1)$ triple $\{1, 3, 8\}$, mentioned earlier, can be extended into a $D(1)$-Diophantine quadruple by including the additional integer 120 in the set, i.e. that $\{1, 3, 8, 120\}$ is a $D(1)$-Diophantine quadruple. This was the first Diophantine quadruple of integers ever found. It was later proven by Euler that there are an infinite number of integral (all integer) ($D(1)$-)Diophantine quadruples. He also extended the quadruple to a *rational* quintuple with the addition of 777480/8288641, so

$$\left\{1, 3, 8, 120, \frac{777480}{8288641}\right\}$$

is a $D(1)$-Diophantine quintuple of rational numbers (see [19]). In 1997, Dujella generalized Euler's result, showing that any rational Diophantine quadruple the product of whose elements is not equal to 1 can be extended to a rational Diophantine quintuple [23]. In 2018, He, Togbé and Ziegler showed in [37] that there is **no** (integer) $D(1)$-Diophantine quintuple.

1.6 Transcendental Numbers

Recall that a complex number x is called algebraic if it is a solution to a polynomial equation,

$$a_m x^m + \cdots + a_1 x + a_0 = 0,$$

with coefficients $a_1, \ldots, a_m$ in $\mathbb{Z}, a_m \neq 0$. m is called the *degree* of this polynomial. For any given algebraic number α, there is a uniquely determined *minimal polynomial*, which can be defined as the *monic* (meaning $a_m = 1$), *irreducible* (meaning that the polynomial is not a product of multiple polynomials of smaller degree) polynomial with coefficients in $\mathbb{Q}$ of smallest degree. Algebraic numbers with the same minimal polynomial are called *conjugates*. The minimal polynomial, for example, of the golden ratio φ is the previously listed $x^2 - x - 1$. φ and $\overline{\varphi}$ are conjugates.

The algebraic numbers are closed in the sense that any solution to a polynomial equation with algebraic coefficients will also be algebraic. At first glance it seems as though algebraic numbers might tell the whole story, and indeed it wasn't until 1844 - more than two millennia after the existence of irrational numbers was established - that Liouville would prove the existence of complex numbers that are not algebraic. We call any complex number that is not algebraic a *transcendental* number.

The first numbers proven to be transcendental were all specifically constructed for the task by Liouville. Cantor (1845-1918) provided an alternative construction in 1874 [9]. One year prior to that, French mathematician Charles Hermite had finally demonstrated that Euler's number e is transcendental, marking the first number proven to be transcendental that wasn't constructed solely to be so. We can think of e as the infinite sum,

$$e = 1 + \frac{1}{1} + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2 \cdot 1} + \cdots + \frac{1}{n!} + \ldots,$$

where $n! = n \times (n-1) \times \cdots \times 3 \times 2 \times 1$ is the product of the first n positive integers, usually called n factorial. e arises naturally in many forms throughout mathematics. For instance, any complex number can be written $re^{i\theta}$ where r and θ are real numbers.

In 1882, the irrational number π, well-known as the ratio between a circle's circumference and its diameter, was shown to be transcendental by German mathematician Ferdinand von Lindemann (1852-1939). (Proofs of the transcendence of e

and π can be found in [32].) Approximations of π have existed for thousands of years - as far back as ancient Babylon and Egypt [33] - but it had only been proven to be irrational in 1761 by Johann Lambert (see [32] for a proof). The number pervades mathematics, appearing frequently in forms seemingly far-distant from the circle. For instance, a result due to Euler states that the infinite sum of the inverses of positive integer squares is equal to an expression in π,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots = \frac{\pi^2}{6}.$$

While Liouville and Cantor's constructed numbers are not without their appeal, I would consider e and π to be much more elegant - aside from perhaps their decimal expansions. They feature prominently in our representations of the patterns of the universe (e.g. many areas of mathematics and physics), and depending on one's philosophical disposition could even be said to be crucial to its as yet hidden and mysterious inner workings. Games played with these transcendental numbers are in that sense potentially played for more than purely mathematical points. Whether one finds that appealing or unappealing is another matter.

As mentioned earlier, polynomials with algebraic coefficients have algebraic solutions, however the same does not hold of transcendental numbers. The polynomial $\pi x - \pi$, for example, has transcendental coefficients but integer root $x = 1$. Many seemingly-unassuming numbers as yet evade categorization as algebraic or transcendental. While π and e are transcendental, we do not know whether π^e is algebraic or transcendental. Similarly, it is unknown whether $\pi + e$ or πe are algebraic or transcendental, though we know that at least one of them must be transcendental.

1.7 Alan Baker and Linear Forms in Logarithms

In 1900, David Hilbert (1862-1943) published his influential list of 23 problems whose solutions he believed would be of great interest and importance in the shaping of mathematics in the coming century. The seventh problem on that list (in English: "irrationality and transcendence of certain numbers"), posed the question:

Given $0, 1 \neq \alpha$ algebraic, and β irrational algebraic, is α^β transcendental?

Despite Hilbert's belief that this problem would outlast the notorious Riemann hypothesis (which was in fact his eighth problem), it was resolved by Gelfond and Schneider (independently) in only 1934. The answer was yes. Contained within the proof of this result - which would come to be known as the Gelfond-Schneider theorem - were the seeds of a new and exciting method of attack on large swathes of unsolved problems in number theory. Gelfond and Schneider both proved that for algebraic $\alpha_1, \alpha_2, \beta_1, \beta_2$, with $\alpha_i \neq 0$ and $\log \alpha_1, \log \alpha_2$ linearly independent over $\mathbb{Q}$ (linear dependence/independence will be explained in the next paragraph), we have

$$\Lambda = \beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 \neq 0.$$

The above quantity Λ is an example of what we call a *linear form in logarithms* - more specifically in this case a linear form in two logarithms. A *linear form in logarithms* of algebraic numbers is an expression of the form

$$L = \beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n,$$

where $\alpha_1, \ldots, \alpha_n, \beta_0, \ldots, \beta_n$ are algebraic numbers and log is any determination of the logarithm - which we usually take to be the logarithm base e (often called the *natural logarithm* and written ln). In 1935, Gelfond gave a lower bound on the absolute value of the linear form in logarithms Λ, and speculated that a generalization of his result would be an incredibly useful number theoretic tool. Some three decades later he would be proven right (see [9], [31]).

Our results on linear forms in logarithms are conditional on the linear independence of the logarithms $\log \alpha_1, \ldots, \log \alpha_n$ over $\mathbb{Q}$ (or equivalently the *multiplicative independence* of the algebraic numbers $\alpha_1, \ldots, \alpha_n$). Real or complex numbers $\alpha_1, \ldots, \alpha_n$ are called *linearly dependent* over the rationals $\mathbb{Q}$ provided there exist rational numbers $r_1, \ldots, r_n$ (with at least one r_i nonzero) such that

$$r_1\alpha_1 + \cdots + r_n\alpha_n = 0.$$

If this is not the case, we call $\alpha_1, \ldots, \alpha_n$ *linearly independent* over the rationals.

In 1962, Alan Baker (1939-2018) finished his PhD thesis on the theory of transcendental numbers, and continued his investigations as a research fellow at Cambridge University from 1964 to 1968. In 1966 and 1967, he published his now famous papers, *Linear Forms in the Logarithms of Algebraic Numbers I,II* and III ([26], [27], [28]). He proved a generalization of the theorem of Gelfond and Schneider (Hilbert's seventh problem): that if $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are algebraic numbers with $\alpha_i \neq 0, 1$ and $1, \beta_1, \ldots, \beta_n$ linearly independent over $\mathbb{Q}$, then the number $\alpha_1^{\beta_1} \cdot \cdots \cdot \alpha_n^{\beta_n}$ is transcendental. In the process, he proved that for algebraic β_0, we must have the following result on linear forms in logarithms of algebraic numbers:

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n \neq 0.$$

Moreover, Baker found that one could bound the absolute value of these linear forms in logarithms of algebraic numbers away from zero. As Gelfond predicted, this idea would have an almost immediate impact on number theory. In 1970, Baker was awarded the Fields medal due in large part to his work on linear forms in logarithms.

Consider a particular type of Diophantine equation (called a Thue equation after Norwegian mathematician Axel Thue (1863-1922)),

$$f(x,y) = a_d x^d + a_{d-1}x^{d-1}y + \cdots + a_1 xy^{d-1} + a_0 y^d = r, \quad 0 \neq r \in \mathbb{Z},$$

where $f(x,1)$ is an irreducible polynomial in x and $d \geq 3$. Thue himself showed in [34] that such an equation has only finitely many solutions in integers x and y. Baker's work went further, showing that not only could one bound the number of solutions, one could also bound their size (see [35]). He showed that given such an equation $f(x,y) = r$, there exists a number B depending only on r, the degree of f and its coefficients, such that for any solution $f(x_0, y_0) = r$, we must have

$$\max\{|x_0|, |y_0|\} \leq B.$$

Due to Baker's contribution, we need in theory only check a finite number of integer solutions in order to completely solve any such Diophantine equation. In practice, the bounds we are left with when using linear forms in logarithms are often too large to adequately facilitate a brute force method, and additional tools and finesse inevitably come into play.

One common tactic is to use what is called the *Baker-Davenport reduction* method (first seen in [30]). If we know that a certain equation has no solutions above a certain bound M, careful application of Baker-Davenport reduction can sometimes allow us to lower this bound drastically. The following result is a variation of the original lemma of Baker and Davenport due to Dujella and Pethő (see [3], Lemma 5a) that will be applied later.

Lemma 1.1 (Variation on a Lemma of Baker and Davenport)**.** *Assume that κ and μ are real numbers and M is a positive integer. Let P/Q be the convergent of the continued fraction expansion of κ such that $Q > 6M$ and let*

$$\eta = ||\mu Q|| - M \cdot ||\kappa Q||,$$

where $||\cdot||$ denotes the distance from the nearest integer. If $\eta > 0$, then there is no solution of the inequality

$$0 < j\kappa - k + \mu < AB^{-j}$$

in integers j and k with

$$\frac{\log(AQ/\eta)}{\log B} \leq j \leq M.$$

Here, the distance from a real number x to the nearest integer, denoted $||x||$ is the minimum of the quantities $|x-n|$ and $1-|x-n|$ where n is the largest integer that is less than or equal to x. For example, if $x = 55/12$ then $|x-n| = |55/12-4| = 7/12$ and $1-|x-n| = 5/12$, so $||x|| = 5/12$.

The reader may be unfamiliar with the term 'convergent of a continued fraction'. The following series of paragraphs offer a brief explanation. For more on continued fractions, see [6] and [9].

Any real number $\theta \in \mathbb{R}$ may be written as a *regular continued fraction*,

$$\theta = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \frac{1}{\ddots}}},$$

where a_0 is an integer, and $a_1, a_2, \dots$ are positive integers which we call the *partial quotients* of θ. We can write θ in abbreviated notation as $\theta = [a_0; a_1, a_2, \dots]$. Continued fraction representations of numbers can be thought of as an alternative to the common decimal expansion, and depending on the situation they can be more useful.

A real number is rational if and only if its continued fraction representation eventually terminates, leaving us with a *finite regular continued fraction*,

$$\frac{p}{q} = [a_0; a_1, \dots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}.$$

Every infinite regular continued fraction corresponds to a unique irrational number and each irrational number has a unique infinite regular continued fraction. If we take the continued fraction representation of a real number θ (which may be irrational), $\theta = [a_0; a_1, a_2, \dots]$, and truncate it after each term, then we obtain successively more accurate rational approximations of θ:

$$\frac{p_0}{q_0} = [a_0], \frac{p_1}{q_1} = [a_0; a_1], \frac{p_2}{q_2} = [a_0; a_1, a_2], \dots, \frac{p_n}{q_n} = [a_0; a_1, a_2, \dots, a_n].$$

We call these rational approximations the *convergents* of θ, and call $\frac{p_n}{q_n} = [a_0, a_1, a_2, \dots, a_n]$ its nth convergent.[1]

[1]Note that both the numerator and the denominator of the convergents will continue to grow with n, so no matter how large the integer M is in the above variation of the Baker-Davenport reduction lemma, one has an infinite number of choices for q bigger than $6M$.

For example, the golden ratio φ satisfies the equation $\varphi^2 - \varphi - 1 = 0$, which can be rearranged to obtain $\varphi = 1 + \frac{1}{\varphi}$. Repeated substitution

$$\varphi = 1 + \frac{1}{\varphi} = 1 + \cfrac{1}{1 + \cfrac{1}{\varphi}} = \cdots = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}$$

shows that the continued fraction representation of the golden ratio, $\varphi = \frac{1+\sqrt{5}}{2}$ is $[1; 1, 1, 1, \dots]$ where 1 repeats infinitely many times. Its first and second convergents are

$$\frac{p_1}{q_1} = [1; 1] = 1 + \frac{1}{1} = 2, \quad \frac{p_2}{q_2} = [1; 1, 1] = 1 + \cfrac{1}{1 + \cfrac{1}{1}} = 1 + \frac{1}{2} = \frac{3}{2},$$

which the reader may notice are ratios of Fibonacci numbers. Indeed, the nth convergent, $\frac{p_n}{q_n}$ of the golden ratio is equal to $\frac{F_{n+2}}{F_{n+1}}$.

The approximations of irrational numbers given by the convergents of its continued fraction are the 'best' approximations in the sense that for any convergent $\frac{p_r}{q_r}$ of an irrational number α, there is no rational number with denominator smaller than or equal to q_r that better approximates α. The following result (see the proof of Theorem 22 in [6]) gives bounds for how well a convergent approximates an irrational number in terms of the subsequent partial quotient in its continued fraction representation. It will be listed as a lemma here so that it can be easily used later in the paper.

Lemma 1.2. *Let α be an irrational number, $\frac{p_r}{q_r}$ its rth convergent and a_{r+1} its $r+1$st partial quotient. The following inequality holds:*

$$\frac{1}{q_r^2(a_{r+1}+2)} < \left|\alpha - \frac{p_r}{q_r}\right| \le \frac{1}{q_r^2 a_{r+1}}.$$

This means that the larger the next partial quotient a_{r+1}, the better $\frac{p_r}{q_r}$ is as an approximation of α. As a result, the golden mean $[1; 1, 1, 1, 1, \dots]$ is the irrational number most poorly approximable by rationals.

The following result due to Legendre establishes a condition which can determine whether a given rational approximation of an irrational number is a convergent of the irrational's continued fraction (see [9]).

Theorem 1.3 (Legendre). *Let α be an irrational number. Let p, q be integers such that $q \geq 1$ and*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}.$$

Then $\frac{p}{q}$ is a convergent of α.

This approach of Baker's using lower bounds on linear forms in logarithms was quickly extended to more classes of Diophantine equations as well as other types of number-theoretic problems. The first instance of this occurred in a context that should be familiar to the reader. Recall that Fermat found the first Diophantine quadruple of integers by adding 120 to the $D(1)$-Diophantine triple $\{1, 3, 8\}$. In 1969, Baker and his former doctoral supervisor, Harold Davenport (1907-1969), showed that 120 is **the only** integer that extends the triple $\{1, 3, 8\}$ to a $D(1)$-Diophantine quadruple ([30]).[2] Note that 120 is not a Fibonacci number, so there is no $D(1)$ diophantine quadruple consisting of $1, 3, 8$ and a fourth Fibonacci number. In 1999, Dujella generalized this result, proving that for integer k, the only integer that extends the $D(1)$ triple $\{F_{2k}, F_{2k+2}, F_{2k+4}\}$ to a quadruple is $4F_{2k+1}F_{2k+2}F_{2k+3}$ (see [36]).

This leads into the main result, which concerns $D(9)$ and $D(64)$-Diophantine triples. Recall that the set $\{F_{2n}, L_r^2 F_{2n+2r}, F_{2n+4r}\}$ is a $D(F_{2r}^2)$-Diophantine triple. Substituting $r = 2$ tells us that $\{F_{2n}, 9F_{2n+4}, F_{2n+8}\}$ is a $D(9)$-Diophantine triple and $r = 3$ tells us $\{F_{2n}, 16F_{2n+6}, F_{2n+12}\}$ is a $D(64)$-Diophantine triple. The aim of this paper will be to prove the following theorems.

Theorem 1.4. *If $\{F_{2n+8}, 9F_{2n+4}, F_k\}$ is a $D(9)$ Diophantine Triple, then if $n > 1$, we must have $k = 2n$. If $n = 1$, we have $F_1 = 1 = F_2$, so $k = 1$ and $k = 2$ are both solutions.*

Theorem 1.5. *If $\{16F_{2n+6}, F_{2n+12}, F_k\}$ is a $D(64)$ Diophantine Triple and $3 \mid n$, then we must have $k = 2n$.*

This pair of theorems means that for $r = 2, 3$, the only Fibonacci number that extends the pair $\{L_r^2 F_{2n+2r}, F_{2n+4r}\}$ to a $D(F_{2r}^2)$-Diophantine triple is F_{2n} (aside from the aforementioned exceptions).

[2] The reader might also recall the more famous result of Fermat, Fermat's last theorem, mentioned in section 1.5. Andrew Wiles, who proved that result, was actually the mathematical descendent of Baker (and Davenport). Wiles' doctoral supervisor, John Coates, was in turn the doctoral student of Baker.

2 Residual Preliminaries

In the previous section I tried to build up linear forms in logarithms in a historical context. In doing so, I was able to integrate many preliminary results that might otherwise have been included in a long list. In order to be comprehensive however, a few definitions and results still need to be stated.

2.1 A Pair of Lemmas on Linear Forms in Logarithms

For any non-zero algebraic number γ of degree d - meaning its minimal polynomial has degree d - over $\mathbb{Q}$ whose minimal polynomial over $\mathbb{Z}$ is $a\Pi_{j=1}^{d}(X-\gamma^{(j)})$, we denote by

$$h(\gamma) = \frac{1}{d}\Big(\log a + \sum_{j=1}^{d} \log\, \max\left(1, |\gamma^{(j)}|\right)\Big)$$

its absolute logarithmic height. Here the $\gamma^{(j)}$ run through all the solutions to the minimal polynomial of γ, i.e. γ and all its conjugates. Given the minimal polynomial of an algebraic number γ, we can obtain its minimal polynomial over $\mathbb{Z}$ by multiplying by the lowest common multiple of the rational coefficients.

The following two lemmas are identical to the first two listed in Section 2 of [2]. The first, due to Mateev, establishes a bound on a linear form in three logarithms. The second, due to Laurent, establishes a bound on a linear form in two logarithms. See also [3], [7], and [8].

Lemma 2.1. *Let Λ be a linear form in logarithms of multiplicatively independent totally real algebraic numbers $\alpha_1, \ldots, \alpha_N$ with rational integer coefficients $b_1, \ldots, b_N$ ($b \neq 0$). Let $h(\alpha_j)$ denote the absolute logarithmic height of α_j for $1 \leq j \leq N$. Define the numbers D, A_j ($1 \leq j \leq N$) and E by $D := [\mathbb{Q}(\alpha_1, \ldots, \alpha_N) : \mathbb{Q}]$, $A_j = \max\{Dh(\alpha_j), |\log \alpha_j|\}, E = \max\{1, \max\{|b_j|A_j/A_N; 1 \leq j \leq N\}\}$. Then*

$$\log|\Lambda| > -C(N)C_0W_0D^2\Omega,$$

where

$$C(N) := \frac{8}{(N-1)!}(N+2)(2N+3)(4e(N+1))^{N+1},$$
$$C_0 := \log\left(e^{4.4N+7}N^{5.5}D^2\log(eD)\right),$$
$$W_0 := \log\left(1.5eED\log(eD)\right),\ \Omega = A_1 \ldots A_N.$$

Lemma 2.2. *Let $\gamma_1 > 1$ and $\gamma_2 > 1$ be two real multiplicatively independent algebraic numbers, $b_1, b_2 \in \mathbb{Z}$ not both 0 and*

$$\Lambda = b_2 \log \gamma_2 - b_1 \log \gamma_1.$$

Let $D := [\mathbb{Q}(\gamma_1, \gamma_2) : \mathbb{Q}]$. Let

$$h_i \geq \max\left\{h(\gamma_i), \frac{|\log \gamma_i|}{D}, \frac{1}{D}\right\} \; for \; i = 1, 2, \; b' \geq \frac{|b_1|}{Dh_2} + \frac{|b_2|}{Dh_1}.$$

Then

$$\log |\Lambda| \geq -17.9 \cdot D^4 \Big(\max\Big\{ \log b' + 0.38, \frac{30}{D}, 1\Big\}\Big)^2 h_1 h_2.$$

2.2 Index of a Field Extension

In order to implement the above lemmas on linear forms in logarithms, the quantities $[\mathbb{Q}(\alpha_1, \ldots, \alpha_N) : \mathbb{Q}]$ and $[\mathbb{Q}(\gamma_1, \gamma_2) : \mathbb{Q}]$ must be known. For algebraic $\alpha_1, \ldots, \alpha_n$, we call $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ an algebraic extension field of $\mathbb{Q}$, and it consists of all numbers that can be written as a quotient f/g where f and g are polynomials evaluated at $\alpha_1, \ldots, \alpha_n$ with coefficients in $\mathbb{Q}$,

$$\mathbb{Q}(\alpha_1, \ldots, \alpha_n) = \left\{\frac{f(\alpha_1, \ldots, \alpha_n)}{g(\alpha_1, \ldots, \alpha_n)} : g \neq 0\right\}.$$

Let $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$. The quantity $[K : \mathbb{Q}]$ is a positive integer that we call the *degree* of the algebraic extension field K over the field $\mathbb{Q}$. It can be defined as the minimum number $m \leq n$ of elements of K, call them $\lambda_1, \ldots, \lambda_m \in K$, such that for *any* element γ of K, we can find rational numbers $r_1, \ldots, r_m$ such that

$$\gamma = r_1\lambda_1 + \cdots + r_m\lambda_m.$$

For a simple algebraic extension $\mathbb{Q}(\alpha)$, the degree $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ is equal to the degree of the minimal polynomial of α. See an algebra textbook for a more rigorous treatment of fields.

2.3 Two More Lemmas

The following two lemmas by the author will be useful in applying 2.1.

Lemma 2.3. *$F_{2n+4}F_{2n+8}$ is neither a square nor 5 times a square.*

Proof. To see that $F_{2n+4}F_{2n+8}$ is not a square, simply note that $F_{2n+4}F_{2n+8} + 1 = F_{2n+6}^2$. Suppose that $F_{2n+4}F_{2n+8}$ is 5 times a square, so $F_{2n+4}F_{2n+8} = 5y^2$ for some integer y. Then we must have that $5y^2 + 1 = F_{2n+6}^2$. We examine the solutions of the Pellian equation

$$X^2 - 5Y^2 = 1, \tag{1}$$

with our interest lying in those solutions $X + Y\sqrt{5}$ for which $X = F_{2n+6}$ for some positive integer n.

This Pellian equation has fundamental solution $9 + 4\sqrt{5} = \alpha^6$, where here $\alpha = \dfrac{1+\sqrt{5}}{2}$. Hence all solutions to equation (1) have the form $(9+4\sqrt{5})^j$ for some $j \in \mathbb{Z}^+$. If we let $X_j + Y_j\sqrt{5} = (9+4\sqrt{5})^j$, then $X_j = \dfrac{(9+4\sqrt{5})^j + (9+4\sqrt{5})^{-j}}{2} = \dfrac{\alpha^{6j} + \alpha^{-6j}}{2}$. Since $F_k = \dfrac{\alpha^k - (\frac{-1}{\alpha})^k}{\sqrt{5}}$, our aim becomes to solve the equation

$$\frac{\alpha^{6j} + \alpha^{-6j}}{2} = \frac{\alpha^{2n+6} - \alpha^{-2n-6}}{\sqrt{5}} \tag{2}$$

for $n, j \in \mathbb{Z}^+$.

We see here that for $n \geq 1$

$$\begin{aligned} \frac{\alpha^{2n+4} + \alpha^{-2n-4}}{2} &< \frac{\alpha^4}{2}\alpha^{2n} + \frac{1}{2} < 8\alpha^{2n} - 1 \\ &< \frac{\alpha^6\alpha^{2n} - \alpha^{-2n-6}}{\sqrt{5}} = \frac{\alpha^{2n+6} - \alpha^{2n-6}}{\sqrt{5}}, \end{aligned}$$

hence we must have $j > \frac{n+2}{3}$. However,

$$\frac{\alpha^{2n+6} + \alpha^{-2n-6}}{2} > \frac{\alpha^{2n+6} + \alpha^{-2n-6}}{\sqrt{5}} > \frac{\alpha^{2n+6} - \alpha^{-2n-6}}{\sqrt{5}},$$

which means $j < \frac{n+3}{3}$.

The bounds $\frac{n+2}{3} < j < \frac{n+3}{3}$ mean that j cannot be an integer, which is a contradiction. Thus there is no solution to the Pellian equation (1) wherein X is a Fibonacci number with index $2n+6, n \in \mathbb{Z}^+$, which means that $F_{2n+4}F_{2n+8}$ is neither a square nor 5 times a square. □

Lemma 2.4. *$F_{2n+6}F_{2n+12}$ is neither a square nor 5 times a square.*

Proof. To see that it is not itself a square, observe that $F_{2n+6}F_{2n+12} + 4 = F_{2n+9}^2$, and the difference between any two nonzero, nonidentical squares is either odd or bigger than 4. Suppose that $F_{2n+6}F_{2n+12}$ is 5 times a square, so $F_{2n+6}F_{2n+12} = 5y^2$ for some integer y. We're interested in solutions to the Pellian equation

$$X^2 - 5Y^2 = 4 \tag{3}$$

with $X = F_{2n+9}$ for some $n \in \mathbb{Z}^+$. Since the fundamental solution of $X^2 - 5Y^2 = 1$ is $9 + 4\sqrt{5}$, a theorem of Nagell (see [12]) gives us that all classes of solutions to equation (3) have fundamental solution $u + v\sqrt{5}$ with

$$0 < v \le \frac{4}{\sqrt{2(9+1)}}\sqrt{4} = \frac{4}{\sqrt{5}} < 2 \text{ and}$$
$$0 \le |u| \le \sqrt{\frac{1}{2}(9+1)\cdot 4} = 2\sqrt{5} < 5,$$

so all solutions to (3) are of the form

$$(\pm 3 + \sqrt{5})(9 + 4\sqrt{5})^j$$

for some integer $j \ge 1$. If we let $V_j + U_j\sqrt{5} = (9 + 4\sqrt{5})^j$, then this means that we have solutions $X_j + Y_j\sqrt{5}$ where $X_j = \pm 3V_j + 5U_j$ and $Y_j = V_j \pm 3U_j$. This means that we want to find solutions (j, n) to the equation

$$X_j = F_{2n+9} = \frac{\alpha^{2n+9} - (\frac{-1}{\alpha})^{2n+9}}{\sqrt{5}} = \frac{\alpha^{2n+9} + \alpha^{-2n-9}}{\sqrt{5}}.$$

Since $(9 + 4\sqrt{5}) = \alpha^6$, we have $V_j = \frac{\alpha^{6j}+\alpha^{-6j}}{2}$, our aim is to solve the following

$$\pm 3(\frac{\alpha^{6j} + \alpha^{-6j}}{2}) + 5(\frac{\alpha^{6j} - \alpha^{-6j}}{2\sqrt{5}}) = \frac{\alpha^{2n+9} + \alpha^{-2n-9}}{\sqrt{5}}.$$

Noting that $-3\sqrt{5} + 5$ and $-3\sqrt{5} - 5$ are both less than 0 and will never yield a solution, we obtain the following after cancelling denominators:

$$(3\sqrt{5} + 5)\alpha^{6j} + (3\sqrt{5} - 5)\alpha^{6j} = 2\alpha^{2n+9} + 2\alpha^{-2n-9}. \tag{4}$$

We see from a brief observation that

$$(3\sqrt{5}+5)\alpha^{2n+6} + (3\sqrt{5}-5)\alpha^{-2n-6} > 2.76\alpha^{2n+9} + 2.76\alpha^{-2n-9} > 2\alpha^{2n+9} + 2\alpha^{-2n-9},$$

meaning that $j < \frac{n+3}{3}$, and

$$(3\sqrt{5}+5)\alpha^{2n+4}+(3\sqrt{5}-5)\alpha^{-2n-4} < 1.06\alpha^{2n+9}+18.95\alpha^{2n-9} < 2\alpha^{2n+9} < 2\alpha^{2n+9}+2\alpha^{-2n-9}$$

which gives us $\frac{n+2}{3} < j < \frac{n+3}{3}$, contradicting $j \in \mathbb{Z}$. Thus we see that $F_{2n+6}F_{2n+12}$ is neither a square nor 5 times a square. □

One may note here that the previous two results become almost trivial given the veracity of a particular outstanding conjecture. It is known by Carmichael's Theorem [40] that every Fibonacci number aside from $F_6 = 8$ and $F_{12} = 144$ contains at least one prime factor not appearing in any previous number in the sequence, a primitive prime factor. It is conjectured that primitive prime factors will always appear to the first power [38], [39]. This of course would tell us that, aside from a few low-indexed exceptions, we have F_nF_m is neither a square nor 5 times a square.

3 Lemma on Pellian Equations

In this section, we prove a lemma regarding Pellian equations that will be crucial in setting up the proof of the main result. The lemma is an expanded version of Lemma 1 in [1], which was for $D(4)$ triples, and its proof is very similar. However, there was a small mistake in the proof found in [1] which is amended here. Due to the use of this lemma in [2], this section will serve to legitimize the results in that paper as well.

Let l be a positive integer and $\{a, b, c\}$ be a $D(l^2)$ Diophantine triple, i.e. there exist positive integers r, s, t such that

$$ab + l^2 = r^2, \quad ac + l^2 = s^2, \quad \text{and} \quad bc + l^2 = t^2.$$

Lemma 3.1. *Let $\{a, b, c\}$ is a $D(l^2)$ triple with $a < b < a\left(4 + \frac{4}{l^2}\right)$ and assume that if l is an odd prime then $l \mid ab$ and if l is not prime then $l^2 \mid a$ or $l^2 \mid b$. (If $l = 2$ these divisibility conditions are unnecessary). Then all solutions of the equation*

$$at^2 - bs^2 = l^2(a - b) \tag{5}$$

are of the form

$$t\sqrt{a} + s\sqrt{b} = (\pm l\sqrt{a} + l\sqrt{b})\left(\frac{r + \sqrt{ab}}{l}\right)^{\nu},$$

where $\nu \in \mathbb{Z}^+$.

Proof. As previously mentioned, the proof is very similar to that of Lemma 1 in [1]; however, allowances are made for $l > 2$ and a mistake in the proof of that Lemma is fixed.

Define $s' = \dfrac{rs - at}{l}$, $t' = \dfrac{rt - bs}{l}$ and $c' = \dfrac{(s')^2 - l^2}{a}$. The triple $\{a, b, c'\}$ is also a $D(l^2)-$Diophantine triple. In the case of $l = 2$ (the lemma found in [1]), $2 \mid (rs - at)$ and $2 \mid (rt - bs)$ no matter whether a and b are both odd or not. For $l > 2$ this may not be true, hence our divisibility requirement. Since if l is prime, $l \mid ab$ and if not, $l^2 \mid a$ or $l^2 \mid b$, we must have that $l \mid r$, and in addition that $l \mid s$ or $l \mid t$, thus s' and t' are always integers. Moreover, since

$$t\sqrt{a} + s\sqrt{b} = (t'\sqrt{a} + s'\sqrt{b})\Big(\frac{r + \sqrt{ab}}{k}\Big),$$

(t', s') belongs to the *same class* of solutions of (5) as (t, s), thus we can replace $c = c'$ and follow the process again with the triple $\{a, b, c'\}$ while always remaining in the same class of solutions. This will be the key to our proof. The above information means that if we let

$$t_\nu^{\pm}\sqrt{a} + s_\nu^{\pm}\sqrt{b} = (\pm l\sqrt{a} + l\sqrt{b})\Big(\frac{r + \sqrt{ab}}{l}\Big)^\nu,$$

then if after a certain number of times repeating the process of finding c' and replacing $c = c'$ in our triple, we have that $(t', s') = (\pm l, l)$, or equivalently, if $c' = a + b \pm 2r$, then we must have that $(t, s) = (t_\nu^{\pm}, s_\nu^{\pm})$ for some positive integer ν. Thus at this point the proof will be complete.

We begin with a pair of useful facts:

Remark. *s' is always positive.*

To see this, observe the Pellian equation:

$$bs^2 - at^2 = l^2(b - a)$$

obtained by multiplying (5) by -1. Multiplying by a, we get

$$abs^2 - a^2t^2 = (r^2 - l^2)s^2 - a^2t^2 = r^2s^2 - a^2t^2 - l^2s^2 = l^2a(b - a)$$

which means that

$$(rs - at)(rs + at) = l^2a(b - a) + l^2s^2.$$

Since r, s, a, t, l are all positive and $0 < a < b$, we must have that $rs - at = ls' > 0$.

Next we show that

Remark. $c' \geq 0$.

If l is prime and $l \mid a$, then $l \mid ab + l^2 = r^2$, which means $l \mid r$ and similarly, $l \mid s$. If l also divides b, then $l \mid t$, and so $l^2 \mid rs - at$, meaning $s' \geq l$, and so $c' \geq 0$. If $l \nmid b$, then $ab + l^2 = r^2$ implies $l^2 \mid a$, giving $l^2 \mid rs - at$, and again $c' \geq 0$. A similar process proves the remark in the case l prime and $l \mid b$.

If l is composite and $l^2 \mid a$, then $l^2 \mid ab + l^2 = r^2$ and $l^2 \mid ac + l^2 = s^2$. So $l^2 \mid rs - at$, which means that

$$l \leq \frac{rs - at}{l} = s',$$

and $c' = \dfrac{(s')^2 - l^2}{a} \geq 0$.

Similarly if $l^2 \mid b$, then $l^2 \mid ab + l^2 = r^2$ and $l^2 \mid bc + l^2 = t^2$. So $l^2 \mid rt - bs$, which means that

$$l \leq \left| \frac{rt - bs}{l} \right| = |t'|.$$

Therefore $c' = \dfrac{(t')^2 - l^2}{b} \geq 0$.

If $l = 2$, we show that $c' \geq 0$ regardless of whether it divides a or b. If $2 \mid ab$, then the proof is the same as above since we have l prime and $l \mid ab$. Suppose that both a and b are odd and suppose for a contradiction that $c' < 0$. This means that $s'^2 - 4 < 0$, which is equivalent to $s' < 2$. Since s' is always positive, this must mean that $s' = 1$, so we set $\dfrac{rs - at}{2} = 1$. Multiplying by 2 and adding at to each side, then squaring, we get $r^2 s^2 = (2 + at)^2$, giving $a^2bc + 4ab + 4ac + 16 = 4 + 4at + a^2bc + 4a^2$. We subtract a^2bc, divide by 4 and rearrange to obtain

$$-3 = a(b + c - a - t).$$

Since we assumed that a and b are both $1 \mod 2$, this must mean that $c - t \equiv 1 \mod 2$. However because of the fact that $b \equiv 1 \mod 2$, we must have that $t^2 = bc + 4 \equiv c \mod 2$, and so $t \equiv c \mod 2$. Therefore $a(b + c - a - t) \equiv 0 \mod 2$ and we have obtained a contradiction. Hence $c' \geq 0$ when $l = 2$.

3.1 Case when $0 \leq c' < b$

If $c' = 0$, then $s' = t' = l$. So $c = a + b + c' + \frac{2}{l^2}(abc' + rs't') = a + b + 2r$, and the proof is complete.

Suppose that $0 < c' < b$. Define $r' = (s')' = \dfrac{rs' - at'}{l}$ and $b' = (c')' = \dfrac{(r')^2 - l^2}{a}$. Then $b' = (c')' = a + b + c' + \dfrac{2}{l^2}(abc' - rs't')$. We have that $\{a, (c')', c', b\}$ is a regular $D(l^2)$-quadruple. Indeed

$$\begin{aligned} ab + l^2 &= r^2, & ac' + l^2 &= (s')^2, & bc' + l^2 &= (t')^2 \\ ab' + l^2 &= (r')^2, & bb' + l^2 &= (q')^2, & b'c' + l^2 &= u^2 \end{aligned}$$

where

$$u = \frac{s't' - rc'}{l} \quad \text{and} \quad q' = (t')' = \frac{rt' - bs'}{l} \text{ both} \in \mathbb{Z}.$$

Because $\{a, c', b\}$ is a $D(l^2)$-triple, we have

$$ac' + l^2 = (s')^2, \quad ab + l^2 = r^2, \quad bc' + l^2 = q^2,$$

and

$$\begin{aligned} r &= \frac{(s')r' + aq'}{l}, & z &= \frac{(s')q' + cr'}{l} \\ r' &= \frac{(s')r - aq}{l}, & q' &= \frac{(s')q - c'r}{l} \end{aligned}$$

It can be seen that

$$\begin{aligned} ab + l^2 = r^2 = \left(\frac{(s')r' + aq'}{l}\right)^2 &= \frac{1}{l^2}((s')^2(ab' + l^2) + a^2(b'c' + l^2) + 2(s')ar'q') \\ &= \frac{1}{l^2}((s')^2ab' + l^2ac' + l^4 + a^2(b'c' + l^2) + 2(s')r'aq'), \end{aligned}$$

so

$$\begin{aligned} b &= \frac{1}{l^2}(s')^2b' + c' + \frac{1}{l^2}ab'c' + a + \frac{2}{l^2}(s')r'q' \\ &> \frac{2}{l^2}ab'c' + c' + a + \frac{2}{l^2}\sqrt{ac'}\sqrt{ab'}\sqrt{b'c'} \\ &= \frac{4}{l^2}ab'c'. \end{aligned}$$

Hence

$$b' < \frac{l^2 b}{4ac'} < \frac{\frac{l^2}{4}(4+\frac{4}{l^2})}{ac'} = \frac{l^2+1}{c'},$$

which means $b'c' < l^2+1$. But since $\{a, b', c'\}$ is a $D(l^2)$-triple, $(c')' = b' = 0$, so $c' = a + b - 2r$.

3.2 We may assume $c' < b$

The proof proceeds by showing that we can assume $c' < b$. It is here that the mistake in [1] arose. The proof of Lemma 1 in that paper for $D(4)$ used a version of inequalities (3.7) and (3.8) found in [10], modified for D(4) triples in order to show that it could be assumed that $c' < r^2$

$$ac' = (s')^2 - 4 < \frac{s^2}{r^2} - 4 = \frac{ac+4}{r^2} - 4 < \frac{ac}{r^2}.$$

In [10], the above inequality was shown to hold for $D(1)$ triples provided we assume that $b \leq a + c$, but in the case of $D(4)$ triples this inequality will never hold. In particular,

$$s' < \frac{s}{r} \tag{6}$$

is never true. To see this, note that we have

$$\begin{aligned} 4s &= s(r^2 - ab) = r^2 s - art + art + abs \\ &= r(rs - at) + a(rt - bs) \\ &= \frac{4(rs' + at')}{2}. \end{aligned}$$

So $s = \dfrac{rs' + at'}{2}$, which means inequality (6) is

$$rs' < \frac{rs' + at'}{2},$$

which is equivalent to

$$rs' < at'.$$

Since s' is always positive the inequality will always fail when $t' \leq 0$, so we assume that $t' > 0$. Squaring both sides, we obtain the equivalent inequality

$$r^2(s')^2 < a^2(t')^2. \tag{7}$$

Substituting $r^2 = ab + 4, (s')^2 = ac' + 4, (t')^2 = bc' + 4$ into (7), we get

$$r^2(s')^2 = a^2bc' + 4ab + 4ac' + 16 < a^2(t')^2 = a^2bc' + 4a^2$$

so our original inequality is equivalent to

$$ab + ac' + 4 < a^2$$

which is never true since $a < b$.

Now let $\{a, b, c\}$ be a $D(l^2)$-Diophantine triple with r, s, t, c', s', t' defined as above. If $c < b$ then we are done, since $c = \overline{c}'$ for the triple $\left\{a, b, \overline{c} = a+b+c+\frac{2}{l^2}(abc+rst)\right\}$ which falls under the previous case (since $0 \leq \overline{c}' < b$).

Suppose that $c \geq b > a > 0$. We want to show that we can assume $c' < b$. To do this, we suppose that $c' \geq c$, and show that a contradiction arises. This means that if $c \geq b$ then we must have $c' < c$, meaning that we can keep on replacing $c' = c$ and repeating with the new triple and eventually we will obtain $c' < b$.

If $c' \geq c$, then

$$\begin{aligned} c' \geq & a + b + c' + \frac{2}{l^2}(abc' + rs't') \\ \text{meaning} \quad & a + b + \frac{2}{l^2}(abc' + rs't') \leq 0 \\ \text{which implies} \quad & t' < 0. \end{aligned}$$

$t' < 0$ means we must have $rt - bs < 0$ and so

$$\begin{aligned} & r^2t^2 < b^2s^2 \\ \text{which means that} \quad & (ab + l^2)(bc + l^2) < b^2(ac + l^2), \\ \text{giving} \quad & ab + bc + l^2 < b^2 \\ \text{which then gives} \quad & c < b - a, \end{aligned}$$

contradicting our assumption that $c \geq b$.

Therefore any $D(l^2)$ Diophantine triple $\{a, b, c\}$ with $a < b < 4(1 + \frac{4}{l^2})$ can, through repetition of this process of taking $c' = c$, be reduced to a triple for which $0 \leq c' < b$, and we have established that the Pellian equation (5) stemming from such triples has solutions in the class of $(\pm l, l)$. $\square$

4 Proving Theorem 1.4

The next two sections follow a process which resembles that contained in [2], where a similar result on $D(4)$ triples of the form $\{F_k, 4F_{2n+4}, F_{2n+6}\}$ was proven.

4.1 Setting up the $D(9)$ Diophantine Triple

Let us begin by setting up a Pellian equation. Given that $\{F_{2n+8}, 9F_{2n+4}, F_k\}$ is a $D(9)$ Diophantine triple, we must have that

$$F_{2n+8}F_k + 9 = X^2, \text{ and } 9F_{2n+4}F_k + 9 = Y^2.$$

We subtract $9F_{2n+4}$ times the first equation from F_{2n+8} times the second to obtain

$$F_{2n+8}Y^2 - 9F_{2n+4}X^2 = 9(F_{2n+8} - 9F_{2n+4}). \tag{8}$$

Since $3 \mid 9F_{2n+4}$, $F_{2n+8} < 9F_{2n+4} < \frac{40}{9}F_{2n+8}$, and $9F_{2n+4}F_{2n+8} + 9 = (3F_{2n+6})^2$, Lemma 3.1 tells us that all solutions $Y\sqrt{F_{2n+8}} + X\sqrt{9F_{2n+4}}$ of this equation are given by

$$Y\sqrt{F_{2n+8}} + 3X\sqrt{F_{2n+4}} = (\pm 3\sqrt{F_{2n+8}} + 9\sqrt{F_{2n+4}})(F_{2n+6} + \sqrt{F_{2n+4}F_{2n+8}})^j.$$

Now we define the sequences $\{V_j\}_{j=1}^{\infty}$ and $\{U_j\}_{j=1}^{\infty}$ by

$$V_j + U_j\sqrt{F_{2n+4}F_{2n+8}} := (F_{2n+6} + \sqrt{F_{2n+4}F_{2n+8}})^j.$$

This gives us

$$\begin{aligned} X &= X_j = 3V_j \pm F_{2n+8}U_j, \text{ and} \\ Y &= Y_j = \pm V_j + 9F_{2n+4}U_j. \end{aligned}$$

Substituting these expressions into our earlier equations, we obtain

$$\begin{aligned} F_{2n+8}F_k + 9 = X^2 &= (3V_j \pm F_{2n+8}U_j)^2, \text{ and} \\ 9F_{2n+4}F_k + 9 = Y^2 &= (\pm 3V_j + 9F_{2n+4}U_j)^2. \end{aligned}$$

This gives us alternative expressions for F_k

$$F_k = \frac{9V_j^2 - 9}{F_{2n+8}} + F_{2n+8}U_j^2 \pm 6U_jV_j \text{ and } F_k = \frac{V_j^2 - 1}{F_{2n+4}} + 9U_j^2F_{2n+4} \pm 6U_jV_j,$$

which together give us

$$F_k = \pm 6U_jV_j + U_j^2(F_{2n+8} + 9F_{2n+4}). \tag{9}$$

We call this sequence

$$C_j^{\pm} := \pm 6U_jV_j + U_j^2(F_{2n+8} + 9F_{2n+4}), \tag{10}$$

and aim to solve $F_k = C_j^{\pm}$ for positive integer j and k. We remark here that

$$\begin{aligned} C_1^- &= -6U_1V_1 + U_1^2(F_{2n+8} + 9F_{2n+4}) = -6F_{2n+6} + F_{2n+8} + 9F_{2n+4} \\ &= F_{2n+5} - 4F_{2n+6} + 9F_{2n+4} = 2F_{2n+4} - 3F_{2n+3} = F_{2n} \end{aligned}$$

and that

$$\begin{aligned} C_1^+ &= 6F_{2n+6} + F_{2n+8} + 9F_{2n+4} = 12F_{2n+6} + F_{2n} \\ &= F_{2n+10} + F_{2n+7} + 2F_{2n+5} + 6F_{2n+4} + F_{2n} \\ &= F_{2n+11} + F_{2n+5} + F_{2n+3} + 2F_{2n} \end{aligned}$$

and therefore $F_{2n+11} < C_1^+ < F_{2n+12}$, thus we assume $j \geq 2$. In addition, because $X_1^+ > X_1^- > 0$ and

$$\begin{aligned} X_{j+1}^{\pm} &= 3F_{2n+6}(3V_j \pm F_{2n+8}U_j) + F_{2n+8}(\pm V_j + 9F_{2n+4}U_j) \\ &= (9F_{2n+6} \pm F_{2n+8})V_j + 3F_{2n+8}(3F_{2n+4} \pm F_{2n+6})U_j \\ &> 3V_j \pm F_{2n+8}U_j = X_j^{\pm}, \end{aligned}$$

and since we're looking for solutions such that $F_k = \dfrac{(X_j)^2 - 9}{F_{2n+8}}$ we may assume that $k > 2n$ when $j \geq 2$.

Define $\beta_n := F_{2n+6} + \sqrt{F_{2n+6}^2 - 1}$. Then we have

$$V_j = \frac{\beta_n^j + \beta_n^{-j}}{2}, \text{ and } U_j = \frac{\beta_n^j - \beta_n^{-j}}{2\sqrt{F_{2n+4}F_{2n+8}}}.$$

So we can rewrite $C_j^{\pm}$ as

$$C_j^{\pm} = \pm 6\frac{\beta_n^{2j} - \beta_n^{-2j}}{4\sqrt{F_{2n+6}^2 - 1}} + (F_{2n+8} + 9F_{2n+4}) \cdot \frac{\beta_n^{2j} + \beta_n^{-2j} - 2}{4(F_{2n+6}^2 - 1)}$$

$$= \frac{\pm 6\beta_n^{2j}}{4\sqrt{F_{2n+6}^2 - 1}} + \frac{(F_{2n+8} + 9F_{2n+4})\beta_n^{2j}}{4(F_{2n+6}^2 - 1)} - \frac{\pm 6\beta_n^{-2j}}{4\sqrt{F_{2n+6}^2 - 1}}$$
$$+ \frac{(F_{2n+8} + 9F_{2n+4})\beta_n^{-2j}}{4(F_{2n+6}^2 - 1)} - \frac{(F_{2n+8} + 9F_{2n+4})}{2(F_{2n+6}^2 - 1)}.$$

If we define the sequence $\gamma_n^\pm$ by

$$\gamma_n^\pm := \frac{\pm 6}{4\sqrt{F_{2n+6}^2 - 1}} + \frac{(F_{2n+8} + 9F_{2n+4})}{4(F_{2n+4}F_{2n+8})},$$

then we have

$$C_j^\pm = \beta_n^{2j}\gamma_n^\pm - \frac{(F_{2n+8} + 9F_{2n+4})}{2(F_{2n+6}^2 - 1)} + \beta_n^{-2j}\gamma_n^\mp,$$

and our problem can be expressed as finding solutions $j \geq 2$ and $k > 2n$ to the equation

$$\beta_n^{2j}\gamma_n^\pm - \frac{(F_{2n+8} + 9F_{2n+4})}{2(F_{2n+6}^2 - 1)} + \beta_n^{-2j}\gamma_n^\mp = \frac{\alpha^k - \overline{\alpha}^k}{\sqrt{5}}. \tag{11}$$

4.2 A Linear Form in Three Logarithms

We begin by finding bounds for $\gamma_n^\pm$.

Lemma 4.1. *The following bounds apply to* $\gamma_n^\pm$*,*

$$0.011\alpha^{-2n-4} < \gamma_n^- < 0.013\alpha^{-2n-4}$$
$$2.574\alpha^{-2n-4} < \gamma_n^+ < 2.585\alpha^{-2n-4}.$$

Proof. We have

$$\sqrt{\gamma_n^\pm} = \frac{3}{2\sqrt{F_{2n+8}}} \pm \frac{1}{2\sqrt{F_{2n+4}}}$$
$$= \frac{3}{2\sqrt{(\alpha^{2n+8} - \alpha^{-2n-8})/\sqrt{5}}} \pm \frac{1}{2\sqrt{(\alpha^{2n+4} - \alpha^{-2n-4})/\sqrt{5}}}$$
$$= \frac{5^{1/4}\alpha^{-n-2}}{2}\left(\frac{3}{\alpha^2\sqrt{(1 - 1/\alpha^{4n+16})}} \pm \frac{1}{\sqrt{(1 - 1/\alpha^{2n+8})}}\right)$$

As a result of the Taylor series of $(1-x)^{1/2}$, we have for $0 < x < 1$

$$1 + \frac{1}{2}x < \frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \cdots < 1 + \frac{x}{2}\Big(\frac{1}{1-x}\Big)$$

so

$$\frac{3}{\alpha^2}\Big(1+\frac{1}{2}\alpha^{-4n-16}\Big) < \frac{3}{\alpha^2\sqrt{(1-1/\alpha^{4n+16})}} < \frac{3}{\alpha^2}\Big(1+\frac{\alpha^{-4n-16}}{2(1-\alpha^{-4n-16})}\Big),$$

hence we must have

$$1.1458980 < \frac{3}{\alpha^2} < \frac{3}{\alpha^2\sqrt{(1-1/\alpha^{4n+16})}} < 1.14599721$$

and similarly

$$1 < \frac{1}{\sqrt{(1-1/\alpha^{2n+8})}} < 1.0041.$$

We then obtain bounds for $\sqrt{\gamma_n^{\pm}}$

$$0.141798 < \frac{2}{5^{1/4}\alpha^{-n-2}}\sqrt{\gamma_n^-} < 0.1499721$$

and

$$2.145898 < \frac{2}{5^{1/4}\alpha^{-n-2}}\sqrt{\gamma_n^+} < 2.15001,$$

and finally the following bounds on $\gamma_n^{\pm}$:

$$0.011\alpha^{-2n-4} < \gamma_n^- < 0.013\alpha^{-2n-4}$$
$$2.574\alpha^{-2n-4} < \gamma_n^+ < 2.585\alpha^{-2n-4}.$$

□

We now define a linear form in three logarithms, Λ by

$$\Lambda := 2j\log\beta_n - k\log\alpha + \log(\sqrt{5}\gamma_n^{\pm})$$

and proceed with the following lemma.

Lemma 4.2. *$0 < \Lambda < 1162\beta_n^{-2j}$ for $j \geq 2$.*

Proof. We can turn (11) into

$$\beta_n^{2j}\gamma_n^{\pm} - \frac{\alpha^k}{\sqrt{5}} = \frac{(F_{2n+8}+9F_{2n+4})}{2(F_{2n+6}^2-1)} - \frac{\overline{\alpha}^k}{\sqrt{5}} - \beta_n^{-2j}\gamma_n^{\mp}$$

$\Lambda = \log\sqrt{5}\gamma_n^{\pm}\beta_n^{2j}\alpha^{-k} > 0$ if and only if $\sqrt{5}\gamma_n^{\pm}\beta_n^{2j}\alpha^{-k} > 1$. Therefore in order to show that $\Lambda > 0$ we will assume, for a contradiction, that $\beta_n^{2j}\gamma_n^+ \leq \dfrac{\alpha^k}{\sqrt{5}}$. This would mean that

$$\frac{\sqrt{5}}{\alpha^k} \leq \frac{\beta_n^{-2j}}{\gamma_n^{\pm}} \leq \frac{\beta_n^{-2j}}{\gamma_n^-},$$

and because $\frac{1}{F_{2n+4}} < \frac{9}{F_{2n+8}}$, and by (11) and our assumption, this gives us the following inequality

$$\begin{aligned}\frac{1}{F_{2n+4}} &< \frac{1}{2F_{2n+4}} + \frac{9}{2F_{2n+8}} = \frac{F_{2n+8}+9F_{2n+4}}{2(F_{2n+6}^2-1)}\\ &\leq \beta_n^{-2j}\gamma_n^{\pm} + \frac{\overline{\alpha}^k}{\sqrt{5}} < \beta_n^{-2j}\Big(\gamma_n^+ + \frac{1}{5\gamma_n^-}\Big),\end{aligned}$$

which we apply below along with the bounds for $\gamma_n^{\pm}$ obtained in 4.1.

$$\begin{aligned}F_{2n+4}^jF_{2n+8}^j = (F_{2n+6}^2-1)^j &< \Big(2F_{2n+6}^2 - 1 + 2F_{2n+6}\sqrt{F_{2n+6}^2-1}\Big)^j = \beta_n^{2j}\\ &< F_{2n+4}\Big(\gamma_n^+ + \frac{1}{5\gamma_n^-}\Big) < F_{2n+4}(2.585\alpha^{-2n-4} + 18.2\alpha^{2n+4})\end{aligned}$$

And $F_{2n+4}^{j-1}F_{2n+8}^j < 2.585\alpha^{-2n-4} + 18.182\alpha^{2n+4}$ is only true if $j < 2$, which is a contradiction. Hence $\beta_n^{2j}\gamma_n^+ > \frac{\alpha^k}{\sqrt{5}}$ and so $\Lambda > 0$. Now we have

$$\begin{aligned}\Big|\alpha^k 5^{-1/2}\beta_n^{-2j}(\gamma_n^{\pm})^{-1} - 1\Big| &< \frac{1}{\beta_n^{2j}\gamma_n^{\pm}}\Big(\frac{F_{2n+8}+9F_{2n+4}}{2(F_{2n+6}^2-1)} + \frac{1}{\sqrt{5}\alpha^k}\Big)\\ &< \frac{1}{\beta_n^{2j}\gamma_n^{\pm}}\Big(\frac{2}{F_{2n+4}} + \frac{1}{\sqrt{5}\alpha^{2n+1}}\Big)\\ &< \frac{581}{\beta_n^{2j}} < \frac{1}{2},\end{aligned}$$

so by the fact that

$$|e^{\Lambda} - 1| < \frac{1}{2} \text{ implies that } |\Lambda| < 2|e^{\Lambda}-1| \tag{12}$$

we must have that $\Lambda < 1162\beta_n^{-2j}$. □

Using these two lemmas we prove a useful proposition

Proposition 4.3. *If equation* (9) *has a positive integer solution* (j, k) *with* $j > 1$ *then*

$$j < 1.15 \cdot 10^{12}(2n+7)\log(78j(2n+7))$$

In order to prove this proposition, we will apply Lemma 2.1 to the linear form in three logarithms Λ as defined above.

$$\Lambda := 2j \log \beta_n - k \log \alpha + \log (\sqrt{5}\gamma_n^{\pm})$$

We take

$$N = 3, \quad D = 4, \quad b_1 = 2j, \quad b_2 = -k, \quad b_3 = 1,$$
$$\alpha_1 = \beta_n, \quad \alpha_2 = \alpha, \quad \alpha_3 = \sqrt{5}\gamma_n^{\pm}.$$

The lemma calls for $\alpha_1, \alpha_2, \alpha_3$ to be multiplicatively independent (i.e. that their logarithms be linearly independent). We have that $\alpha_2 \in \mathbb{Q}(\sqrt{5})$ and $\alpha_1, \alpha_3^2 \in \mathbb{Q}(\sqrt{F_{2n+4}F_{2n+8}})$, and by Lemma 2.3, $F_{2n+4}F_{2n+8}$ is neither a square nor 5 times a square. Therefore, writing $F_{2n+4}F_{2n+8} = du^2$ for $u, d \in \mathbb{Z}$, $5 \neq d$ square-free, since no non-zero power of α_2 can be in $\mathbb{Q}(\sqrt{d})$, if $\alpha_1, \alpha_2, \alpha_3$ are multiplicatively dependent we must have that α_1 and α_3^2 are multiplicatively dependent. Since α_1 is a unit in $\mathbb{Q}(\sqrt{d})$, we must have that $\alpha_3^2 = 5(\gamma_n^{\pm})^2$ is a unit, but the norm of $5(\gamma_n^{\pm})^2$ is

$$25(\gamma_n^+\gamma_n^-)^2 = 25\Big(\frac{9F_{2n+4} - F_{2n+8}}{4F_{2n+4}F_{2n+8}}\Big)^4 < 1,$$

so this is never an integer for any n, therefore α_3^2 is not a unit for any n.

The absolute logarithmic heights for α_1 and α_2 are

$$h(\alpha_1) = h(\beta_n) = \frac{1}{2}\log \beta_n, \quad h(\alpha_2) = h(\alpha) = \frac{1}{2}\log \alpha.$$

For α_3, since

$$(x - \gamma_n^+)(x - \gamma_n^-) = x^2 - 2\Big(\frac{F_{2n+8} + 9F_{2n+4}}{4(F_{2n+4}F_{2n+8})}\Big)x + \Big(\frac{9F_{2n+4} - F_{2n+8}}{4F_{2n+4}F_{2n+8}}\Big)^2,$$

clearing denominators we get the minimal polynomial

$$16F_{2n+4}^2F_{2n+8}^2x^2 - 8(F_{2n+8}^2F_{2n+4} + 9F_{2n+4}^2F_{2n+8})x + (9F_{2n+4} - F_{2n+8})^2,$$

and since $|\gamma_n^{\pm}| \leq |\gamma_n^+| < 2.585\alpha^{-6} < 1$, and $F_\lambda < \alpha^\lambda/\sqrt{5}$ for positive even λ we have

$$h(\gamma_n^{\pm}) = \frac{1}{2}\log (16F_{2n+4}^2F_{2n+8}^2) = \log (4F_{2n+4}F_{2n+8}) < (4n + 12)\log \alpha + \log 4/5.$$

Hence we can take

$$\begin{aligned} h(\alpha_3) = h(\sqrt{5}\gamma_n^{\pm}) \leq h(\sqrt{5}) + h(\gamma_n^{\pm}) &< \frac{1}{2}\log 5 + (4n + 12)\log \alpha + \log 4/5 \\ &< 2\log \alpha + (4n + 12)\log \alpha = (4n + 14)\log \alpha \end{aligned}$$

and finally since we need $A_i \geq D \cdot h(\alpha_i)$, we take

$$A_1 = 2\log\beta_n, \quad A_2 = 2\log\alpha, \quad A_3 = 8(2n+7)\log\alpha.$$

Next, since $\alpha^{\lambda-2} \leq F_\lambda \leq \alpha^{\lambda-1}$, we see that

$$\beta_n < 2F_{2n+8} < 2\alpha^{2n+5} < \alpha^{2n+7}$$

and in addition,

$$\begin{aligned}
\alpha^{k-1} &< 2\alpha^{k-2} < 2F_k \leq 12U_jV_j + 2U_j^2(F_{2n+8} + 9F_{2n+4}) \\
&< 20U_jV_j + F_{2n+4}F_{2n+8}U_j^2 < (V_j + U_j\sqrt{F_{2n+4}F_{2n+8}})^2 - V_j^2 \\
&< (V_j + U_j\sqrt{F_{2n+4}F_{2n+8}})^2 = (F_{2n+6} + \sqrt{F_{2n+4}F_{2n+8}})^{2j} \\
&< (2F_{2n+6})^{2j} < (2\alpha^{2n+5})^{2j} < \alpha^{2j(2n+7)}.
\end{aligned}$$

Due to the results above, we can take

$$\begin{aligned}
E &= \max\{1, \max\{|b_j|A_j/A_N; 1 \leq j \leq N\}\} \\
&\leq \max\left\{2j, k, 1, \frac{2j\log\beta_n}{\log\alpha}, 4j\log\beta_n, \frac{k\log\alpha}{\log\beta_n}, 2k\log\alpha, 4(2n+7), \frac{4(2n+7)\log\alpha}{\log\beta_n}\right\} \\
&= \max\left\{k, \frac{2j\log\beta_n}{\log\alpha}, 4(2n+7)\right\} \leq 2j(2n+7)
\end{aligned}$$

and

$$\begin{aligned}
C(3) &= \frac{8}{2}(5)(9)(16e)^4 < 6.45 \times 10^8 \\
C_0 &= \log e^{20.2}3^{5.5}(16)\log(4e) < 30 \\
W_0 &= \log(1.5eE \cdot 4\log 4e) < \log(78j(2n+7)) \\
\Omega &= (2\log\beta_n)(2\log\alpha)(8(2n+7)\log\alpha).
\end{aligned}$$

Proof of Proposition 4.3. Applying Lemma 2.1 and combining this with the previous lemma, we see,

$$2j\log\beta_n - \log 1162 < -\log|\Lambda| < 1.434 \cdot 10^{11}(2n+7)(\log\beta_n)(\log(78j(2n+7))),$$

therefore

$$j < 1.15 \cdot 10^{12}(2n+7)\log(78j(2n+7)),$$

as desired. □

4.3 Linear Form in Two Logarithms

Using $j = 1, k = 2n$ in Λ we define the linear form in three logarithms, Λ_0, by

$$\Lambda_0 := 2\log\beta_n - 2n\log\alpha + \log(\sqrt{5}\gamma_n^{\pm}).$$

We will use Λ_0 to help form a linear form in two logarithms later, first we find the following upper bound on Λ_0

Lemma 4.4. $|\Lambda_0| < 9473\beta_n^{-2}$.

Proof. Assume for now that $n \geq 2$. After substituting the one known solution, $j = 1, k = 2n$, into our equation (11) from earlier, it becomes

$$\beta_n^2\gamma_n^{\pm} - \frac{\alpha^{2n}}{\sqrt{5}} = \frac{(F_{2n+8} + 9F_{2n+4})}{2(F_{2n+6}^2 - 1)} - \frac{\overline{\alpha}^{2n}}{\sqrt{5}} - \beta_n^{-2}\gamma_n^{\mp}.$$

If $\beta_n^2\gamma_n^{\pm} \leq \frac{\alpha^{2n}}{\sqrt{5}}$, then $\frac{\alpha^{-2n}}{\sqrt{5}} \leq \frac{1}{5\beta_n^2\gamma_n^{\pm}}$ and

$$\begin{aligned}\left|\alpha^{2n}5^{-1/2}\beta_n^{-2}(\gamma_n^{\pm})^{-1} - 1\right| &< \frac{\beta_n^{-2}\gamma_n^{\mp} + \frac{\alpha^{-2n}}{\sqrt{5}}}{\beta_n^2\gamma_n^{\pm}} \\ &< \frac{\gamma_n^{\mp} + \frac{1}{5\gamma_n^{\pm}}}{\beta_n^4\gamma_n^{\pm}} \\ &< \frac{235.3 + 1656.2\alpha^{4n+8}}{\beta_n^4} \\ &< \beta_n^{-2}(235.3/(55 + 12\sqrt{21})^2 + 1656.2) \\ &< 1657\beta_n^{-2} < \frac{1}{2}.\end{aligned}$$

If $\beta_n^2\gamma_n^{\pm} > \frac{\alpha^{2n}}{\sqrt{5}}$, then

$$\begin{aligned}\left|\alpha^{2n}5^{-1/2}\beta_n^{-2}(\gamma_n^{\pm})^{-1} - 1\right| &< \frac{1/(2F_{2n+4}) + 1/F_{2n+4}}{\beta_n^2\gamma_n^{\pm}} \\ &< \frac{3}{2F_{2n+4}\beta_n^2\gamma_n^{\pm}} < 306\beta_n^{-2} < \frac{1}{2}.\end{aligned}$$

For $n = 1$, $|\Lambda_0| = 2\log(21 + \sqrt{440}) - 2\log\alpha + \log(\sqrt{5}\gamma_1^{+}) < 9473\beta_n^{-2}$. In every case we have (by inequality (12)) $|\Lambda_0| < 9473\beta_n^{-2}$. □

We now form the aforementioned linear form in two logarithms, which will help to give us a hard bound on j. Define Λ_1 by

$$\Lambda_1 := K \log \alpha - (j-1) \log (5/4),$$

where $K = (2j-1)(2n+6) - k - 6$. We have the following bound on Λ_1

Lemma 4.5. $|\Lambda_1| < (9j + 17042)\alpha^{-4n-12}$.

Proof. Note first that

$$\begin{aligned}
\beta_n &= F_{2n+6} + \sqrt{F_{2n+6}^2 - 1} = \frac{(F_{2n+6} + \sqrt{F_{2n+6}^2 - 1})^2}{F_{2n+6} + \sqrt{F_{2n+6}^2 - 1}} \\
&= \frac{2F_{2n+6}^2 + 2F_{2n+6}\sqrt{F_{2n+6}^2 - 1} - 1}{F_{2n+6} + \sqrt{F_{2n+6}^2 - 1}} = 2F_{2n+6} - \frac{1}{F_{2n+6} + \sqrt{F_{2n+6}^2 - 1}} \\
&= 2F_{2n+6}\left(1 - \frac{1}{2F_{2n+6}(F_{2n+6} + \sqrt{F_{2n+6}^2 - 1})}\right)
\end{aligned}$$

and $2F_{2n+6} = \frac{2}{\sqrt{5}}\left(\alpha^{2n+6} - \overline{\alpha}^{2n+6}\right) = \frac{2}{\sqrt{5}}\alpha^{2n+6}\left(1 - \frac{1}{\alpha^{4n+12}}\right)$. So if we define δ_n by

$$\delta_n = \left(1 - \frac{1}{2F_{2n+6}(F_{2n+6} + \sqrt{F_{2n+6}^2 - 1})}\right)\left(1 - \frac{1}{\alpha^{4n+12}}\right),$$

then $\log \beta_n = \log(\frac{2}{\sqrt{5}}) + (2n+6)\log \alpha + \log \delta_n$ and we can see that

$$\begin{aligned}
\Lambda - \Lambda_0 &= \left(2j \log \beta_n - k \log \alpha + \log(\sqrt{5}\gamma_n^{\pm})\right) - \left(2 \log \beta_n - 2n \log \alpha + \log(\sqrt{5}\gamma_n^{\pm})\right) \\
&= (2j-2)\log \beta_n - (k-2n)\log \alpha \\
&= (2j-2)\log(2/\sqrt{5}) + K \log \alpha + (2j-2)\log \delta_n,
\end{aligned}$$

where $K = (2j-1)(2n+6) - k - 6$. Hence we have $\Lambda_1 = \Lambda - \Lambda_0 - (2j-2)\log \delta_n$. We can bound $|\log \delta_n|$ by taking

$$\begin{aligned}
|\log \delta_n| &\le \left|\log\left(1 - \frac{1}{2F_{2n+6}(F_{2n+6} + \sqrt{F_{2n+6}^2 - 1})}\right)\right| + \left|\log\left(1 - \frac{1}{\alpha^{4n+12}}\right)\right| \\
&< \frac{1}{F_{2n+6}\left(F_{2n+6} + \sqrt{F_{2n+6}^2 - 1}\right)} + \frac{1}{\alpha^{4n+12}} < \frac{1}{2\alpha^{4n+8}} + \frac{1}{\alpha^{4n+12}} \\
&< \frac{9}{2\alpha^{4n+12}}.
\end{aligned}$$

Here we have used the triangle inequality and (12). This then gives us

$$|\Lambda_1| \leq |\Lambda| + |\Lambda_0| + |2j-2||\log \delta_n| < \frac{9473}{\beta_n^2} + \frac{9(j-1)}{\alpha^{4n+12}},$$

and since

$$\beta_n = F_{2n+6} + \sqrt{F_{2n+6}^2 - 1} > 2\alpha^{2n+4}, \text{ we must have}$$
$$\beta_n^2 > \frac{4}{7}\alpha^{4n+12}.$$

Therefore

$$|\Lambda_1| < (9j + 17042)\alpha^{-4n-12}.$$

□

Lemma 4.6. *If equation* (9) *has a positive integer solution* (j, k) *with* $j > 1$*, then*

$$j < 3.55 \times 10^{19}, \text{ and } \quad n < 246806.$$

Proof. We apply Lemma 2.2. Let

$$D = 2, \quad \gamma_1 = \frac{5}{4}, \quad \gamma_2 = \alpha, \quad b_1 = (j-1), \quad b_2 = K.$$

In addition, take $h_1 = \log 5, h_2 = \frac{1}{2}$. By the previous lemma, we have

$$\begin{aligned} K &< \frac{(j-1)\log(5/4) + (9j + 17042)\alpha^{-4n-12}}{\log \alpha} \\ &< 0.4638(j-1) + 0.0085j + 16.0466 < 0.48j + 15.59, \end{aligned}$$

and because

$$\frac{|b_1|}{Dh_2} + \frac{|b_2|}{Dh_1} = (j-1) + \frac{K}{2\log 5} < 1.15j + 3.85,$$

Lemma 2.2 gives us a lower bound on Λ_1,

$$\log|\Lambda_1| > -17.9 \cdot 8 \cdot \log 5 \cdot \Big(\max\big\{ \log(1.15j + 3.85) + 0.38, 15 \big\} \Big)^2.$$

From the previous lemma, we see that

$$\log|\Lambda_1| < -(4n+12)\log \alpha + \log(9j + 17042).$$

A combination of these two bounds yields

$$n < 119.75\Big(\max\Big\{\log(1.15j + 3.85) + 0.38, 15\Big\}\Big)^2 + 0.52\log(9j + 17042).$$

If

$$\log(1.15j + 3.85) + 0.38 \leq 15$$

then $j < 1943527$ and $n < 26952$. Otherwise,

$$n < 119.75\Big(\log(1.15j + 3.85) + 0.38\Big)^2 + 0.52\log(9j + 17042)$$

and we substitute this bound into Proposition 4.3 to obtain

$$\begin{aligned} j <& 1.15 \cdot 10^{12}(2(119.75\Big(\log(1.15j + 3.85) + 0.38\Big)^2 + 0.52\log(9j + 17042)) + 7) \\ &\times \log(78j(2(119.75\Big(\log(1.15j + 3.85) + 0.38\Big)^2 + 0.52\log(9j + 17042)) + 7)), \end{aligned}$$

which means that $j < 3.55 \times 10^{19}$ and so $n < 246806$. □

4.4 Refining Our Bounds

We improve our bounds on n and j in this section before applying Baker-Davenport reduction on those bounds in the next section. Lemma 4.5 gives us

$$|K\log\alpha - (j-1)\log(5/4)| < (9j + 17042)\alpha^{-4n-12}.$$

Thus we can divide by $j - 1$ to get

$$\Big|\frac{\log(5/4)}{\log\alpha} - \frac{K}{j-1}\Big| < \frac{9j + 17042}{(j-1)\alpha^{4n+12}\log\alpha}.$$

Assume that

$$\frac{9j + 17042}{(j-1)\alpha^{4n+12}\log\alpha} < \frac{1}{2(j-1)^2}. \tag{13}$$

By above,

$$\Big|\frac{\log(5/4)}{\log\alpha} - \frac{K}{j-1}\Big| < \frac{1}{2(j-1)^2}.$$

By Theorem 1.3, we must have that $\frac{K}{(j-1)}$ is a convergent in the simple continued fraction expansion of $\log(5/4)/\log\alpha$. Since the denominator of the 46th convergent

$$\frac{25158053660121411107}{54253653513327093513}$$

is greater than the upper bound of 3.55×10^{19} we established for j, we can use the denominator of the 45th convergent

$$\frac{4460457560349832575}{9619031832089360168}$$

which is bigger than 9.6×10^{18} to obtain the following lower bound,

$$\left|\frac{\log(5/1)}{\log\alpha} - \frac{K}{j-1}\right| \geq \left|\frac{\log(5/1)}{\log\alpha} - \frac{4460457560349832575}{9619031832089360168}\right| > 1.9 \times 10^{39}.$$

Combining these bounds

$$1.9 \times 10^{39} < \frac{9j + 17042}{(j-1)\alpha^{4n+12}\log\alpha} < 17060\alpha^{-4n-12}(\log\alpha)^{-1}$$

gives us $n < 49$. We also know from Lemma 1.2 that if $\frac{p_r}{q_r}$ is the rth convergent of $\frac{\log(5/4)}{\log\alpha}$, then

$$\left|\frac{\log(5/4)}{\log\alpha} - \frac{p_r}{q_r}\right| > \frac{1}{(a_{r+1}+2)q_r^2},$$

where a_{r+1} is the $(r+1)$st partial quotient of $\frac{\log(5/4)}{\log\alpha}$. Therefore for $2 \leq r \leq 45$,

$$\min\left\{\frac{1}{(a_{r+1}+2)(j-1)^2}\right\} < \frac{9j + 17042}{(j-1)\alpha^{4n+12}\log\alpha}.$$

Since we have that $\max\{a_{r+1} : 2 \leq r \leq 45\} = a_{36} = 49$, we have

$$\alpha^{4n+12} < 51(j-1)(9j+17042)(\log\alpha)^{-1}.$$

If (13) does not hold, i.e., if

$$\frac{9j + 17042}{(j-1)\alpha^{4n+12}\log\alpha} \geq \frac{1}{2(j-1)^2},$$

then we have

$$\alpha^{4n+12} \leq 2(j-1)(9j+17042)(\log\alpha)^{-1}.$$

In either case,

$$\alpha^{4n+12} < 51(j-1)(9j+17042)(\log\alpha)^{-1} < 444352j^2.$$

This leads to the following proposition.

Proposition 4.7. *If equation* (9) *has a positive integer solution* (j,k) *with* $j > 1$*, then*

$$n < 1.04\log j + 3.76$$

When we combine this with the bound for j in Proposition 4.3, we get

$$j < 1.15\cdot 10^{12}(2.08\log j + 14.52)\log\left(78j(2.08\log j + 14.52)\right),$$

which implies

Lemma 4.8. *If equation* (9) *has a positive integer solution* (j,k) *with* $j > 1$*, then* $j < 4.63\times 10^{15}$ *and* $n < 42$*.*

4.5 Reduction of the Bounds

We now use the reduction method of Baker and Davenport described in Lemma 4.3 to bring the bounds for j and n down to something more computationally manageable. We then use a procedure written in Maple$^{\text{TM}}$ to check all remaining possibilities.

We know that

$$0 < 2j\log\beta_n - k\log\alpha + \log\left(\sqrt{5}\gamma_n^{\pm}\right) < 1162\beta_n^{-2j}.$$

In order to apply Baker-Davenport reduction, we consider

$$\kappa = \frac{2\log\beta_n}{\log\alpha},\quad \mu = \frac{\log\left(\sqrt{5}\cdot\gamma_n^{\pm}\right)}{\log\alpha},\quad A = \frac{1162}{\log\alpha},\quad B = \beta_n^2,\quad M = 4.63\times 10^{15}.$$

We then used a set of procedures written in Maple to undertake the computations (see Appendix A). In all cases, we obtained $j \leq 6$ and therefore $1 \leq n \leq 6$. So we have the following result.

Lemma 4.9. *If equation* (9) *has a positive integer solution* (j,k) *with* $j > 1$*, then* $j \leq 6$ *and* $n \leq 6$*.*

Applying this result to equation (10) in order to prove Theorem 1.1, we see that no combination of n and j with $1 \leq n \leq$, $2 \leq j \leq$ yields a Fibonacci number. We have already established that $F_{11} < C_1^+ < F_{12}$, which means our only solution is $C_1^- = F_{2n}$. When $n = 1$, $F_{2n} = F_2 = 1 = F_1$, giving us one extra solution in that case.

5 Proving Theorem 1.5

5.1 Setting up the $D(64)$ Diophantine Triple

We now wish to show the same thing for the $D(64)$-diophantine triple $\{F_k, F_{2n+12}, 16F_{2n+6}\}$. We wish to show that for this triple the only value k can take is $2n$. Note that in this case, we have $a = 16F_{2n+6}$ and $b = F_{2n+12}$ in Lemma 3.1, and $64 \mid a$ if we assume that $3 \mid n$, which gives us that all solutions of the equation $4Y\sqrt{F_{2n+6}} + X\sqrt{F_{2n+12}}$ are given by

$$4Y\sqrt{F_{2n+6}} + X\sqrt{F_{2n+12}} = (\pm 32\sqrt{F_{2n+6}} + 8\sqrt{F_{2n+12}})\left(\frac{F_{2n+9}\sqrt{F_{2n+6}F_{2n+12}}}{2}\right)^j,$$

where $j \geq 0$.

Again we define sequences $(U_j)_{j\geq 0}$ and $(V_j)_{j\geq 0}$, this time by

$$V_j + U_j\sqrt{F_{2n+6}F_{2n+12}} := (F_{2n+9} + \sqrt{F_{2n+6}F_{2n+12}})^j.$$

This gives us

$$y = y_j = \pm 8V_j + 2U_jF_{2n+12} \text{ and } x = x_j = 8V_j \pm 32F_{2n+6}U_j,$$

so we get

$$F_k = \pm 32U_jV_j + 4U_j^2(F_{2n+12} + 16F_{2n+6}). \quad (14)$$

If we let $C_j^{(\pm)} := \pm 32U_jV_j + 4U_j^2(F_{2n+12} + 16F_{2n+6})$ for $j \geq 1$ then our goal now is to solve

$$F_k = C_j^{(\pm)} \quad (15)$$

for some positive j and k.

As in the previous section, the equation has the solution $C_j^{(\pm)} = F_{2n}$, so to prove our result we must prove that there are no other solutions. We can note immediately that

$$F_{2n+14} < C_1^{(+)} < F_{2n+15},$$

so we suppose for a contradiction that $j \geq 2$. In this case we have

$$\beta_n := \frac{F_{2n+9} + \sqrt{F_{2n+6}F_{2n+12}}}{2},$$

so we can express

$$V_j := \frac{\beta_n^j + \beta_n^j}{2} \text{ and } U_j := \frac{\beta_n^j - \beta_n^j}{2\sqrt{F_{2n+9}^2 - 4}}.$$

We find that $C_j^{(\pm)} = \beta_n^{2j}\gamma_n^{(\pm)} - 2\Big(\frac{F_{2n+12} + 16F_{2n+6}}{F_{2n+9}^2 - 4}\Big) + \beta_n^{-2j}\gamma_n^{(\mp)}$ where

$$\gamma_n^{(\pm)} := \pm\frac{8}{\sqrt{F_{2n+9}^2 - 4}} + \frac{F_{2n+12} + 16F_{2n+6}}{F_{2n+9}^2 - 4}.$$

So we must solve:

$$C_j^{(\pm)} = \beta_n^{2j}\gamma_n^{(\pm)} - 2\left(\frac{F_{2n+12} + 16F_{2n+6}}{F_{2n+9}^2 - 4}\right) + \beta_n^{-2j}\gamma_n^{(\mp)} = \frac{\alpha^k - \overline{\alpha}^k}{\sqrt{5}}.$$

5.2 A Linear Form in Three Logarithms (2)

Lemma 5.1. *The following bounds apply to* $\gamma_n^{\pm}$*,*

$$0.00694321\alpha^{-2n-4} < \gamma_n^- < 0.00733800\alpha^{-2n-4}$$
$$8.45276900\alpha^{-2n-4} < \gamma_n^+ < 8.46635831\alpha^{-2n-4}.$$

Using the same identity from earlier, we get

$$\sqrt{\gamma_n^{(\pm)}} = \left(\frac{4}{\sqrt{F_{2n+12}}} \pm \frac{1}{\sqrt{F_{2n+6}}}\right) = 5^{1/4}\alpha^{-n-3}\left(\frac{4}{\alpha^3\sqrt{1 - \frac{1}{\alpha^{4n+24}}}} \pm \frac{1}{\sqrt{1 - \frac{1}{\alpha^{4n+12}}}}\right),$$

giving us

$$1 + \frac{1}{2}\left(\frac{1}{\alpha^{4n+24}}\right) < \frac{1}{\sqrt{1 - \frac{1}{\alpha^{4n+24}}}} < 1 + \frac{\frac{1}{\alpha^{4n+24}}}{2(1 - \frac{1}{\alpha^{4n+24}})} < \frac{4}{\alpha^3} \cdot \frac{1}{\sqrt{1 - \frac{1}{\alpha^{4n+24}}}},$$

so

$$0.9442719 < \frac{4}{\alpha^3}\frac{1}{\sqrt{1 - \frac{1}{\alpha^{4n+24}}}} < 0.9442765.$$

Thus:

$$1 < \frac{1}{\sqrt{1-\frac{1}{\alpha^4 n+24}}} < 1 + \frac{\frac{1}{\alpha^{4n+12}}}{2(1-\frac{1}{\alpha^{4n+12}})} < 1.00155765.$$

Rearranging

$$1.9442719 < \frac{\sqrt{\gamma_n^+}}{5^{1/4}\alpha^{-n-3}} < 1.94583415$$

and

$$-0.0572858 < \frac{\sqrt{\gamma_n^-}}{5^{1/4}\alpha^{-n-3}} < -0.0557234$$

gives us the result. □

We define a linear form in logarithms, $\Lambda := 2j\log(\beta_n) - k\log(\alpha) + \log(\sqrt{5}(\gamma_n^{(\pm)})$.

Lemma 5.2. $0 < \Lambda < 3868\beta_n^{-2j}$ *for* $j \geq 2$.

Proof. We have

$$\beta_n^{2j}\gamma_n^{(\pm)} - \frac{\alpha^k}{\sqrt{5}} = \frac{2(F_{2n+12}+16F_{2n+6})}{F_{2n+9}^2-4} - \beta_n^{-2j}\gamma_n^{(\mp)} - \frac{\overline{\alpha}^k}{\sqrt{5}}.$$

Suppose for a contradiction that $\beta_n^{2j}\gamma_n^{(\pm)} \leq \dfrac{\alpha_n^k}{\sqrt{5}}$, thus

$$\frac{\sqrt{5}}{\alpha^k} \leq \frac{\beta_n^{-2j}}{\gamma_n^{(\pm)}} \leq \frac{\beta_n^{-2j}}{\gamma_n^{(-)}},$$

so

$$\begin{aligned}\frac{1}{F_{2n+6}} < \frac{2}{F_{2n+6}} + \frac{32}{F_{2n+12}} &= \frac{2(F_{2n+12}+16F_{2n+6})}{F_{2n+9}^2-4}\\ &< \beta_n^{-2j}\gamma_n^{(\mp)} + \frac{\overline{\alpha}^k}{\sqrt{5}} < \beta_n^{-2j}\gamma_n^{(+)} + \frac{1}{\sqrt{5}\alpha^k}\\ &\leq \beta_n^{-2j}\gamma_n^{(+)} + \frac{1}{\sqrt{5}}\left(\frac{\beta_n^{-2j}}{\sqrt{5}\gamma_n^{(-)}}\right) = \beta_n^{-2j}\left(\gamma_n^{(+)} + \frac{1}{5\gamma_n^{(-)}}\right).\end{aligned}$$

We have

$$F_{2n+6}^j F_{2n+12}^j < \beta_n^{2j} < F_{2n+6}\left(\gamma_n^{(+)} + \frac{1}{5\gamma_n^{(-)}}\right) < F_{2n+6}(8.47\alpha^{-2n-6} + 28.81\alpha^{2n+6}),$$

which gives

$$F_{2n+6}^{j-1}F_{2n+12}^{j} < 8.47\alpha^{-2n-6} + 28.81\alpha^{2n+6},$$

which is a contradiction. Therefore $\beta_n^{2j}\gamma_n^{(\pm)} - \dfrac{\alpha^k}{\sqrt{5}} > 0$, so $\Lambda > 0$.

Furthermore,

$$\begin{aligned}
|\alpha^k 5^{-1/1}\beta_n^{-2j}(\gamma_n^{(\pm)})^{-1} - 1| &= \frac{1}{\beta_n^{2j}\gamma_n^{(\pm)}}\left|\frac{\alpha^k}{\sqrt{5}} - \beta_n^{2j}\gamma_n^{(\pm)}\right| = \frac{1}{\beta_n^{2j}\gamma_n^{(\pm)}}\left(\beta_n^{2j}\gamma_n^{(\pm)} - \frac{\alpha^k}{\sqrt{5}}\right)\\
&= \frac{1}{\beta_n^{2j}\gamma_n^{(\pm)}}\left[\frac{2(F_{2n+12}+16F_{2n+6})}{F_{2n+9}^2-4} - \beta_n^{-2j}\gamma_n^{(\mp)} - \frac{\overline{\alpha}^k}{\sqrt{5}}\right]\\
&< \frac{1}{\beta_n^{2j}\gamma_n^{(\pm)}}\left[\frac{2(F_{2n+12}+16F_{2n+6})}{F_{2n+9}^2-4} + \frac{1}{\alpha^k\sqrt{5}}\right]\\
&\leq \frac{1}{\beta_n^{2j}\gamma_n^{(\pm)}}\left[\frac{2(F_{2n+12}+16F_{2n+6})}{F_{2n+9}^2-4} + \frac{1}{\alpha^{2n+1}\sqrt{5}}\right].
\end{aligned}$$

We assume k is odd (for k even the inequality is clear), and $k \geq 2n$ from earlier. We then have that the above:

$$\begin{aligned}
&= \frac{1}{\beta_n^{2j}\gamma_n^{(\pm)}}\left[\frac{2}{F_{2n+6}} + \frac{32}{F_{2n+12}} + \frac{1}{\alpha^{2n+1}\sqrt{5}}\right]\\
&< \frac{144.026}{\beta_n^{2j}}\alpha^{2n+6}\left[\frac{2\sqrt{5}}{\alpha^{2n+6} - \frac{1}{\alpha^{2n+6}}} + \frac{32\sqrt{5}}{\alpha^{2n+12} - \frac{1}{\alpha^{2n+12}}} + \frac{1}{\alpha^{2n+1}\sqrt{5}}\right]\\
&= \frac{144.026}{\beta_n^{2j}}\left[\frac{2\sqrt{5}}{1 - \frac{1}{\alpha^{4n+12}}} + \frac{32\sqrt{5}}{\alpha^6 - \frac{1}{\alpha^{4n+18}}} + \frac{\alpha^5}{\sqrt{5}}\right]\\
&\leq \frac{144.026}{\beta_n^{2j}}\left[\frac{2\sqrt{5}}{1 - \frac{1}{\alpha^{16}}} + \frac{32\sqrt{5}}{\alpha^6 - \frac{1}{\alpha^{22}}} + \frac{\alpha^5}{\sqrt{5}}\right] < \frac{1934}{\beta_n^{2j}} < \frac{1}{2}.
\end{aligned}$$

So we have $|\Lambda| < \dfrac{3868}{\beta_n^{2j}}$ □

Proposition 5.3. *If* (14) *has a positive integer solution* (j,k) *with* $j > 1$*, then*

$$j < 1.16 \cdot 10^{10}(2n+9)\log(156j(n+5)).$$

Proof. We take

$$N = 3, \quad D = 4, \quad b_1 = 2j, \quad b_2 = -k, \quad b_3 = 1,$$
$$\alpha_1 = \beta_n, \quad \alpha_2 = \alpha, \quad \alpha_3 = \sqrt{5}\gamma_n^{(\pm)}.$$

We have from Lemma 2.4 that $\alpha_1, \alpha_2, \alpha_3$ are multiplicatively independent.

We have

$$h(\alpha_1) = h(\beta_n) = \frac{1}{2}\log \beta n, \text{ and } H(\alpha_2) = h(\alpha) = \frac{1}{2}\log \alpha.$$

We now calculate $h(\alpha_3)$.

$$\begin{aligned}(X-\gamma_n^{(+)})(X-\gamma_n^{(-)}) &= X^2 - (\gamma_n^{(+)}+\gamma_n^{(-)})X + \left(\frac{F_{2n+12}-16F_{2n+6}}{F_{2n+6}F_{2n+12}}\right)^2 \\ = F_{2n+6}^2F_{2n+12}^2X^2 &- 2(F_{2n+12}^2F_{2n+6}+16F_{2n+12}F_{2n+6}^2)X + (F_{2n+12}-16F_{2n+6})^2,\end{aligned}$$

so

$$\begin{aligned}h(\gamma_n^{(\pm)}) &= \frac{1}{2}\left[\log\left(F_{2n+6}^2F_{2n+12}^2\right) + \log(1) + \log(1)\right] = \log\left(F_{2n+6}F_{2n+12}\right) \\ &< \log\left(\frac{\alpha^{2n+6}\alpha^{2n+12}}{5}\right) = (4n+18)\log(\alpha) + \log\left(\frac{1}{5}\right).\end{aligned}$$

Thus

$$\begin{aligned}h(\alpha_3) &= h(\sqrt{5}\gamma_n^{(\pm)}) \le h(\sqrt{5}) + h(\gamma_n^{(\pm)}) \\ &< \frac{1}{2}\log(5) + (4n+18)\log(\alpha) + \log\left(\frac{1}{5}\right) \\ &= (4n+18)\log(\alpha).\end{aligned}$$

We have

$$\begin{aligned}A_1 &= \max\{2\log(\beta_n), |\log(\beta_n)|\} = 2\log(\beta_n), \\ A_2 &= \max\{2\log\alpha, |\log\alpha|\} = 2\log\alpha, \\ A_3 &= \max\{4(4n+18)\log(\alpha), |\log(\sqrt{5}\gamma_n^{(\pm)})|\} = 8(2n+9)\log(\alpha).\end{aligned}$$

As $\alpha^{l-2} \le F_l \le \alpha^{l-1}$, we have that $\beta_n < 2F_{2n++9} < 2\alpha^{2n+8} < \alpha^{2(n+5)}$.
Moreover,

$$\begin{aligned}\alpha^{k-1} &< 2\alpha^{k-2} < 2F_k \le 64U_jV_j + 8U_j^2(F_{2n+12}+16F_{2n+6}) \\ &< (V_j + U_j\sqrt{F_{2n+6}F_{2n+12}})^2 = \left(\frac{F_{2n+9}+\sqrt{F_{2n+9}^2-4}}{2}\right)^{2j} \\ &< F_{2n+9}^{2j} < (\alpha^{2n+8})^{2j} < \alpha^{4j(n+4)}.\end{aligned}$$

Let

$$\begin{aligned}
E &\leq 4j(n+5) \\
C(3) &< 6.45 \times 10^8 \\
C_0 &< 30 \\
W_0 &= \log(1.5eE4\log(4e)) < \log(156j(n+5)) \\
\Omega &= A_1A_2A_3 \leq 32(2n+9)\log(\alpha)\log(\beta_n).
\end{aligned}$$

This gives by Lemma 2.1

$$\begin{aligned}
\log|\Lambda| &> -C(N)C_0W_0D^2\Omega \\
&\geq -6.45 \times 10^8 \cdot 30\log(156j(n+5))(32(2n+9)(\log(\alpha))^2(\log(\beta_n))).
\end{aligned}$$

So

$$\begin{aligned}
\log\Lambda &< \log\frac{3868}{\beta_n^{2j}} = \log(3868) - 2j\log(\beta_n) \\
j &< 1.16 \times 10^{10}(2n+9)\log(156j(n+5)).
\end{aligned}$$

□

5.3 Linear Form in Two Logarithms (2)

Lemma 5.4. *Define $\Lambda_0 = 2\log\beta_n - 2n\log\alpha + \log(\sqrt{5}\gamma_n^{(\pm)})$. We have that*

$$|\Lambda_0| < \frac{270830}{\beta_n^2}.$$

Proof. Assume that $n \geq 2$. We have

$$\beta_n^2\gamma_n^{(\pm)} - \frac{\alpha^{2n}}{\sqrt{5}} = \frac{2(F_{2n+12} + 16F_{2n+6})}{F_{2n+9}^2 - 4} - \beta_n^{-2}\gamma_n^{(\mp)} - \frac{\alpha^{-2n}}{\sqrt{5}}.$$

If $\dfrac{1}{\beta_n^2\gamma_n^{(\pm)}} \geq \alpha^{-2n}\sqrt{5}$, then $\dfrac{1}{5\beta_n^2\gamma_n^{(\pm)}} \geq \dfrac{\alpha^{-2n}}{\sqrt{5}}$, and we have

$$\begin{aligned}
|\alpha^{2n}5^{-\frac{1}{2}}\beta_n^{-2}(\gamma_n^{(\pm)})^{-1} - 1| &= \frac{1}{\beta_n^2\gamma_n^{(\pm)}}\left|\frac{\alpha^{2n}}{\sqrt{5}} - \beta_n^2\gamma_n^{(\pm)}\right| \\
&< \frac{\beta_n^{-2}\gamma_n^{(\mp)} + \dfrac{\alpha^{-2n}}{\sqrt{5}}}{\beta_n^2\gamma_n^{(\pm)}} < \frac{\gamma_n^{(\mp)} + \dfrac{1}{5\gamma_n^{(\pm)}}}{\beta_n^4\gamma_n^{(\pm)}} \\
&< \frac{8.46635831 + \dfrac{\alpha^{4n+12}}{5(0.00694321)}}{0.00694321\beta_n^{4n}} \\
&< \frac{1219.4 + 4148.7\alpha^{4n+12}}{\beta_n^4} \\
&\leq \frac{\dfrac{1219.4}{\beta_n^2} + \dfrac{4148.7\alpha^{4n+12}}{\beta_n^2}}{\beta_n^2} < 4149\beta_n^{-2}.
\end{aligned}$$

If $\beta_n^2\gamma_n^{(\pm)} > \dfrac{\alpha^{2n}}{\sqrt{5}}$, then

$$\begin{aligned}
|\alpha^{2n}5^{-\frac{1}{2}}\beta_n^{-2}(\gamma_n^{(\pm)})^{-1} - 1| &= \frac{1}{\beta_n^2\gamma_n^{(\pm)}}(\beta_n^2\gamma_n^{(\pm)} - \alpha^{2n}5^{-\frac{1}{2}}) \\
&< \frac{1}{\beta_n^2\gamma_n^{(\pm)}}\frac{2(F_{2n+12} + 16F_{2n+6})}{F_{2n+9}^2 - 4} = \frac{\left(\dfrac{2}{F_{2n+6}} + \dfrac{32}{F_{2n+12}}\right)}{\beta_n^2\gamma_n^{(\pm)}} \\
&< \frac{\left(\dfrac{2}{F_{2n+6}} + \dfrac{32}{F_{2n+12}}\right)}{\beta_n^2(0.00694321)\alpha^{-2n-6}} = \frac{\dfrac{2\alpha^{2n+6}}{F_{2n+6}} + \dfrac{32\alpha^{2n+6}}{F_{2n+12}}}{\beta_n^2(0.00694321)} \\
&< \frac{2\alpha^2 + \dfrac{32}{\alpha^4}}{\beta_n^2(0.00694321)} < \frac{9.91}{\beta_n^2(0.00694321)} < \frac{1428}{\beta_n^2}.
\end{aligned}$$

So $\Lambda_0 \leq 2\log\beta_1 - 2\log\alpha + \log(\sqrt{5}\gamma_1^{(+)}) < \dfrac{270830}{\beta_n^2}$. In all cases we have

$$|\Lambda_0| < \frac{270830}{\beta_n^2}.$$

□

Lemma 5.5. *Let*

$$\Lambda_1 := K\log\alpha - (j-1)\log\left(\frac{5}{4}\right),\ \textit{where}\ K := (2j-1)(2n+9)-k-9.$$

Then we have

$$|\Lambda_1| < \frac{10(j+5493)}{\alpha^{4n+14}}.$$

Proof. We know

$$\begin{aligned}\beta_n = F_{2n+9} + \sqrt{F_{2n+9}^2 - 4} &= 2F_{2n+9} - \frac{4}{F_{2n+9}+\sqrt{F_{2n+9}^2-4}} \\ &= 2F_{2n+9}\left(1 - \frac{4}{2F_{2n+9}(F_{2n+9}+\sqrt{F_{2n+9}^2-4})}\right)\end{aligned}$$

and

$$2F_{2n+9} = \frac{2}{\sqrt{5}}(\alpha^{2n+9} - \overline{\alpha}^{2n+9}) = \frac{2}{\sqrt{5}}\alpha^{2n+9}\left(1 + \frac{1}{\alpha^{4n+18}}\right).$$

Define

$$\delta_n := \left(1 - \frac{4}{2F_{2n+9}(F_{2n+9}+\sqrt{F_{2n+9}^2-4})}\right)\left(1+\frac{1}{\alpha^{4n+18}}\right).$$

Then $\log(\beta_n) = \log\left(\frac{2}{\sqrt{5}}\right) + (2n+9)\log\alpha + \log\delta_n$.

$$\begin{aligned}\Lambda - \Lambda_0 =& (2j-2)\log(\beta_n) - (k-2n)\log(\alpha) \\ =& (2j-2)\log\left(\frac{2}{\sqrt{5}}\right) + (2j-2)(2n+9)\log(\alpha) \\ &+ (2j-2)\log(\delta_n) - (k-2n)\log(\alpha) \\ =& (2j-2)\log(\delta_n) + K\log(\alpha) - (j-1)\log\left(\frac{5}{4}\right).\end{aligned}$$

So $\Lambda_1 = \Lambda - \Lambda_0 - (2j-2)\log(\delta_n)$. By the triangle inequality we have

$$|\log(\delta_n)| \le \left|\log\left(1 - \frac{4}{2F_{2n+9}(F_{2n+9}+\sqrt{F_{2n+9}^2-4})}\right)\right| + \left|\log\left(1+\frac{1}{\alpha^{4n+18}}\right)\right|,$$

so

$$|\log(\delta_n)| < \frac{4}{F_{2n+9}(F_{2n+9}+\sqrt{F_{2n+9}^2-4})} + \frac{1}{\alpha^{4n+18}}$$

$$< \frac{4}{\alpha^{4n+14}} + \frac{1}{\alpha^{4n+18}} < \frac{5}{\alpha^{4n+14}},$$

and

$$|\Lambda_1| \leq |\Lambda| + |\Lambda_0| + |2j-2||\log \delta_n| < \frac{3868}{\beta_n^2} + \frac{270830}{\beta_n^2} + \frac{10(j-1)}{\alpha^{4n+14}}.$$

Clearly

$$\beta_n = F_{2n+9}\left(1+\sqrt{1-\frac{4}{F_{2n+9}^2}}\right) \geq F_{2n+9}\left(1+\frac{89^2-4}{89^2}\right) > \frac{\alpha^{2n+9}}{\sqrt{5}}\left(1+\sqrt{\frac{7917}{89^2}}\right).$$

Thus

$$\beta_n^2 > \alpha^{4n+18}\left(1+\frac{\sqrt{7917}}{89}\right)^2 > \frac{3}{4}\alpha^{4n+18} > 5\alpha^{4n+14},$$

and we have $|\Lambda_1| < \dfrac{274698}{\beta_n^2} + \dfrac{10(j-1)}{\alpha^{4n+14}} < \dfrac{10(j+5493)}{\alpha^{4n+14}}$. □

5.4 Refining Our Bounds (2)

Lemma 5.6. *If* (14) *has a positive integer solution* (j,k) *with* $j>1$ *then*

$$j < 6.25 \times 10^{21} \text{ and } n < 37024766.$$

Proof. We apply Lemma 2.1. We have

$$D = 2, \quad \gamma_1 = \frac{5}{4}, \quad \gamma_2 = \alpha, \quad b_1 = (j-1), \quad b_2 = K.$$

We take $h_1 = \log(5)$, and $h_2 = \frac{1}{2}$. We have $K\log(\alpha) - (j-1)\log(\frac{5}{4}) < \frac{10(j+5493)}{\alpha^{4n+14}}$, so

$$K < \frac{(j-1)\log\left(\frac{5}{4}\right) + \frac{10(j+5493)}{\alpha^{4n+14}}}{\log(\alpha)} < 19.3 + 0.47j.$$

Thus we take

$$\frac{|b_1|}{Dh_2} + \frac{|b_2|}{Dh_1} = (j-1) + \frac{K}{2log(5)} \leq 4.996 + 1.15j.$$

We let $b' = 5 + 1.15j$. Hence

$$\log|\Lambda_1| > (-17.9)\cdot 16 \cdot (\max\{\log(5+1.15j)+0.38, 15\})^2 \cdot \log(5) \cdot \frac{1}{2}.$$

The previous Lemma gives: $\log|\Lambda_1| < -(4n+14)\log(\alpha) + \log(10(j+5493))$, so we have

$$(-17.9)\cdot 16\cdot (\max\{\log(5+1.15j)+0.38, 15\})^2\cdot \log(5)\cdot \frac{1}{2} < \log|\Lambda_1|$$

$$< -(4n+14)\log(\alpha) + \log(10(j+5493)).$$

Rearranging this we get

$$n < 120(\max\{log(5+1.15j)+0.38, 15\})^2 + 0.52\cdot \log(10(j+5493)).$$

If $\log(1.15j+5)+0.38 < 15$, then $j < 1943957$ and $n < 27009$. Otherwise

$$n < 120(\log(5+1.15j)+0.38)^2 + 0.52\log(10(j+5493)).$$

Combining this with $j < 1.16\times 10^{12}(2n+9)\log(156j(n+5))$ gives $j < 6.25\times 10^{21}$ and $n < 37024766$. □

Proposition 5.7. *If* (14) *has a positive integer solution* (j,k), *with* $j > 1$, *then*

$$n < 1.04\cdot \log j + 4.6.$$

Proof. We have that $|K\log(\alpha) - (j-1)\log\left(\frac{5}{4}\right)| < 10(j+5493)\alpha^{-(4n+14)}$. Hence

$$\left|\frac{\log(5/4)}{\log(\alpha)} - \frac{K}{(j-1)}\right| < \frac{10(j+5493)}{(j-1)\alpha^{4n+14}\log(\alpha)}.$$

First, assume

$$\frac{10(j+5493)}{(j-1)\alpha^{4n+14}\log(\alpha)} < \frac{1}{2(j-1)^2}. \tag{16}$$

Then

$$\left|\frac{\log(5/4)}{\log(\alpha)} - \frac{k}{(j-1)}\right| < \frac{1}{2(j-1)^2}.$$

The denominator of the 49th convergent of $\frac{\log(5/4)}{\log(\alpha)}$ is greater than 6.25×10^{21}, our upper bound of j. The 48th convergent gives the lower bound:

$$\left|\frac{\log(5/4)}{\log(\alpha)} - \frac{K}{j-1}\right| > 4\times 10^{-44}.$$

So

$$4 \times 10^{-44} < \frac{10(j+5493)}{(j-1)\alpha^{4n+14}\log(\alpha)} < 5513 \cdot \alpha^{-4n-14}(\log(\alpha))^{-1}.$$

This gives $\alpha^{4n} < \dfrac{4 \cdot 10^{44} \cdot (5513)}{\alpha^{14}\log(\alpha)}$, so $n < \dfrac{\log \dfrac{4 \times 10^{44} \cdot (5513)}{\alpha^{14}\log(\alpha)}}{4 \cdot \log(\alpha)}$ gives us $n < 55$.

Since we know that $\dfrac{p_r}{q_r}$ is the rth convergent, $\left|\dfrac{\log(5/4)}{\log(\alpha)} - \dfrac{p_r}{q_r}\right| > \dfrac{1}{(a_{r+1}+2)q_r^2}$, and since $\max\{a_{r+1} : 2 \le r \le 47\} = a_{36} = 49$,

$$\frac{1}{51(j-1)^2} < \frac{10(j+5493)}{(j-1)\alpha^{4n+14}\log(\alpha)} \implies \alpha^{4n+14} < \frac{510(j-1)(j+5493)}{\log(\alpha)}.$$

If (16) does not hold, we have

$$\frac{1}{2(j-1)^2} \le \frac{10(j+5493)}{(j-1)\alpha^{4n+14}\log(\alpha)} \implies \alpha^{4n+14} \le \frac{20(j-1)(j+5493)}{\log(\alpha)}.$$

In both cases

$$\alpha^{4n+14} < \frac{510(j-1)(j+5493)}{\log(\alpha)} < 1060j(j+5493) < 5823640j^2,$$

so $n < 1.04 \cdot \log(j) + 4.6$. □

Combining the above proposition with 5.3 yields:

Lemma 5.8. *If* (14) *has a positive integer solution* (j,k) *with* $j > 1$, *then*

$$j < 4.88 \times 10^{15} \text{ and } n < 43.$$

5.5 Baker-Davenport Reduction

Using the same method as before, we have obtained our bounds on j and n. All that remains to prove Theorem 1.2 is to apply Baker-Davenport reduction to these bounds.

We have

$$0 < 2j\log\beta_n - k\log\alpha + \log(\sqrt{5}\cdot\gamma_n^{(\pm)}) < 3868\beta_n^{-2j}.$$

To apply Baker-Davenport reduction, consider

$$\kappa = \frac{2\log\beta_n}{\log\alpha}, \quad \mu = \frac{\log(\sqrt{5}\cdot\gamma_n^{(\pm)})}{\log\alpha}, \quad A = \frac{3868}{\log\alpha}, \quad M = 4.88\times 10^{15}$$

We again use procedures written in Maple™ to find $j \le 5$, which in turn gives us $1 \le n \le 5$ (see Appendix B).

Lemma 5.9. *If* (14) *has a positive integer solution* (j,k) *with* $j > 1$*, then*

$$j \le 5 \text{ and } n \le 5.$$

We can very quickly check all possible j between 2 and 5 and n from 1 to 5 in (15) to see that no solution exists. We already saw that $F_{14} < C_1^{(+)} < F_{15}$, so the only possible solution to (15) is $C_1^{(-)} = F_{2n}$. When $n = 1$, we have the solution $F_{2n} = F_2 = 1 = F_1$, so there is an additional solution in this case again.

6 Conclusion

The results in obtained here could certainly be more general. When writing it we did make an attempt to generalize the two main theorems into a result on all Diophantine triples of the form $\{F_{2n+4m}, L_m^2F_{2n+2m}, F_k\}$, and while at this point I don't have any reason to believe that it can't be accomplished, it won't be quite as straightforward as what was done here. For one thing, in the lemma on Pellian equations in section 3, one condition was that $l^2 \mid a$ or $l^2 \mid b$. In order to use this in the case of these general triples, it would need to be shown that $l^2 = F_{2m}^2$ always divides one (or both) of F_{2n+4m} or $L_m^2F_{2n+2m}$. In the $D(64)$ case covered in this paper we were already restricting ourselves to n divisible by 3. For general m, it quickly becomes even more restrictive. In the case of $m = 4$, $F_{2m} = 21$ and $L_m^2 = 49$. For $n \equiv 2 \mod 6$, we have $21^2 \mid L_4^2F_{2n+8}$, but otherwise 9 will not divide $49F_{2n+8}$. A more general version of the lemma on Pellian equations could help expand the results in this paper.

One additional potential problem occurs when applying the lemma of Mateev. In order to show that $\alpha_1, \alpha_2, \alpha_3$ are multiplicatively independent, it must be shown that $F_{2n+2m}F_{2n+4m}$ is neither a square nor 5 times a square. For $m = 2, 3$, this was proven using a theorem of Nagell on Pellian equations (see section 2.4). However, the Pellian equations generated for larger m are not guaranteed to have a single class of solutions. A program written quickly in Maple™ revealed that for $m = 4$ there are in fact no classes of solutions, which makes things easy, and for $m < 15$, there is never more than one class. However when $m = 15$, there are 4 classes of solutions to the relevant Pellian equation. After that the (rudimentary) program became cumbersomely slow. As a result, it seems uncertain whether the proofs of the lemmas in section 2.4 are generalizable to any integer m. A more light-handed approach may be needed.

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A D(9) Baker-Davenport Reduction Calculations

```
> with(combinat) :
> F := n→fibonacci(n) :
> gammaplus := n→ 3/(2·sqrt(F(2·n+4)·F(2·n+8))) + (F(2·n+8) + 9·F(2·n+4))/(4·F(2·n+4)·F(2·n+8)) :
> gammaminus := n→ -3/(2·sqrt(F(2·n+4)·F(2·n+8)))
      + (F(2·n+8) + 9·F(2·n+4))/(4·F(2·n+4)·F(2·n+8)) :
> betan := n→F(2·n+6) + sqrt((F(2·n+6))^2 − 1) :
> a := (1 + sqrt(5))/2 :
> kappan := n→ (2·log(betan(n)))/log(a) :
> muminus := n→ ( log(sqrt(5)·gammaminus(n))/log(a) ) :
> muplus := n→ ( log(sqrt(5)·gammaplus(n))/log(a) ) :
> A := 1162/log(a) :
> Bn := n→(betan(n))^2 :
> M := 4.63·10^15 :
> with(numtheory) :
> with(NumberTheory) :
> cfkapn := n→ContinuedFraction(kappan(n)) :
> fc := proc(n, t)
  local m, i, Q;
  Q := 0;
  i := 0;
  m := cfkapn(n);
  while Q ≤ t do
   i := i + 1;
  Q := Denominator(m, i);
  end do:
  return(Q, i)
  end proc:
> fc(1, 6·M)
                         44045204789851415, 29                         (1)
> etan := proc(n)
  local m, i, t, x, y, j;
  j := 6·M;
  t := -1;
  y := 0;
  x := -1;
  while (evalf(t − y) ≤ 0) or (evalf(x − y) ≤ 0) do
```

```
t := abs(frac(muplus(n)·fc(n,j)[1]) − round(frac(muplus(n)·fc(n,j)[1])));
x := abs(frac(muminus(n)·fc(n,j)[1]) − round(frac(muminus(n)·fc(n,j)[1])));
y := M·(abs(frac(kappan(n)·fc(n,j)[1]) − round(frac(kappan(n)·fc(n,j)[1]))));
j := fc(n,j)[1]
end do:
return(evalf(t − y), evalf(x − y), j)
end proc:
> etan(42)
```

0.0086482577176300636077265821671585594635487205336798334, **(2)**

$3.1305957542690319818574708133665675385746628851 \cdot 10^{-10}$,

1294983495867130172499296 79

```
> bound := proc(n)
local x, y;
```

$$x := \frac{\log\left(\frac{A \cdot etan(n)[3]}{etan(n)[1]}\right)}{\log(Bn(n))};$$

$$y := \frac{\log\left(\frac{A \cdot etan(n)[3]}{etan(n)[2]}\right)}{\log(Bn(n))};$$

```
return(floor(evalf(x)), floor(evalf(y)))
end proc:
> bound(40)
```

0, 1 **(3)**

```
> listn := proc(n)
local i;
for i from 1 to n do
print(bound(i))
end do:
end proc:
> listn(42)
```

6, 6

5, 5

4, 4

3, 3

3, 3

2, 2

2, 2

2, 2

2, 2

1, 1

1, 1

1, 1

1, 1
1, 1
1, 1
1, 1
1, 1
1, 1
1, 1
1, 1
1, 1
1, 1
0, 1
0, 1
1, 1
0, 1
0, 1
0, 1
0, 1
0, 1
0, 1
0, 1
0, 1
0, 1
0, 1
0, 1
0, 1
0, 1
0, 1
0, 1
0, 1
0, 1 **(4)**

>

>

B D(64) Baker-Davenport Reduction Calculations

```
> with(combinat) :
> F := n→fibonacci(n) :
> gammaplus := n→ 8/(sqrt(F(2·n + 6)·F(2·n + 12)))
      + ((F(2·n + 12) + 16·F(2·n + 6)))/(F(2·n + 6)·F(2·n + 12)) :
> gammaminus := n→ -8/(sqrt(F(2·n + 6)·F(2·n + 12)))
      + ((F(2·n + 12) + 16·F(2·n + 6)))/(F(2·n + 6)·F(2·n + 12)) :
> betan := n→ ((F(2·n + 9) + sqrt((F(2·n + 9))^2 − 4)))/2 :
> a := ((1 + sqrt(5)))/2 :
> kappan := n→ ((2·log(betan(n))))/(log(a)) :
> muminus := n→ ((log(sqrt(5)·gammaminus(n)))/(log(a))) :
> muplus := n→ ((log(sqrt(5)·gammaplus(n)))/(log(a))) :
> A := 3868/(log(a)) :
> Bn := n→(betan(n))^2 :
> M := 4.88·10^15 :
> with(numtheory) :
> with(NumberTheory) :
> cfkapn := n→ContinuedFraction(kappan(n)) :
> fc := proc(n, t)
  local m, i, Q;
  Q := 0;
  i := 0;
  m := cfkapn(n);
  while Q ≤ t do
   i := i + 1;
  Q := Denominator(m, i);
  end do:
  return(Q, i)
  end proc:
> fc(1, 6·M)
                    302835779297689803, 33                    (1)
> etan := proc(n)
  local m, i, t, x, y, j;
  j := 6·M;
```

```
t := -1;
y := 0;
x := -1;
while (evalf(t - y) ≤ 0) or (evalf(x - y) ≤ 0) do
t := abs(frac(muplus(n)·fc(n, j)[1]) - round(frac(muplus(n)·fc(n, j)[1])));
x := abs(frac(muminus(n)·fc(n, j)[1]) - round(frac(muminus(n)·fc(n, j)[1])));
y := M·(abs(frac(kappan(n)·fc(n, j)[1]) - round(frac(kappan(n)·fc(n, j)[1]))));
j := fc(n, j)[1]
end do:
return(evalf(t - y), evalf(x - y), j)
end proc:
> etan(43)
```

0.41018902218821349577881091509320177001879931769702987913, **(2)**

3.93611804593920658506665318568192537527773274 10^{-12},

59518701483690965606765821

```
> bound := proc(n)
local x, y;
```

$$x := \frac{\log\left(\frac{A\cdot etan(n)[3]}{etan(n)[1]}\right)}{\log(Bn(n))};$$

$$y := \frac{\log\left(\frac{A\cdot etan(n)[3]}{etan(n)[2]}\right)}{\log(Bn(n))};$$

```
return(floor(evalf(x)), floor(evalf(y)))
end proc:
> bound(43)
```

0, 1 **(3)**

```
> listn := proc(n)
local i;
for i from 1 to n do
print(bound(i))
end do:
end proc:
> listn(43)
```

5, 5

4, 4

4, 4

3, 3

3, 2

2, 2

2, 2

2, 2

2, 1

1, 1
1, 1
1, 1
1, 1
1, 1
1, 1
1, 1
1, 1
1, 1
1, 1
1, 1
1, 1
1, 1
1, 1
1, 1
0, 1
0, 1
0, 1
0, 1
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0, 1
0, 1
0, 1
0, 1
0, 1
0, 1
0, 1
0, 1
0, 1
0, 1
0, 1
0, 1
0, 1 **(4)**

>