

Response Curves of Deterministic and Probabilistic
Cellular Automata in One and Two Dimensions

Andrew Skelton

Department of Mathematics and Statistics

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Faculty of Mathematics and Science,
Brock University,
St. Catharines, Ontario

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Abstract

One of the most important problems in the theory of cellular automata (CA) is determining the proportion of cells in a specific state after a given number of time iterations. We approach this problem using patterns in preimage sets - that is, the set of blocks which iterate to the desired output. This allows us to construct a response curve - a relationship between the proportion of cells in state 1 after n -iterations as a function of the initial proportion. We derive response curve formulae for many two-dimensional deterministic CA rules with L-neighbourhood. For all remaining rules, we find experimental response curves. We also use preimage sets to classify surjective rules. In the last part of the thesis, we consider a special class of one-dimensional probabilistic CA rules. We find response surface formula for these rules and experimental response surfaces for all remaining rules.

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Introduction

Cellular automata (CA) have been used to model real-life phenomena as varied as disease spread [2, 3], forest fires [7, 21], reaction-diffusion systems, traffic flow patterns [17], and many others [22]. One of the most important problems considered in the theory of CA is the so-called forward problem. Given the initial state, determine the state of the CA after n -iterations.

In many models, it is often sufficient to determine some aggregate property of the final state instead of the state itself. For example, the proportion, or density, of cells in a given state is one of such aggregate properties. For disease spread models, this may be the proportion of the population that is infected. For forest fire models, this may be the proportion of trees on fire. In particular, we are interested in how iteration of the CA rule affects this proportion.

The questions we ask then are, given a randomly generated initial binary configuration, how do n -iterations of a given rule affect the proportion of cells in state 1? What is the dependence of this density on the initial density? What is the limiting density? Using signal processing terminology, we want to know the “response curve”, the density of the output as a function of the density of the input.

An important example of the use of response curves in CA theory is the Density Classification Problem (DCP). We say that the initial density of ones is given by ρ and the asymptotic density of ones, that is, the proportion of ones after a large number of rule iterations, is given by $P(1)$. Is it possible to determine a rule such that $P(1) = 0$ if $\rho < 1/2$ and $P(1) = 1$ if $\rho > 1/2$? In this case, the response curve would take the shape of a step function. It is known that a single such rule does not exist [25], but important results [11, 13, 14] have shown this to be possible if one considers the composition of multiple rules, or probabilistic rules.

Many studies of the forward problem have focused on ergodic theory or symbolic dynamics. We approach this problem from another direction. We find patterns on the sets of preimages of a given block and use the structure and cardinalities of such sets to derive response curve formulae. We ask, given a rule, can we determine its response curve? In surprisingly many cases, the response curves can be calculated explicitly. Two classes of CA rules will be considered in this thesis. First we consider a class of two-dimensional rules with so-called L-neighbourhood. Then we will focus

our attention on a class of one-dimensional probabilistic rules for which we were able to derive explicit response surfaces. For all of these rules, we also perform simulations to obtain experimental response curves and surfaces and we note interesting and novel behaviour.

In the course of this research, we also came across the question of surjectivity when considering rules whose preimage sets are balanced. In the theory of cellular automata (CA), the surjectivity of the global map is one of the most extensively studied properties of CA. It is only natural to ask, therefore, for examples of surjective CA.

In the case of one-dimensional CA, such examples are easy to find using the well-known Amoroso-Patt algorithm for determining surjectivity [1]. Using this algorithm it can be shown that among the 88 minimal elementary CA rules, the only surjective rules have Wolfram code numbers 15, 30, 45, 51, 60, 90, 105, 106, 150, 154, 170 and 204.

In two dimensions, however, the situation is much different. It has been shown that the question of surjectivity of two-dimensional cellular automata is undecidable [23]. This, however, is not the only problem. Even if we had an efficient algorithm to determine if a rule is surjective or not, producing a list of all surjective CA using a Moore neighbourhood of 9 sites by applying the algorithm to each rule would still be impossible, because there are $2^{(2^9)} = 2^{512} \approx 1.3 \times 10^{134}$ possible rules to be considered. Nevertheless, in this thesis we show that if one considers binary rules with the smallest “truly” two-dimensional neighbourhood, consisting of the central site, its top neighbour, and its right neighbour, then the exhaustive list of surjective rules can be determined. This can be done by a direct computerized search for violation of balance conditions to eliminate non-surjective rules, and then proving that the remaining rules are surjective. Using this method, we found that among the 88 minimal L-shaped 2D binary rules, 16 rules are surjective. A similar method was applied to rules with a von Neumann neighbourhood, where we were able to determine surjectivity for all rules with the exception of four “hard” cases.

Definitions and Basic Properties

Let $\mathcal{G} = \{0, 1, \dots, N-1\}$, $N \in \mathbb{N}^*$, be called a *symbol set* and let d be the *dimension* of the cellular automata. We denote $\mathcal{S}_d = \mathcal{G}^{\mathbb{Z}^d}$ be the set of all d -dimensional infinite lattices, or *configurations*, over \mathcal{G} . We will first consider one-dimensional cellular automata ($d = 1$), and then two-dimensional cellular automata ($d = 2$).

2.1 One-dimensional Cellular Automata

A *block of length n* is an ordered set $b_0 b_1 \dots b_{n-1}$, where $n \in \mathbb{N}$, $b_i \in \mathcal{G}$. We let \mathcal{B}_n denote the set of all blocks of length n over \mathcal{G} and \mathcal{B} be the set of all blocks (of any finite length) over \mathcal{G} .

If we consider a *neighbourhood* $\mathcal{N} = \mathcal{B}_{2r+1}$, for $r \in \mathbb{N}$, a mapping $f : \mathcal{N} \mapsto \mathcal{B}_1 = \mathcal{G}$ will be called a *cellular automaton rule of radius r* . Corresponding to f (also called a *local mapping*) we define a *global mapping* $F : \mathcal{S}_1 \rightarrow \mathcal{S}_1$ such that $(F(s))_i = f(s_{i-r}, \dots, s_i, \dots, s_{i+r})$ for any $s \in \mathcal{S}_1$.

A *block evolution operator* corresponding to f is a mapping $\mathbf{f} : \mathcal{B}_n \mapsto \mathcal{B}_{n-2r}$ defined as follows. Let $r \in \mathbb{N}$ be the radius of f , and let $b = b_0 b_1 \dots b_{n-1} \in \mathcal{B}_n$ where $n \geq 2r+1 > 0$. Then the mapping \mathbf{f} is defined in terms of the concatenation $\mathbf{f}(b) = \{f(b_i, b_{i+1}, \dots, b_{i+2r})\}_{i=0}^{n-2r-1}$. Note that if $b \in \mathcal{B}_{2r+1}$ then $\mathbf{f}(b) = f(b)$.

When we consider the case $\mathcal{G} = \{0, 1\}$ and $r = 1$ rules, we denote these cellular automata to be *elementary cellular automata* (or *elementary rules*). In this case, when $b \in \mathcal{B}_3$, then $\mathbf{f}(b) = f(b)$. We denote as *basic blocks*, the set $\mathcal{B}_3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$.

The *composition of two one-dimensional rules* f, f' can be now defined in terms of their corresponding global mappings F and F' as $(F \circ F')(s) = F(F'(s))$, where $s \in \mathcal{S}_1$. We will often examine the effect of composing a rule with itself n -times, which we denote $F^n(s)$ or $\mathbf{f}^n(b)$, where appropriate.

We denote by $\mathbf{f}^{-1}(b)$, the set of all *n -step preimages* of block b under the rule f , that is, all blocks c such that $\mathbf{f}^n(c) = b$. The number of n -step preimages of a block b under f , denoted $\text{card}[\mathbf{f}^{-n}(b)]$, is defined as the number of elements of the set $\mathbf{f}^{-n}(b)$.

We will often require the use of Kolmogorov consistency conditions [8] to simplify

our analysis. For any block $b = b_0 b_1 \dots b_k$, the probability of occurrence of block b is given in terms of the occurrence probabilities (see Section 3.1) of block b right-augmented (or left-augmented) with each possible element of \mathcal{G} so that

$$P(b_0 \dots b_k) = P(b_0 \dots b_k 0) + P(b_0 \dots b_k 1) + \dots + P(b_0 \dots b_k (N-1)). \quad (2.1)$$

For example, when $k = 2$ and $\mathcal{G} = \{0, 1\}$ (i.e. $N = 2$), (2.1) allows us to obtain the following system of four consistency conditions

$$\begin{aligned} P(000) + P(001) &= P(00) = P(000) + P(100), \\ P(010) + P(011) &= P(01) = P(001) + P(101), \\ P(100) + P(101) &= P(10) = P(010) + P(110), \\ P(110) + P(111) &= P(11) = P(011) + P(111). \end{aligned} \quad (2.2)$$

From system (2.2), we obtain the following identities

$$\begin{aligned} P(001) &= P(100), \\ P(011) &= P(110), \\ P(010) + P(011) &= P(001) + P(101). \end{aligned} \quad (2.3)$$

We will often make use of these conditions (and others derived from (2.1) for larger values of k) in future analysis.

For one-dimensional cellular automata the following number scheme is used [31]. An elementary rule f is assigned a *Wolfram number* W , between 0 and 255, as follows

$$W(f) = \sum_{a_0, a_1, a_2 \in \{0,1\}} f(a_0 a_1 a_2) 2^{4a_0 + 2a_1 + a_2}. \quad (2.4)$$

2.2 Two-dimensional Cellular Automata

The above definitions can be easily generalized to the case when $d = 2$. There are many choices for a local neighbourhood that can be used for two-dimensional cellular automata. We will focus our attention on two particular neighbourhoods: the L-neighbourhood and the von Neumann neighbourhood.

2.2.1 L-neighbourhood

We first consider the smallest truly two-dimensional neighbourhood, in which the local function takes as an argument the previous time step value of the current site,

the site above and to the right of the current site. Blocks in two dimensions will be defined as regions of 2D lattice in the shape of isosceles right triangles. We define the set of triangular blocks of size r as the set consisting of elements

$$\begin{array}{c} b_{1,r} \\ \vdots \\ b_{1,1} \dots b_{r,1}, \end{array} \quad (2.5)$$

where each $b_{i,j} \in \mathcal{G}$. This set will be denoted by \mathcal{T}_r . The set of blocks \mathcal{T}_2 will be denoted *basic blocks*.

If we define an L -neighbourhood as a block $b = \begin{smallmatrix} b_0 \\ b_1 & b_2 \end{smallmatrix} \in \mathcal{T}_2$, then we may define the local mapping of a *two-dimensional cellular automaton with L -shaped neighbourhood* as $g : \mathcal{T}_2 \rightarrow \mathcal{G}$. Similarly as in one dimension, g has a corresponding global mapping, $G : \mathcal{G}^{\mathbb{Z}^2} \rightarrow \mathcal{G}^{\mathbb{Z}^2}$ such that, for any $i, j \in \mathbb{Z}$,

$$(G(s))_{i,j} = g \left(\begin{smallmatrix} s_{i,j+1} \\ s_{i,j} & s_{i+1,j} \end{smallmatrix} \right),$$

for any $s \in \mathcal{G}^{\mathbb{Z}^2}$. The block evolution operator $\mathbf{g} : \mathcal{T}_r \rightarrow \mathcal{T}_{r-1}$ will be defined as a function which transforms triangular block (2.5) into another block

$$\begin{array}{c} c_{1,r-1} \\ \vdots \\ c_{1,1} \dots c_{r-1,1}, \end{array}$$

where $c_{i,j} = g \left(\begin{smallmatrix} b_{i,j+1} \\ b_{i,j} & b_{i+1,j} \end{smallmatrix} \right)$ for $i \in \{1, \dots, r-1\}$, $j \in \{1, \dots, r-i\}$.

When we wish to refer to a block from \mathcal{T}_n contained in an infinite configuration $s \in \mathcal{G}^{\mathbb{Z}^2}$ and placed at $(0,0)$, we will use the following notation:

$$s[n] = \begin{array}{ccc} & s_{i,j+n-1} & \\ & \vdots & \ddots \\ & s_{i,j} & \dots s_{i+n-1,j} \end{array}.$$

If we consider the case when $\mathcal{G} = \{0,1\}$ with a local mapping $f \left(\begin{smallmatrix} a_0 \\ a_1 & a_2 \end{smallmatrix} \right) \rightarrow \mathcal{G}$, then many of the 256 possible rules are related to each other by the group of 4 transformations $D_1 \times S_2$, where D_1 is the dihedral group with a single reflectional symmetry and S_2 denotes all permutations of the elements in $\{0,1\}$. The class of

four equivalent rules is written in terms of local mappings as

$$\begin{aligned} g \left(\begin{smallmatrix} a_0 \\ a_1 \end{smallmatrix} a_2 \right) &= g \left(\begin{smallmatrix} a_0 \\ a_1 \end{smallmatrix} a_2 \right), & \bar{g} \left(\begin{smallmatrix} a_0 \\ a_1 \end{smallmatrix} a_2 \right) &= 1 - g \left(\begin{smallmatrix} 1-a_0 \\ 1-a_1 \end{smallmatrix} 1-a_2 \right), \\ g_{ref} \left(\begin{smallmatrix} a_0 \\ a_1 \end{smallmatrix} a_2 \right) &= g \left(\begin{smallmatrix} a_2 \\ a_1 \end{smallmatrix} a_0 \right), & \bar{g}_{ref} \left(\begin{smallmatrix} a_0 \\ a_1 \end{smallmatrix} a_2 \right) &= 1 - g \left(\begin{smallmatrix} 1-a_2 \\ 1-a_1 \end{smallmatrix} 1-a_0 \right). \end{aligned}$$

For two-dimensional elementary cellular automata with L-neighbourhood, we adapt the numbering system used in [31]. An elementary rule g is assigned a *Wolfram number* W , between 0 and 255, as follows

$$W(g) = \sum_{a_0, a_1, a_2 \in \{0,1\}} g \left(\begin{smallmatrix} a_0 \\ a_1 \end{smallmatrix} a_2 \right) 2^{4a_0+2a_1+a_2}. \quad (2.6)$$

Since each elementary rule is a member of a group of (not necessarily distinct) equivalent rules, it is convenient to consider only a *minimal rule* for each group whose Wolfram number is given by

$$W_{min}(g) = \min \{ W(g), W(\bar{g}), W(g_{ref}), W(\bar{g}_{ref}) \}. \quad (2.7)$$

A comprehensive list of all 88 elementary rules, their equivalencies and minimal representatives can be found in [31].

Again, we will use Kolmogorov consistency conditions, adapted from (2.1) to the L-neighbourhood, to simplify our analysis. In the case when $r = 2$ (2.5) and $\mathcal{G} = \{0,1\}$, we obtain the following consistency conditions

$$\begin{aligned} P(0) &= P \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} 0 \right) + P \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} 1 \right) + P \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} 0 \right) + P \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} 1 \right), \\ &= P \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} 0 \right) + P \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} 1 \right) + P \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} 0 \right) + P \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} 1 \right), \\ &= P \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} 0 \right) + P \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} 0 \right) + P \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} 0 \right) + P \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} 0 \right), \\ & \quad (2.8) \\ P(1) &= P \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} 0 \right) + P \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} 1 \right) + P \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} 0 \right) + P \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} 1 \right), \\ &= P \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} 0 \right) + P \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} 1 \right) + P \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} 0 \right) + P \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} 1 \right), \\ &= P \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} 1 \right) + P \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} 1 \right) + P \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} 1 \right) + P \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} 1 \right). \end{aligned}$$

Equating, when appropriate, each pair of conditions in system (2.8) gives us six identities that we may use in future analysis. We note in passing that due the geometry of the neighbourhood, these identities contain more terms than those for one dimension. Thus, it is necessary to determine five basic block probabilities before the remainder may be obtained using consistency identities.

2.2.2 von Neumann neighbourhood

We now provide definitions for a commonly used two-dimensional neighbourhood, the von Neumann neighbourhood, in which the state of a cell at the next time step depends on the current value of the cell, and the current values of the cells above, below, to the left and right of the given cell.

Blocks in two dimensions will be defined as diamond-shaped regions of 2D lattice. We define the set of diamond blocks of size r as the set consisting of elements

$$\begin{array}{ccccc}
 & & b_{0,(r-1)} & & \\
 & \ddots & \vdots & \ddots & \\
 b_{-(r-1),0} & \dots & b_{0,0} & \dots & b_{(r-1),0}, \\
 & \ddots & \vdots & \ddots & \\
 & & b_{0,-(r-1)} & &
 \end{array} \quad (2.9)$$

where each $b_{i,j} \in \mathcal{G}$ (see Figure 4.2a). This set will be denoted by \mathcal{D}_r .

If we define an *von Neumann-neighbourhood* as a block $b = \begin{smallmatrix} b_0 \\ b_1 \ b_2 \ b_3 \\ b_4 \end{smallmatrix} \in \mathcal{D}_2$, then we may define the local mapping of a *two-dimensional cellular automaton with von Neumann neighbourhood* as $h : \mathcal{D}_2 \rightarrow \mathcal{G}$. As in one dimension, h has a corresponding global mapping, $H : \mathcal{G}^{\mathbb{Z}^2} \rightarrow \mathcal{G}^{\mathbb{Z}^2}$ such that, for any $i, j \in \mathbb{Z}$,

$$(H(s))_{i,j} = h \left(\begin{array}{ccc} & s_{i,j+1} & \\ s_{i-1,j} & s_{i,j} & s_{i+1,j} \\ & s_{i,j-1} & \end{array} \right),$$

for any $s \in \mathcal{G}^{\mathbb{Z}^2}$. The block evolution operator $\mathbf{h} : \mathcal{D}_r \rightarrow \mathcal{D}_{r-1}$ will be defined as a function which transforms diamond block (2.9) into another block

$$\begin{array}{ccccc}
 & & c_{0,(r-2)} & & \\
 & \ddots & \vdots & \ddots & \\
 c_{-(r-2),0} & \dots & c_{0,0} & \dots & c_{(r-2),0}, \\
 & \ddots & \vdots & \ddots & \\
 & & c_{0,-(r-2)} & &
 \end{array}$$

where $c_{i,j} = h \left(\begin{array}{ccc} & b_{i,j+1} & \\ b_{i-1,j} & b_{i,j} & b_{i+1,j} \\ & b_{i,j-1} & \end{array} \right)$ for all $i \in \{-(r-2), \dots, (r-2)\}$ and all $j \in \{-(r-2-|i|), \dots, (r-2-|i|)\}$.

If we consider the case when $\mathcal{G} = \{0,1\}$ with a von Neumann local mapping $g \left(\begin{array}{ccc} a_0 & & \\ a_1 & a_2 & a_3 \\ a_4 & & \end{array} \right) \rightarrow \mathcal{G}$, then many of the 2^{2^5} possible rules are related to each other by the group of 16 transformations $D_4 \times S_2$, where D_4 is the dihedral group consisting

of symmetries of a square and S_2 denotes all permutations of the elements in $\{0, 1\}$. The class of sixteen equivalent rules is written in terms of local mappings as

$$\begin{aligned}
g \left(\begin{smallmatrix} a_0 \\ a_1 \ a_2 \ a_3 \\ a_4 \end{smallmatrix} \right) &= g \left(\begin{smallmatrix} a_0 \\ a_1 \ a_2 \ a_3 \\ a_4 \end{smallmatrix} \right), & \bar{g} \left(\begin{smallmatrix} a_0 \\ a_1 \ a_2 \ a_3 \\ a_4 \end{smallmatrix} \right) &= 1 - g \left(\begin{smallmatrix} 1-a_0 \\ 1-a_1 \ 1-a_2 \ 1-a_3 \\ 1-a_4 \end{smallmatrix} \right), \\
g_{M_x} \left(\begin{smallmatrix} a_0 \\ a_1 \ a_2 \ a_3 \\ a_4 \end{smallmatrix} \right) &= g \left(\begin{smallmatrix} a_4 \\ a_1 \ a_2 \ a_3 \\ a_0 \end{smallmatrix} \right), & \bar{g}_{M_x} \left(\begin{smallmatrix} a_0 \\ a_1 \ a_2 \ a_3 \\ a_4 \end{smallmatrix} \right) &= 1 - g \left(\begin{smallmatrix} 1-a_4 \\ 1-a_1 \ 1-a_2 \ 1-a_3 \\ 1-a_0 \end{smallmatrix} \right), \\
g_{M_y} \left(\begin{smallmatrix} a_0 \\ a_1 \ a_2 \ a_3 \\ a_4 \end{smallmatrix} \right) &= g \left(\begin{smallmatrix} a_0 \\ a_3 \ a_2 \ a_1 \\ a_4 \end{smallmatrix} \right), & \bar{g}_{M_y} \left(\begin{smallmatrix} a_0 \\ a_1 \ a_2 \ a_3 \\ a_4 \end{smallmatrix} \right) &= 1 - g \left(\begin{smallmatrix} 1-a_0 \\ 1-a_3 \ 1-a_2 \ 1-a_1 \\ 1-a_4 \end{smallmatrix} \right), \\
g_{R_2} \left(\begin{smallmatrix} a_0 \\ a_1 \ a_2 \ a_3 \\ a_4 \end{smallmatrix} \right) &= g \left(\begin{smallmatrix} a_4 \\ a_3 \ a_2 \ a_1 \\ a_0 \end{smallmatrix} \right), & \bar{g}_{R_2} \left(\begin{smallmatrix} a_0 \\ a_1 \ a_2 \ a_3 \\ a_4 \end{smallmatrix} \right) &= 1 - g \left(\begin{smallmatrix} 1-a_4 \\ 1-a_3 \ 1-a_2 \ 1-a_1 \\ 1-a_0 \end{smallmatrix} \right), \\
g_{R_1} \left(\begin{smallmatrix} a_0 \\ a_1 \ a_2 \ a_3 \\ a_4 \end{smallmatrix} \right) &= g \left(\begin{smallmatrix} a_4 \\ a_1 \ a_2 \ a_0 \\ a_3 \end{smallmatrix} \right), & \bar{g}_{R_1} \left(\begin{smallmatrix} a_0 \\ a_1 \ a_2 \ a_3 \\ a_4 \end{smallmatrix} \right) &= 1 - g \left(\begin{smallmatrix} 1-a_4 \\ 1-a_1 \ 1-a_2 \ 1-a_0 \\ 1-a_3 \end{smallmatrix} \right), \\
g_{R_3} \left(\begin{smallmatrix} a_0 \\ a_1 \ a_2 \ a_3 \\ a_4 \end{smallmatrix} \right) &= g \left(\begin{smallmatrix} a_3 \\ a_0 \ a_2 \ a_4 \\ a_1 \end{smallmatrix} \right), & \bar{g}_{R_3} \left(\begin{smallmatrix} a_0 \\ a_1 \ a_2 \ a_3 \\ a_4 \end{smallmatrix} \right) &= 1 - g \left(\begin{smallmatrix} 1-a_3 \\ 1-a_0 \ 1-a_2 \ 1-a_4 \\ 1-a_1 \end{smallmatrix} \right), \\
g_{D_x} \left(\begin{smallmatrix} a_0 \\ a_1 \ a_2 \ a_3 \\ a_4 \end{smallmatrix} \right) &= g \left(\begin{smallmatrix} a_3 \\ a_4 \ a_2 \ a_0 \\ a_1 \end{smallmatrix} \right), & \bar{g}_{D_x} \left(\begin{smallmatrix} a_0 \\ a_1 \ a_2 \ a_3 \\ a_4 \end{smallmatrix} \right) &= 1 - g \left(\begin{smallmatrix} 1-a_3 \\ 1-a_4 \ 1-a_2 \ 1-a_0 \\ 1-a_1 \end{smallmatrix} \right), \\
g_{D_{-x}} \left(\begin{smallmatrix} a_0 \\ a_1 \ a_2 \ a_3 \\ a_4 \end{smallmatrix} \right) &= g \left(\begin{smallmatrix} a_1 \\ a_0 \ a_2 \ a_4 \\ a_3 \end{smallmatrix} \right), & \bar{g}_{D_{-x}} \left(\begin{smallmatrix} a_0 \\ a_1 \ a_2 \ a_3 \\ a_4 \end{smallmatrix} \right) &= 1 - g \left(\begin{smallmatrix} 1-a_1 \\ 1-a_0 \ 1-a_2 \ 1-a_4 \\ 1-a_3 \end{smallmatrix} \right).
\end{aligned}$$

where R_1, R_2, R_3 denote clockwise rotations of 90, 180 and 270 degrees, M_x, M_y denote reflections in the x and y-axes, D_x, D_{-x} denote reflections in the lines $y = x, y = -x$ and \bar{g} denoting a Boolean Conjugation of each of these rules.

For two-dimensional elementary cellular automata with von Neumann neighbourhood, we adapt the numbering system used in [31]. An elementary rule h is assigned a *Wolfram number* W , between 0 and $2^{32} - 1$, as follows

$$W(g) = \sum_{a_0, \dots, a_4 \in \{0,1\}} h \left(\begin{smallmatrix} a_0 \\ a_1 \ a_2 \ a_3 \\ a_4 \end{smallmatrix} \right) \cdot 2^{16a_0 + 8a_1 + 4a_2 + 2a_3 + a_4}. \quad (2.10)$$

Since each elementary rule is a member of a group of (not necessarily distinct) equivalent rules, it is convenient to consider only a *minimal rule* for each group whose Wolfram number is the smallest amongst the set of 16 equivalent rules. It can be shown that there are 270,754,432 minimal rules.

Distance between configurations

When we wish to quantify the distance between two-dimensional configurations, we use the following metric.

Proposition 2.1. *For $s, t \in \mathcal{G}^{\mathbb{Z}^2}$ and $i, j \in \mathbb{Z}$, the following satisfies all axioms of a metric:*

$$d(s, t) = \begin{cases} \frac{1}{1 + \min_{i, j \in \mathbb{Z}} (\max\{|i|, |j|\} : s_{i, j} \neq t_{i, j})} & \text{if } s \neq t \\ 0 & \text{if } s = t \end{cases}. \quad (2.11)$$

Proof. Among the four required properties of a metric, d obviously satisfies non-negativity, identity of indiscernibles, and symmetry. We will only prove the triangle inequality, i.e., that for any $s, t, r \in \mathcal{G}^{\mathbb{Z}^2}$ we have $d(s, t) \leq d(s, r) + d(r, t)$.

Assume that $d(s, t) = \frac{1}{k+1}$ and $d(s, r) = \frac{1}{k^*+1}$. We now consider two cases:

- If $k^* \leq k$, then $\frac{1}{k^*+1} \geq \frac{1}{k+1}$. From the definition of d , one concludes that in this case $d(r, t) = \frac{1}{k^*+1}$, and therefore

$$d(s, r) + d(r, t) = \frac{1}{k^*+1} + \frac{1}{k^*+1} \geq \frac{1}{k+1} = d(s, t).$$

- if $k^* > k$, then, again from the definition of d , $d(r, t) = \frac{1}{k+1}$, and thus

$$d(s, r) + d(r, t) = \frac{1}{k^*+1} + \frac{1}{k+1} > \frac{1}{k+1} = d(s, t).$$

□

Deterministic Cellular Automata

In this chapter ¹, we will focus our attention only on the “simplest” two-dimensional cellular automata, that is, cellular automata with L-neighbourhood. We attempt to find response curves for all 88 minimal binary rules. In 14 cases, the response curve formula was independent of the number of iterations n . We were able to find explicit response curves for each of these rules.

In 12 further cases, the response curve formula was dependent on the number of iterations. We found that in each of these cases, the density decreased until reaching some limiting asymptotic density. Many of these rules approached an asymptotic configuration consisting entirely of cells in state 0. We found that some rules, like Rule 138 presented below, approach some fixed density which is dependent on the initial density. The most interesting case was Rule 130 and we present a full and complete analysis of the derivation of the response curve in both the one and two-dimensional cases.

For other rules, were able to determine the density when the density of ones in the initial configuration was 0.5. This included the special case of surjective rules for which a more detailed analysis is presented in Chapter 4. We were also able to provide density conjectures for an additional set of rules.

In many cases, we were also able to find response curves for the eight basic blocks. Some of these formulae are very interesting, including the formulae for Rule 130 and those for Rule 168 which depend on Catalan numbers. We may use these basic block density formulae to prove results relating to rule emulation. In fact, we were able to determine that the set of 26 rules for which a full response curve could be derived are those which asymptotically emulate either the shift or identity rule. We also consider all rules which finitely emulate other rules and present all known relationships.

¹A version of this chapter (and portions of Chapter 4) has been accepted for publication. H. Fuks and A. Skelton. Response Curves and Preimage Sequences of Two-Dimensional Cellular Automata. *CA-CSC proc.*, 2011.

3.1 Response Curves

Suppose that we start with an initial configurations in which a certain proportion of sites is in state 1. The simplest way to achieve this is to set each site to be in state 1 with probability ρ , and 0 with probability $1 - \rho$, doing so independently for all sites. This means that the probability that a randomly selected site to be in state 1 is ρ . Suppose that we apply n -iterates of some CA rule to such configuration. What is the probability that in the resulting configuration, the state of a randomly selected site is 1? In order to formulate this problem more precisely, we will use the concept of probability measure, similarly as done in [16] for one-dimensional CA.

First we define the Bernoulli measure in two dimensions. If $C \subset \mathcal{S}_2$ is a subset consisting of all elements of \mathcal{S}_2 taking the same values on a finite subset E of \mathbb{Z}^2 , then we define $\mu_\rho(C) = \rho^j(1 - \rho)^{|E|-j}$, where j is the number of points in E at which C takes the value 1. Then one can show that μ_ρ can be extended in a unique way to all subsets of \mathcal{S}_2 from σ -algebra generated by all sets C so that

$$\mu_\rho(\emptyset) = 0,$$

$$\mu_\rho(\mathcal{S}_2) = 1,$$

$$C_1 \subset C_2 \implies \mu_\rho(C_1) \leq \mu_\rho(C_2), \text{ and}$$

$$\mu_\rho\left(\bigcup_k C_k\right) = \sum_k \mu_\rho(C_k),$$

for all countable collections $\{C_k\}$ of pairwise disjoint subsets of \mathcal{S}_2 . Such an extension will be called a *Bernoulli measure* μ_ρ and is obviously a probability measure. We will consider now a special set C associated with blocks in \mathcal{T}_r .

Given a block $b \in \mathcal{T}_r$, we define a *cylinder set given by b* , $C_{i,j}(b)$, as the set of all configurations in which block b is fixed and placed at coordinate (i, j) aligned at the lower-left element of b . We define a *measure* of such a cylinder set, $\mu[C_{i,j}(b)]$, to be the probability of occurrence of block b placed as above. Since one can show that μ_ρ defined above is translationally invariant we may drop the indices i, j . For $\rho \in [0, 1]$, the Bernoulli measure is a measure where all sites are independently set to 1 with probability ρ , and to 0 with probability $1 - \rho$. In such case, for $b \in \mathcal{T}_r$,

$$\mu_\rho[C(b)] = \rho^j(1 - \rho)^{(r^2+r)/2-j}, \quad (3.1)$$

where j is a number of cells in state 1 in b .

We now consider the action of the global mapping G on the measure of a cylinder set given by block b , which yields

$$(G\mu_\rho)[C(b)] = \mu_\rho[G^{-1}(C(b))]. \quad (3.2)$$

Considering instead n iterations of G , we obtain

$$(G^n\mu_\rho)[C(b)] = \mu_\rho[G^{-n}(C(b))]. \quad (3.3)$$

If we let $\mathbf{g}^{-n}(b)$ be the set of all n -step preimages of block b , that is, the set of all blocks a such that $\mathbf{g}^n(a) = b$, then we can write

$$\mu_\rho[G^{-n}(C(b))] = \sum_{a \in \mathbf{g}^{-n}(b)} \mu_\rho[C(a)]. \quad (3.4)$$

Using the notation $P_n(b) = \mu_\rho[G^{-n}(C(b))]$, we write (3.3) as

$$P_n(b) = \sum_{a \in \mathbf{g}^{-n}(b)} P_0(a). \quad (3.5)$$

If $b = 1$, and if the initial measure is Bernoulli, then in the above formula each $P_0(a)$ depends only on ρ , where $\rho = P_0(1)$. $P_n(1)$ can then be interpreted as the density of 1s in the configuration obtained by iterating the CA rule n -times starting from disordered initial configurations with density of ones equal to ρ .

The plot of $P_n(1)$ versus ρ will be called a *response curve* for each elementary 2D CA. Finally, we denote $P(b)$ to be the *asymptotic density of block b* , which we obtain by taking the limit of $P_n(b)$ as $n \rightarrow \infty$ (if the limit exists). Note that in order to use (3.5) we will need to know the elements of the set $\mathbf{g}^{-n}(b)$ for any n .

3.1.1 Theoretical Response Curves

For 26 minimal rules, we were able to determine an explicit response curve formula. In some cases, we found that the response curve was independent of n , beyond simple period-two behaviour. In other cases, the response curve was dependent on n , and then a separate formula for the asymptotic density could be obtained. We present in detail three examples of each type. In each example, we describe the structure of the preimage sets but, due to space constraints, we omit direct proofs while noting that each case can be proved easily by induction.

Rules with Non-Decaying Density

In each of the following examples, the formula for the asymptotic density is the same as the response curve. We provide detailed analysis for Rules 0, 3 and 42 and the remaining results are presented in Table 3.1.

Proposition 3.1. *The response curve for Rule 0 is $P_n(1) = 0$.*

Proof. No triangular blocks of any size can be mapped under \mathbf{g}_0^n to single block 1. Therefore, $\text{card}[\mathbf{g}_0^{-n}(b)] = 0$ and we apply (3.5) to obtain our result. \square

Proposition 3.2. *The response curve for Rule 42 is $P_n(1) = \rho(1 - \rho)(1 + \rho)$.*

Proof. It can be shown by induction that the only blocks that map to a single 1 under \mathbf{g}_{42}^n are either blocks in \mathcal{T}_n where $b_{n,1} = 1, b_{n-1,2} = 0$ and all other elements are arbitrary, or blocks in \mathcal{T}_n where $b_{n,1} = b_{n-1,2} = 1, b_{n-1,1} = 0$ and all other elements are arbitrary, so that

$$\begin{array}{c} \star \\ \vdots \\ \star \\ \star \cdots \star \star 1 \\ \hline n-1 \end{array} \quad \text{or} \quad \begin{array}{c} \star \\ \vdots \\ \star \\ \star \cdots \star 0 1 \\ \hline n-1 \end{array},$$

where \star denotes an element in an arbitrary state. Using (3.1) we determine the initial probability of occurrence of blocks of either type and using (3.5), conclude that $P_n(1) = \rho(1 - \rho) + \rho^2(1 - \rho)$, which simplifies to the desired result. An experimental curve confirming this result is presented in Figure A.3i. \square

Proposition 3.3. *The response curve for Rule 3 is*

$$P_n(1) = \begin{cases} (1 - \rho)^2 & \text{if } n \text{ even,} \\ \rho(1 + \rho - \rho^2) & \text{if } n \text{ odd.} \end{cases}$$

Proof. Since Rule 3 has period-2 behaviour, we must consider cases when n is odd and when n is even. When n is odd, the only blocks that map to a single 1 under n -iterations of \mathbf{g}_3 are blocks in \mathcal{T}_n where $b_{1,(n+1)/2} = b_{1,(n+3)/2} = 0$ and all other elements are arbitrary, so that

Rules	$P_n(1)$	$P_n^{(s)}(1)$
0	0	0
2	$\rho(1 - \rho)^2$	1/8
3	$(1 - \rho)^2$ $\rho(1 + \rho - \rho^2)$	5/8 (n odd) 1/4 (n even)
4	$\rho(1 - \rho)^2$	1/8
5	$(1 - \rho)^2$ $\rho(1 + \rho - \rho^2)$	5/8 (n odd) 1/4 (n even)
10	$\rho(1 - \rho)$	1/4
12	$\rho(1 - \rho)$	1/4
51	$1 - \rho$ (n odd) ρ (n even)	1/2
34	$\rho(1 - \rho)$	1/4
42	$\rho(1 - \rho)(1 + \rho)$	3/8
51	$1 - \rho$ (n odd) ρ (n even)	1/2
76	$\rho(1 - \rho)(1 + \rho)$	3/8
170	ρ	1/2
204	ρ	1/2

Table 3.1: Rules with Non-Decaying Density

entirely of ones. We use (3.1) to find the initial probability of this block and (3.5) produces our result. \square

We can now find the asymptotic density under Rule 128 as

$$P(1) = \lim_{n \rightarrow \infty} P_n(1) = \begin{cases} 0 & \text{if } \rho \neq 1, \\ 1 & \text{if } \rho = 1. \end{cases}$$

Proposition 3.5. *The response curve for Rule 32 is $P_n(1) = \rho^{n+1}(1 - \rho)^n$.*

Proof. Under rule 32, the only blocks that map to a single 1 under \mathbf{g}_{32}^n are of the form

$$n-1 \begin{bmatrix} 1 & & & & & & & & & & \\ 0 & \ddots & & & & & & & & & \\ \star & \ddots & \ddots & & & & & & & & \\ \vdots & \ddots & \ddots & \ddots & & & & & & & \\ \star & \cdots & \star & 0 & 1 & & & & & & \end{bmatrix}.$$

Using (3.1), we can determine the initial probability of occurrence of blocks of this type and using (3.5), we obtain our result. \square

We now find the asymptotic density as $P(1) = \lim_{n \rightarrow \infty} P_n(1) = 0$.

Proposition 3.6. *The response curve for Rule 138 is $P_n(1) = \frac{\rho^{2n+2} + \rho}{\rho + 1}$.*

Proof. The only blocks that map to a single 1 under \mathbf{g}_{138}^n are comprised entirely of arbitrary elements in the top $n - 1$ rows, and have their lower two rows of the form

$$\begin{array}{c} \star \cdots \star 0 1 \cdots 1 1 \\ \star \cdots \star \star \underbrace{1 \cdots 1 1 1}_i \end{array},$$

where i ranges from 1 to $n + 1$. Using (3.1) we can determine the initial probability of occurrence of blocks for each possible value of i . Summing over all i and using (3.5), we conclude that

$$\begin{aligned} P_n(1) &= \rho^{2n+1} + \sum_{i=1}^n \rho^{2i-1} (1 - \rho) \\ &= \rho^{2n+1} + \frac{1 - \rho}{\rho} \sum_{i=1}^n (\rho^2)^i \\ &= \rho^{2n+1} + \frac{1 - \rho}{\rho} \left(\frac{\rho^2 (\rho^{2n} - 1)}{\rho^2 - 1} \right), \end{aligned}$$

which simplifies to our desired result. \square

Again, we can find the asymptotic density (confirmed in Figure A.5j) as

$$P(1) = \lim_{n \rightarrow \infty} P_n(1) = \begin{cases} \frac{\rho}{1 + \rho} & \text{if } \rho \neq 1, \\ 1 & \text{if } \rho = 1. \end{cases}$$

While the response curve is continuous, the asymptotic density has a discontinuity at $\rho = 1$, which corresponds to an initial condition consisting entirely of ones.

3.1.2 Experimental Response Curves

For those rules for which an explicit response curve formula could not be derived, we were able to obtain experimental response curves. A random initial configuration of 250000 elements was iterated $n = 1000/\rho(1 - \rho)$ times when $\rho \in (0, 1)$ and $n = 100000$ when $\rho = 0, 1$, with periodic boundary conditions and density was averaged over the last 10 configurations. We average our results over 10 iterations from different initial conditions. Such experimental curves are presented in Appendix A. We note in passing that the example shown in Figure A.6e (Rule 168), exhibits a response curve resembling a “phase transition”, that is, discontinuity of the derivative. No 1D elementary rules exhibit such behaviour.

Rules	$P_n(1)$	$P(1)$
8	$\rho^{n+1}(1-\rho)^n$	0
32	$\rho^{n+1}(1-\rho)^n$	0
40	$2^n \rho^{n+1}(1-\rho)^n$	0
72	$2^n \rho^{n+1}(1-\rho)^n$	0
128	$\rho^{(n^2+3n+2)/2}$	0 if $\rho \neq 1$ 1 if $\rho = 1$
130	see 3.4, 3.5 and [18]	
132	see Rule 130	
136	ρ^{n+1}	0 if $\rho \neq 1$ 1 if $\rho = 1$
138	$\frac{\rho^{2n+2}+\rho}{\rho+1}$	$\frac{\rho}{1+\rho}$ if $\rho \neq 1$ 1 if $\rho = 1$
140	$\frac{\rho^{2n+2}+\rho}{\rho+1}$	$\frac{\rho}{1+\rho}$ if $\rho \neq 1$ 1 if $\rho = 1$
160	ρ^{n+1}	0 if $\rho \neq 1$ 1 if $\rho = 1$
162	$\frac{\rho^{2n+2}+\rho}{\rho+1}$	$\frac{\rho}{1+\rho}$ if $\rho \neq 1$ 1 if $\rho = 1$

Table 3.2: Rules with Decaying Density

3.2 Response Sequences

In the special case when $\rho = 1/2$, the probability of any block of a given size is equally likely and (3.5) can be expressed as

$$P_n(b) = 2^{-(r+n+1)(r+n)/2} \text{card} [\mathbf{g}^{-n}(b)], \quad (3.6)$$

where $\text{card} [\mathbf{g}^{-n}(b)]$ denotes the number of elements in the set of all n -step preimages of block b . If we want to indicate that we consider the special case of $\rho = 1/2$, we will use the notation $P_n^{(s)}(b)$, and the sequence of $P_n^{(s)}(b)$ for $n = 0, 1, 2, \dots$ will be called a *response sequence*. In order to use eq. (3.6) we need only know the number of elements in the set of n -step preimages of b .

3.2.1 Theoretical and Conjectured

In some cases we were unable to determine an explicit expression for the response curve of a given rule, but we were able to derive an explicit formula for $\text{card} [\mathbf{g}^{-n}(1)]$, and thus use (3.6) to obtain a response sequence. For 21 additional rules, we were able to either prove or conjecture a response sequence. These are the class of sur-

jective rules and we prove these results in detail in Chapter 4.

To find response sequences for the remaining rules, we performed an exhaustive search through all potential preimages for each rule. For the L-neighbourhood, the number of potential preimages is $2^{(n^2+3n+2)/2}$, which makes searches for large n impossible. We performed our searches using the Shared Hierarchical Academic Research Computing Network (SHARCNET) and we were able to obtain cardinalities of preimage sets to level $n = 7$. We then attempted to conjecture a formula for the sequence using the first six terms, and checked the conjecture with the seventh sequence term.

Rules 23, 27, 29, 43, 46, 58, 77, 78, 142, 172, 178, 184 each shared the first seven terms of the preimage sequence with the surjective rules in Chapter 4, so that for these rules we conjecture that $P_n^{(s)}(1) = 1/2$. Experimental evidence of this is found in the response curve plots in Appendix A. For all remaining rules, a list of the first seven preimage cardinalities is available upon request. It is interesting to note that some sequences appear to follow obvious patterns for a small number of terms, but larger terms show that this is not the case. For example, for Rule 6, the first six terms are $\{2, 16, 256, 8512, 496000, 64372608, \dots\}$.

3.2.2 Basic Blocks

As a final remark we would like to indicate that in addition to probabilities of 1 after n iterations, it is sometimes possible to compute probabilities of other blocks. If $\rho = 1/2$, it is often possible to compute the number of preimages of these blocks. For example, for 40 of the 88 minimal rules, we were able to find preimage sequences for all eight basic blocks (Table 3.3). In each case, it is only necessary to determine preimage sequences for 5 of the 8 blocks, then we may use Kolmogorov consistency conditions (2.8) to determine the remaining three. In some cases, these formulas are rather striking, such as in the case of rule 130, reported in detail in Section 3.5. Another remarkable example is rule 172, for which we came out with a following conjecture.

Conjecture 3.1. *Under 2D Rule 172 the preimage sequences of basic blocks are given by*

$$\text{card} [\mathbf{g}^{-n}(b)] = \begin{cases} 2^{(n^2+5n)/2} \sum_{k=0}^n \frac{C_k}{4^k} & \text{if } b \in B_1, \\ 2^{(n^2+5n)/2} \left(2 - \sum_{k=0}^n \frac{C_k}{4^k} \right) & \text{if } b \in B_2, \end{cases}$$

where C_k denotes the k -th Catalan number and

$$B_1 = \left\{ \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}, \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}, \begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix} \right\}, \quad B_2 = \left\{ \begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}, \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix} \right\}.$$

Work on a proof of this result is ongoing and will be reported elsewhere.

3.3 Rule emulation

We will now briefly turn our attention to dynamics of 2D rules. When one prints sample spatiotemporal diagrams of 2D rules with L-shaped neighbourhood, one can easily observe that all rules for which density response curves can be calculated theoretically exhibit somewhat “simple” dynamics. A convenient way to describe this “simplicity” is to say that after a few iterations these rules essentially behave like identity or shift. In order to formalize this statement, we need to introduce the concept of emulation [29], first finite and then asymptotic.

3.3.1 Finite Rule Emulation

We say that *Rule X emulates Rule Y at level n* if,

$$\mathbf{g}_X^{n+1}(b) = g_Y(\mathbf{g}_X^n(b)). \quad (3.7)$$

for any block $b \in \mathcal{T}_{n+2}$. We will demonstrate this with an example. Let us consider Rule 76, with a local rule given by

$$g_{76} \left(\begin{smallmatrix} x & \\ y & z \end{smallmatrix} \right) = (1-x)y(1-z) + (1-x)yz + xy(1-z) = y(1-xz). \quad (3.8)$$

Using (3.8), we can determine an algebraic expression for the composition of g_{76} with itself,

$$\begin{aligned} \mathbf{g}_{76}^2(b) &= g_{76} \left(\mathbf{g}_{76} \left(\begin{smallmatrix} x_0 \\ x_1 & x_2 \\ x_3 & x_4 & x_5 \end{smallmatrix} \right) \right) \\ &= x_3(1-x_1x_4)(1-x_1(1-x_0x_2)x_4(1-x_2x_5)) \\ &= x_3(1-x_1x_4) \\ &= g_{204}(\mathbf{g}_{76}(b)), \end{aligned}$$

where we have used the fact that when $x \in \{0, 1\}$, we know that $x^2 = x$. We therefore conclude that Rule 76 emulates identity at level 1. We checked all 88×87 pairs of distinct elementary rules for finite rule emulation. In Figure 3.1, we show all level 1

Rule	$P_n^{(s)}(b_i) \quad (b_i \in \mathcal{B}_3, 0 \leq i \leq 7)$
0	1, 0, 0, 0, 0, 0, 0, 0
2	21/32, 1/8, 3/32, 0, 3/32, 0, 1/32, 0
3	(n odd) 15/32, 5/32, 3/32, 1/32, 3/32, 1/32, 3/32, 1/32 (n even) 3/32, 5/32, 3/64, 5/64, 3/64, 5/64, 3/16, 5/16
4	21/32, 3/32, 1/8, 0, 3/32, 1/32, 0, 0
5	(if n odd) 15/32, 3/32, 5/32, 1/32, 3/32, 3/32, 1/32, 1/32 (if n even) 3/32, 3/64, 5/32, 5/64, 3/64, 3/16, 5/64, 5/16
8	$1 - 11 \cdot 2^{-2n-3}, 3 \cdot 2^{-2n-3}, 3 \cdot 2^{-2n-3}, 2^{-2n-3}, 2^{-2n-1}, 0, 0, 0$
10	3/8, 3/16, 1/8, 1/16, 3/16, 0, 1/16, 0
12	3/8, 1/8, 3/16, 1/16, 3/16, 1/16, 0, 0
27	5/32, 5/32, 3/32, 3/32, 3/32, 3/32, 5/32, 5/32
29	5/32, 3/32, 5/32, 3/32, 3/32, 5/32, 3/32, 5/32
32	$1 - 11 \cdot 2^{-2n-3}, 3 \cdot 2^{-2n-3}, 2^{-2n-1}, 0, 3 \cdot 2^{-2n-3}, 2^{-2n-3}, 0, 0$
34	3/8, 3/16, 3/16, 0, 1/8, 1/16, 1/16, 0
42	7/32, 3/16, 1/8, 3/32, 1/8, 3/32, 5/32, 0
76	7/32, 1/8, 3/16, 3/32, 1/8, 5/32, 3/32, 0
128	$1 - A(12B - 5), A(4B - 3), A(4B - 3), A, A(4B - 3), A, A, A$
130	$1 - 3F/8 - 4AB, F/8, (F - 1)/2 + AB, AB,$ $(F - 1)/2 + AB, AB, (4 - 3F)/8, (4 - 3F)/8$
132	$1 - 3F/8 - 4AB, (F - 1)/2 + AB, F/8, AB,$ $(F - 1)/2 + AB, (4 - 3F)/8, AB, (4 - 3F)/8$
136	$1 - 7 \cdot 2^{-2n-3}, 2^{-n-2} - 2^{-2n-3}, 2^{-n-2} - 2^{-2n-3}, 2^{-n-2} - 2^{-2n-3},$ $2^{-n-1} - 3 \cdot 2^{-2n-3}, 2^{-2n-3}, 2^{-2n-3}, 2^{-2n-3}$
138	$\frac{11}{42} - \frac{5}{24}4^{-n} + \frac{1}{14}8^{-n}, \frac{1}{6} - \frac{1}{24}4^{-n}, \frac{5}{42} + \frac{1}{24}4^{-n} - \frac{1}{14}8^{-n},$ $\frac{5}{42} + \frac{1}{24}4^{-n} - \frac{1}{14}8^{-n}, \frac{5}{21} + \frac{1}{24}4^{-n} - \frac{1}{14}8^{-n}, \frac{1}{8}4^{-n},$ $\frac{1}{21} + \frac{1}{24}4^{-n} + \frac{1}{28}8^{-n}, \frac{1}{21} + \frac{1}{24}4^{-n} + \frac{1}{28}8^{-n}$
140	$\frac{11}{42} - \frac{5}{24}4^{-n} + \frac{1}{14}8^{-n}, \frac{5}{42} + \frac{1}{24}4^{-n} - \frac{1}{14}8^{-n},$ $\frac{1}{6} - \frac{1}{24}4^{-n}, \frac{5}{42} + \frac{1}{24}4^{-n} - \frac{1}{14}8^{-n}, \frac{5}{21} + \frac{1}{24}4^{-n} - \frac{1}{14}8^{-n},$ $\frac{1}{21} + \frac{1}{24}4^{-n} + \frac{1}{28}8^{-n}, \frac{1}{8}4^{-n}, \frac{1}{21} + \frac{1}{24}4^{-n} + \frac{1}{28}8^{-n}$
160	$1 - 7 \cdot 2^{-2n-3}, 2^{-n-2} - 2^{-2n-3}, 2^{-n-1} - 3 \cdot 2^{-2n-3}, 2^{-2n-3},$ $2^{-n-2} - 2^{-2n-3}, 2^{-n-2} - 2^{-2n-3}, 2^{-2n-3}, 2^{-2n-3}$
162	$\frac{11}{42} - \frac{5}{24}4^{-n} + \frac{1}{14}8^{-n}, \frac{1}{6} - \frac{1}{24}4^{-n}, \frac{5}{21} + \frac{1}{24}4^{-n} - \frac{1}{14}8^{-n},$ $\frac{1}{8}4^{-n}, \frac{5}{42} + \frac{1}{24}4^{-n} - \frac{1}{14}8^{-n}, \frac{5}{42} + \frac{1}{24}4^{-n} - \frac{1}{14}8^{-n},$ $\frac{1}{21} + \frac{1}{24}4^{-n} + \frac{1}{28}8^{-n}, \frac{1}{21} + \frac{1}{24}4^{-n} + \frac{1}{28}8^{-n}$
172	$C/8, (2 - C)/8, (2 - C)/8, C/8, C/8, (2 - C)/8, (2 - C)/8, 1/8C$
184	$C/8, (2 - C)/8, C/8, (2 - C)/8, (2 - C)/8, C/8, (2 - C)/8, C/8$

Table 3.3: Response Sequences for Basic Blocks (where we denote for convenience, $A = 2^{(-n^2-5n-6)/2}$, $B = 2^n$, $C = \sum_{k=0}^n \frac{C_k}{4^k}$, and C_k refers to the k -th Catalan number, and $F = \sum_{i=0}^n 2^{-i(i+3)/2}$).

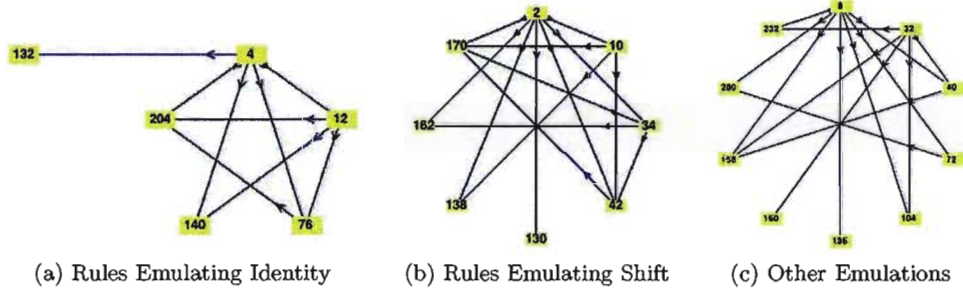


Figure 3.1: Finite Emulation Relations

emulation relations between all minimal elementary 2D rules with L-neighbourhood as directed graphs in which an arrow travels from X to Y if and only if Rule X emulates Rule Y at level 1. In Figure 3.1a are all rules which finitely emulate the identity Rule 204. In Figure 3.1b are all rules which finitely emulate the left shift Rule 170. Finally, in Figure 3.1c are another class of interrelated emulation rules. In addition to the rules in the graph, we also discovered that rules 6, 14, 18 and 50 emulate rules 134, 142, 146 and 178 respectively.

3.3.2 Asymptotic Rule Emulation

In [15], the author defined the following metric to describe the distance between two elementary 1D cellular automata rules. We adapt this and define the following metric to describe the distance between two elementary 2D cellular automata rules with L-neighbourhood, for any $k \geq 1$,

$$d(f, g) = 2^{(-k^2 - 3k - 2)/2} \sum_{b \in B_k} |f(b) - g(b)|. \quad (3.9)$$

We say that *Rule f asymptotically emulates Rule g* if

$$\lim_{n \rightarrow \infty} d(f^{n+1}, g \circ f^n) = 0. \quad (3.10)$$

We now derive a useful equation with which we can calculate the distance between two rules at a given level- n . We define the following function for any block $b \in \mathcal{T}_2$,

$$(f \oplus g)(b) = f(b) + g(b) \pmod{2},$$

which outputs 1 if and only if $f(b) \neq g(b)$. Thus, we can use this function to count the number of blocks on which local mappings f and g differ. Adapting Proposition

3 from [15], we obtain the following proposition (proof in [15]).

Proposition 3.7. *Let f, g be two 2D local mappings with L -neighbourhood, and let $A_0 = (f \oplus g)^{-1}(1)$, and $A_n = \mathbf{f}^{-n}(A_0)$. If $P_n^{(s)}(a)$ is defined under \mathbf{f} , then,*

$$d(\mathbf{f}^{n+1}, g \circ \mathbf{f}^n) = \frac{\text{card}[A_n]}{2^{(n^2+5n+6)/2}} = \sum_{a \in A_0} \frac{\text{card}[\mathbf{f}^{-n}(a)]}{2^{(n^2+5n+6)/2}} = \sum_{a \in A_0} P_n^{(s)}(a).$$

We demonstrate this procedure with an example.

Proposition 3.8. *2D CA Rule 160 asymptotically emulates the identity rule.*

Proof. We start by determining the set A_0 by considering the effect of the local mappings on basic blocks for both rules 160 and 204.

$b \in \mathcal{T}_2$	$\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}$	$\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}$
$\mathbf{g}_{160}(b)$	0	0	0	0	0	1	0	1
$\mathbf{g}_{204}(b)$	0	0	1	1	0	0	1	1

By considering the basic blocks on which Rules 160 and 204 differ, we conclude that the set $A_0 = \{\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}, \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}, \begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\}$. We use the data in Table 3.3 to apply Proposition 3.7 and obtain

$$d(\mathbf{g}_{160}^{n+1}, \mathbf{g}_{204} \circ \mathbf{g}_{160}^n) = 3 \cdot 2^{-n-2} - 4^{-n-1}.$$

Therefore, since the limit of this expression goes to 0, we conclude that Rule 160 emulates identity asymptotically. \square

Table 3.4 shows all known results of rules emulating shift or identity. We can now state our observation expressed at the beginning of this section using the concept of emulation: *all rules included in Tables 3.1 and 3.2 emulate identity or shift either in a final number of steps or asymptotically.*

3.4 Example: Rule 130 (1D)

In the next section ², we will perform a full analysis of two-dimensional Rule 130 with L -neighbourhood. For comparison, we now perform a full analysis on one-dimensional Rule 130. Some of the following results were conjectured in [15]. Using

²A version of this section and the subsequent section has been published. H. Fuk s and A. Skelton. Response curves for cellular automata in one and two dimensions - an example of rigorous calculations. *Journal of Natural Computing Research*, 1:8599, 2010.

Rule f	$d(f^{n+1}, f_{204} \circ f^n)$	$P^{(s)}(1)$
8	$3 \cdot 2^{-2n-3}$	0
32	$5 \cdot 2^{-2n-3}$	0
40	2^{-n-1}	0
72	4^{-n-2}	0
128	$2^{(-n^2-3n-2)/2} - 2^{(-n^2-5n-6)/2}$	0
132	$2^{(-n^2-3n-4)/2}$	$\simeq 0.179$
136	2^{-n-2}	0
140	2^{-2n-3}	$1/3$
160	$3 \cdot 2^{-n-2} - 4^{-n-1}$	0
Rule f	$d(f^{n+1}, f_{170} \circ f^n)$	$P^{(s)}(1)$
130	$2^{(-n^2-3n-4)/2}$	$\simeq 0.179$
138	2^{-2n-3}	$1/3$
162	2^{-2n-3}	$1/3$

Table 3.4: Asymptotic Emulation

the numbering system in (2.4), we know that the local mapping for Rule 130 is given by

$$f(x_0x_1x_2) = \begin{cases} 1 & \text{if } (x_0x_1x_2) = (001) \text{ or } (111), \\ 0 & \text{otherwise.} \end{cases} \quad (3.11)$$

In this section, both the local function f and the corresponding block evolution operator \mathbf{f} will refer to one-dimensional rule 130, unless otherwise noted. We will now calculate the response curve for this rule. Since the only preimages of 1 under rule 130 are blocks 111 and 001, we can calculate the density of ones, from (3.5), as

$$P_n(1) = \sum_{a \in \mathbf{f}^{-n}(1)} P_0(a) = \sum_{a' \in \mathbf{f}^{-n+1}(001)} P_0(a') + \sum_{a' \in \mathbf{f}^{-n+1}(111)} P_0(a'). \quad (3.12)$$

So we must determine the elements of the sets $\mathbf{f}^{-n}(111)$ and $\mathbf{f}^{-n}(001)$. The following two propositions describe these sets.

Proposition 3.9. *The set $\mathbf{f}^{-n}(111)$ consists only of the block $\underbrace{11 \dots 1}_{2n+3}$.*

Proof. Exhaustively checking all 32 potential preimages of 111 one can show that the only length-5 string b such that $\mathbf{f}(b) = 111$ is 11111. Since any block of 1's consists of overlapping blocks 111, its preimage also must consist entirely of ones, and the proof by induction follows. \square

Proposition 3.10. *The set $\mathbf{f}^{-n}(001)$ consists of all blocks of the form*

$$\underbrace{\star \dots \star}_{2n-2i} 1 0 1 \underbrace{1 \dots 1}_{2i} \quad (\text{if } i \text{ is odd}) \quad \text{or} \quad \underbrace{\star \dots \star}_{2n-2i} 0 0 1 \underbrace{1 \dots 1}_{2i} \quad (\text{if } i \text{ is even}),$$

where $i \in \{0 \dots n\}$ and \star denotes an arbitrary value in \mathcal{S} .

Proof. By induction, if $n = 1$, then $i \in \{0, 1\}$, and our formula provides us with the following two types of preimages, $\star\star 001$ or 10111 . One can exhaustively check all 32 blocks of length five to verify that these are the only blocks b such that $\mathbf{f}(b) = 001$.

Now, let us assume that we have the following n -step preimage:

$$\underbrace{\star \dots \star}_{2n-2i} 0 0 1 \underbrace{1 \dots 1}_{2i} \quad \text{where } i \text{ even and } i \in \{2 \dots n\}.$$

We find the preimages of this string, starting from the right and working toward the left. By considering all 16 blocks of length four, we can see that the only string that has image 11 under \mathbf{f} is 1111, so we start with the following preimage (written above the string we are considering, with arrow indicating direction of proceeding):

$$\begin{array}{cccccccccccc} & & & & & & & & \leftarrow & 1 & 1 & 1 & 1 \\ \star & \star & \dots & \star & \star & 0 & 0 & 1 & 1 & \dots & 1 & 1 & 1 \end{array}$$

Continuing to the left, using the rule table of rule 130 we can construct the preimage up to the following point:

$$\begin{array}{cccccccccccc} & & & & & & & & \leftarrow & 1 & 0 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 \\ \star & \star & \dots & \star & \star & 0 & 0 & 1 & 1 & \dots & 1 & 1 & 1 \end{array}$$

From here to the left, all the remaining entries in the preimage are arbitrary:

$$\begin{array}{cccccccccccc} \star & \star & \dots & \star & \star & 1 & 0 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 \\ \star & \star & \dots & \star & \star & 0 & 0 & 1 & 1 & \dots & 1 & 1 & 1 \end{array}$$

We now have the following $(n+1)$ -step preimage:

$$\underbrace{\star \dots \star}_{2(n+1)-2(i+1)} 1 0 1 \underbrace{1 \dots 1}_{2(i+1)} \quad \text{where } (i+1) \text{ is odd and } (i+1) \in \{3 \dots (n+1)\}.$$

To finish the proof, we need to perform similar analysis for two other cases as follows.

- When $i = 0$, a similar argument is used to find all $(n+1)$ -step preimages in which $i = 0, 1$.

$b_i \in \mathcal{B}_3$	000	001	010	011	100	101	110	111
$\text{card}[\mathbf{f}^{-n}(b_i)]$	$4^{n+1} - 3$	$\frac{4^{n+1}-1}{3}$	$\frac{4^{n+1}-1}{3}$	1	$\frac{4^{n+1}-1}{3}$	1	1	1

Table 3.5: Number of Preimages of Basic Blocks of One-Dimensional Rule 130

- When $i \in \{1, \dots, n\}$ and odd, another similar argument can be used to find all $(n+1)$ -step preimages in which $(i+1)$ is even.

We omit the details due to space constraints, but note that all possible $(n+1)$ -step preimages are accounted for in the analysis, thus completing the induction step \square

We may now proceed with enumerating the elements of $\mathbf{f}^{-n}(001)$ to confirm the results in Table 3.3.

Proposition 3.11. *The number of blocks in the set of n -step preimages of block 001 is given by $\text{card}[\mathbf{f}^{-n}(001)] = \frac{4^{n+1} - 1}{3}$.*

Proof. According to Proposition (3.10), for each value of $i \in \{0 \dots n\}$, there are $2n - 2i$ arbitrary values in the preimage, so that there are 2^{2n-2i} of such preimages. Summing over i we obtain

$$\text{card}[\mathbf{f}^{-n}(001)] = \sum_{i=0}^n 2^{2n-2i} = 4^n \sum_{i=0}^n 4^{-i} = \frac{4^{n+1} - 1}{3}.$$

\square

Reasoning as in Proposition (3.9) leads to $\text{card}[\mathbf{f}^{-n}(110)] = \text{card}[\mathbf{f}^{-n}(101)] = 1$.

It turns out that we now have enough information to find the cardinality of the preimage sets of each of the four remaining basic blocks by using consistency conditions in system (2.2). The results are summarized in Table 3.5.

Now, to find an equation for our response curve, we can rewrite the 1D density equation in [16] to give

$$P_n(1) = \sum_{b \in \mathbf{f}^{-n+1}(111)} P_0(b) + \sum_{b \in \mathbf{f}^{-n+1}(001)} P_0(b) = P_{n-1}(111) + P_{n-1}(001). \quad (3.13)$$

We know the structure of the n -step preimage sets for Rule 130 in one dimension, thus we may compute desired probabilities in the above formula. We know from Proposition 3.9 that the preimage of 111 is comprised entirely of ones, hence

$$P_n(111) = P_0(\underbrace{11 \dots 1}_{2n+3}) = \rho^{2n+3}. \quad (3.14)$$

From Proposition 3.10 we find,

$$\begin{aligned}
P_n(001) &= \sum_{i \in \{0 \dots n\}, i \text{ is even}} (1 - \rho)^2 \rho^{2i+1} + \sum_{i \in \{0 \dots n\}, i \text{ is odd}} (1 - \rho) \rho^{2i+2} \\
&= \sum_{k=0}^{\lceil \frac{n-1}{2} \rceil} (1 - \rho)^2 \rho^{4k+1} + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (1 - \rho) \rho^{4k+4} \\
&= \frac{\rho \left(-\rho^{4\lceil (n-1)/2 \rceil + 4} + \rho^{4\lceil (n-1)/2 \rceil + 5} - \rho^{4\lfloor (n+1)/2 \rfloor + 3} + \rho^3 - \rho + 1 \right)}{\rho^3 + \rho^2 + \rho + 1}. \quad (3.15)
\end{aligned}$$

We now use equations (3.13), (3.14) and (3.15) to obtain the equation of the response curve:

$$P_n(1) = \rho^{2n+1} + \frac{\rho \left(-\rho^{4\lceil (n-2)/2 \rceil + 4} + \rho^{4\lceil (n-2)/2 \rceil + 5} - \rho^{4\lfloor n/2 \rfloor + 3} + \rho^3 - \rho + 1 \right)}{\rho^3 + \rho^2 + \rho + 1}. \quad (3.16)$$

By taking the limit as $n \rightarrow \infty$, we obtain the asymptotic response curve,

$$P(1) = \begin{cases} \frac{\rho(\rho^3 - \rho + 1)}{\rho^3 + \rho^2 + \rho + 1} & \text{if } \rho < 1, \\ 1 & \text{if } \rho = 1. \end{cases} \quad (3.17)$$

We performed computer simulations to illustrate the formula (3.17) for the the dependence of $P(1)$ on the initial density ρ . We considered an initial configuration of 5000 cells, and varied the initial density from 0 to 100%, increasing it by a step size of 1%, iterating rule 130 until we reached a fixed density. Results were then averaged over 20 runs for each initial density.

The results are presented in Figure 3.2. As we can see, the response curve calculated above for infinite configurations agrees very well with simulations performed on a finite lattice. As a final remark in this section, let us note that for the special case of $\rho = 1/2$ we obtain

$$P_n^{(s)}(1) = \frac{1}{6} + \frac{1}{3} 4^{-n}. \quad (3.18)$$

We can see that the convergence toward $P^{(s)}(1) = 1/6$ is exponential with exponent $-n$ and this result confirms the conjecture in [15].

We complete our analysis by using our cardinalities of preimage sequences to consider the asymptotic behaviour of Rule 130.

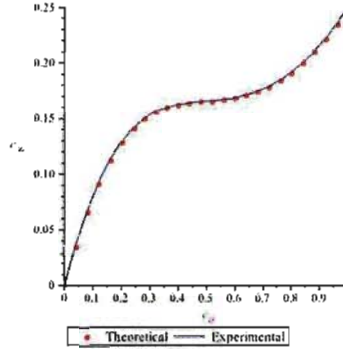


Figure 3.2: Plots of the response curve, i.e., dependence of c_∞ on ρ for one-dimensional rule 130. Circles correspond to computer simulations using a lattice of 5000 sites, as described in the text, while the continuous line represents theoretical curve.

Proposition 3.12. *The one-dimensional version of Rule 130 asymptotically emulates Rule 170 (the right shift rule).*

Proof. Using the results obtained in Section 3.3.2 and considering the basic blocks on which Rules 130 and 170 differ, we conclude that the set $A_0 = \{101, 011\}$. Using Table 3.5, we may use Proposition 3.7 to conclude that

$$d(\mathbf{f}_{130}^{n+1}, \mathbf{f}_{170} \circ \mathbf{f}_{130}^n) = \frac{\text{card}[\mathbf{f}_{130}^{-n}(101)] + \text{card}[\mathbf{f}_{130}^{-n}(011)]}{2^{2n+3}} = 2^{-2n-2}, \quad (3.19)$$

which clearly goes to zero as n becomes large. We can therefore say that the one-dimensional version of Rule 130 asymptotically emulates the right shift Rule 170. \square

3.5 Example: Rule 130 (2D)

We now consider a similar analysis performed on Rule 130 with L-neighbourhood. We define the local mapping as

$$g\left(\begin{smallmatrix} x \\ y \end{smallmatrix} z\right) = \begin{cases} 1 & \text{if } \left(\begin{smallmatrix} x \\ y \end{smallmatrix} z\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} 1\right) \text{ or } \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} 1\right), \\ 0 & \text{otherwise.} \end{cases} \quad (3.20)$$

Since the only preimages of 1 are triangular blocks $\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} 1\right)$ and $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} 1\right)$, we need, similarly as in one dimension, to analyze structure of preimage sets $\mathbf{g}^{-n}\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} 1\right)$ and $\mathbf{g}^{-n}\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} 1\right)$.

thus the lemma is indeed true for $k = 2$. Suppose now that it holds for a given k , and let us consider a binary sequence a_1, a_2, \dots, a_{k+1} . We clearly have

$$Q = q(a_1, a_2, \dots, a_{k+1}) = q(a_1, a_2, \dots, a_k) + q(a_2, a_2, \dots, a_{k+1}) + 1 \pmod{2},$$

hence

$$\begin{aligned} Q &= 2^{k-1} - 1 + \sum_{i=1}^k \binom{k-1}{i-1} a_i + 2^{k-1} - 1 + \sum_{i=2}^{k+1} \binom{k-1}{i-2} a_i + 1 \pmod{2} \\ &= 2^k - 1 + a_1 + \sum_{i=2}^k \binom{k-1}{i-1} a_i + a_{k+1} + \sum_{i=2}^k \binom{k-1}{i-2} a_i \pmod{2}. \end{aligned}$$

Using Pascal's identity, this becomes

$$Q = 2^k - 1 + a_1 + a_k + \sum_{i=2}^k \binom{k}{i-1} \pmod{2}.$$

Using $\binom{k}{0} = \binom{k}{k} = 1$ we can incorporate a_1 and a_{k+1} into the sum, obtaining

$$Q = 2^k - 1 + \sum_{i=1}^{k+1} \binom{k}{i-1} \pmod{2},$$

which is the desired formula for $k + 1$. Proof by induction is therefore complete. \square

Having this lemma, we can now proceed with the sketch of the proof of the proposition.

Proof. Consider first the case of $n = 1$, when the only preimages we obtain are of the form

$$\begin{array}{ccc} \star & & a_1 \\ \star & 0 & , \quad \text{or} \quad a_2 \quad 1 \\ \star & 0 \quad 1 & a_3 \quad 1 \quad 1 \end{array} \quad (3.23)$$

where $a_1 = a_3$ and $a_2 = [1 + a_3 \pmod{2}]$ are given by eq. (3.21) and (3.22) for any $a_3 \in \{0, 1\}$. One can check that these are indeed the only desired preimages by applying g to all blocks \mathcal{T}_3 and verifying that only the blocks of the above form produce $\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$. This means that the block $\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$ can appear in a configuration by two ways, by moving from the left one unit at a time, or by being created from a configuration shown on the right hand side of (3.23).

Consider a number of steps larger than one, denoted by n . A given block may appear in a certain place because it was created in another location, k units to the right, and then moved to the desired place from there in k steps, where $k \leq n$, or because it appeared in this place as a result of the second configuration of eq. (3.23).

Now note that because of Lemma 1, block shown in eq. (3.13) subject to conditions (3.21) and (3.22) will in i iterations produce the block

$$\begin{array}{ccccccc}
 & & & & & & * \\
 & & & & & & \vdots \\
 & & & & & & \ddots \\
 & & & & & & \vdots \\
 & & & & & & * \\
 & & & & & & \vdots \\
 & & & & & & 0 \\
 * & \dots & * & 0 & 1.
 \end{array}$$

$\underbrace{\hspace{10em}}_{n-i}$

Then, after $n - i$ iterations the block $\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$ will move to the left one step at a time, so that in the end block $\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$ will be produced as a result of applying operator \mathbf{g} , n -times to block (3.13). Since i varies from 0 to n this means that indeed all possibilities of arriving from some place on the right ($i = 1, \dots, n - 1$) as well as being “created in place” ($i = n$) are covered. A very similar argument can be used in the following. \square

Proposition 3.14. *The set, $\mathbf{g}^{-n}(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix})$, consists of all blocks in the set $\bigcup_{i=0}^n (A_i \setminus B_i) \cup C$, where, for a fixed value of i , A_i and B_i are, respectively, the sets of all blocks of the form*

$$\begin{array}{ccccccc}
 * & & & & & & * \\
 \vdots & & & & & & \vdots \\
 \vdots & & & & & & \ddots \\
 \vdots & & & & & & \vdots \\
 \vdots & & & & & & * \\
 \vdots & & & & & & \vdots \\
 \vdots & & & & & & b_1 \\
 \vdots & & & & & & \vdots \\
 \vdots & & & & & & b_2 \quad 1 \\
 \vdots & & & & & & \vdots \\
 \vdots & & & & & & b_3 \quad \vdots \quad \ddots \\
 \vdots & & & & & & \vdots \\
 \vdots & & & & & & \vdots \\
 * & \dots & * & b_{i+2} & 1 & \dots & 1,
 \end{array}$$

$\underbrace{\hspace{10em}}_{n-i} \quad \underbrace{\hspace{10em}}_{i+1}$

$$\begin{array}{ccccccc}
 * & & & & & & * \\
 \vdots & & & & & & \vdots \\
 \vdots & & & & & & \ddots \\
 \vdots & & & & & & \vdots \\
 \vdots & & & & & & * \\
 \vdots & & & & & & \vdots \\
 \vdots & & & & & & 1 \\
 \vdots & & & & & & \vdots \\
 \vdots & & & & & & \vdots \\
 \vdots & & & & & & \vdots \\
 * & \dots & * & 1 & \dots & 1,
 \end{array}$$

$\underbrace{\hspace{10em}}_{n-i} \quad \underbrace{\hspace{10em}}_{i+2}$

and C is the set whose only element is the block consisting entirely of ones, where $i \in \{0 \dots n\}$ and for $b_3 \dots b_{i+2}$ we take arbitrary values. If $i = 0$, then $(b_1, b_2) = (1, 1)$ and if $i > 0$, then b_1, b_2 are determined by

$$b_1 = \sum_{j=2}^{i+1} \binom{i}{j-1} b_j \pmod{2}, \quad (3.24)$$

$$b_2 = \sum_{j=2}^i \binom{i}{j-1} b_{j+1} \pmod{2}. \quad (3.25)$$

We then determine the cardinality of the preimage set.

Proposition 3.15. *The number of n -step preimages of block $\begin{smallmatrix} 0 & \\ & 1 \end{smallmatrix}$ is:*

$$\text{card}[\mathbf{g}^{-n}(\begin{smallmatrix} 0 & \\ & 1 \end{smallmatrix})] = 2^{(n^2+5n)/2} \sum_{i=0}^n 2^{-i(i+3)/2}.$$

Proof. For a given value of i , we have $(n+2) + (n+1) + \dots + (i+3) + i = (n(n+5) - i(i+5))/2 + i$ arbitrary values represented by \star . Therefore, for each i , there are $2^{(n(n+5)-i(i+5))/2+i} = 2^{(n(n+5)-i(i+3))/2}$ possible configurations. Summing over all values of $i \in \{0 \dots n\}$ gives the desired result. \square

Similar analysis can be performed for other blocks and we discover that

$$\text{card}[\mathbf{g}^{-n}(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix})] = \text{card}[\mathbf{g}^{-n}(\begin{smallmatrix} 1 & \\ & 0 \end{smallmatrix})], \quad \text{card}[\mathbf{g}^{-n}(\begin{smallmatrix} 0 & \\ & 1 \end{smallmatrix})] = \text{card}[\mathbf{g}^{-n}(\begin{smallmatrix} 1 & \\ & 0 \end{smallmatrix})]. \quad (3.26)$$

We now have enough information to obtain the number of preimages of each of the remaining basic blocks, since, similarly as in one dimensional case, we use consistency conditions in system (2.8). Results are summarized in Table 2, where numbers of preimages for all eight basic blocks are shown.

Using Proposition 3.13 and (3.5) one can therefore compute $P_n(\begin{smallmatrix} 0 & \\ & 1 \end{smallmatrix})$ as follows:

$$P_n(\begin{smallmatrix} 0 & \\ & 1 \end{smallmatrix}) = \sum_{i=0}^n \sum_{a_3 \dots a_{i+2} \in \{0,1\}} P_0(\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}) P_0 \left(\begin{smallmatrix} a_3 \\ a_4 \\ \vdots \\ a_{i+2} \end{smallmatrix} \right) P_0 \left(\begin{smallmatrix} 1 \\ \vdots \\ \underbrace{1 \dots 1}_{i+1} \end{smallmatrix} \right),$$

$b_i \in \mathcal{T}_2$	$\text{card}[\mathbf{g}^{-n}(b_i)]$
$\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$	$2^{(n^2+5n)/2} \left(8 - 3 \sum_{i=0}^n 2^{-i(i+3)/2} \right) - 4 \cdot 2^n$
$\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$	$2^{(n^2+5n)/2} \left(\sum_{i=0}^n 2^{-i(i+3)/2} \right)$
$\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$	$2^{(n^2+5n)/2} \left(-4 + 4 \sum_{i=0}^n 2^{-i(i+3)/2} \right) + 2^n$
$\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$	2^n
$\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$	$2^{(n^2+5n)/2} \left(-4 + 4 \sum_{i=0}^n 2^{-i(i+3)/2} \right) + 2^n$
$\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$	2^n
$\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$	$2^{(n^2+5n)/2} \left(4 - 3 \sum_{i=0}^n 2^{-i(i+3)/2} \right)$
$\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$	$2^{(n^2+5n)/2} \left(4 - 3 \sum_{i=0}^n 2^{-i(i+3)/2} \right)$

Table 3.6: Number of preimages of basic blocks for two-dimensional rule 130

where a_1, a_2 are given by (3.21)-(3.22). Similarly, Proposition 3.14 and (3.5) yield

$$\begin{aligned}
P_n \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) &= \sum_{i=0}^n \sum_{b_3 \dots b_{i+2} \in \{0,1\}} P_0 \left(\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix} \right) P_0 \left(\begin{smallmatrix} b_3 \\ b_4 \\ \vdots \\ b_{i+2} \end{smallmatrix} \right) P_0 \left(\begin{smallmatrix} 1 \\ \vdots \\ \underbrace{1 \dots 1}_{i+1} \end{smallmatrix} \right) \\
&\quad - \sum_{i=0}^n P_0 \left(\begin{smallmatrix} 1 \\ \vdots \\ \underbrace{1 \dots 1}_{i+2} \end{smallmatrix} \right) + P_0 \left(\begin{smallmatrix} 1 \\ \vdots \\ \underbrace{1 \dots 1}_{n+2} \end{smallmatrix} \right),
\end{aligned}$$

where b_1, b_2 are given by (3.24)-(3.25). Since $P_n(1) = P_{n-1} \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) + P_{n-1} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right)$, and

$$P_0 \left(\begin{smallmatrix} 1 \\ \vdots \\ \underbrace{1 \dots 1}_n \end{smallmatrix} \right) = \rho^{n(n+1)/2},$$

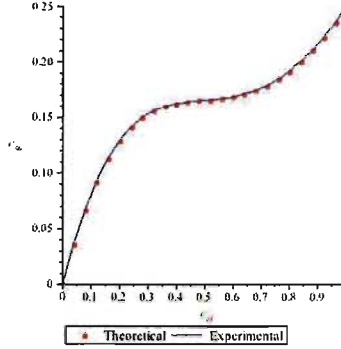


Figure 3.3: Plots of the response curve, i.e., dependence of c_n on ρ for two-dimensional rule 130. Circles correspond to computer simulations, while the continuous line represents theoretical curve for $n = 12$.

we obtain

$$\begin{aligned}
 P_n(1) = & \sum_{i=0}^{n-1} \sum_{a_3 \dots a_{i+2} \in \{0,1\}} P_0 \left(\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix} \right) P_0 \left(\begin{smallmatrix} a_3 \\ a_4 \\ \vdots \\ a_{i+2} \end{smallmatrix} \right) \rho^{(i+1)(i+2)/2} \\
 & + \sum_{i=0}^{n-1} \sum_{b_3 \dots b_{i+2} \in \{0,1\}} P_0 \left(\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix} \right) P_0 \left(\begin{smallmatrix} b_3 \\ b_4 \\ \vdots \\ b_{i+2} \end{smallmatrix} \right) \rho^{(i+1)(i+2)/2} \\
 & - \sum_{i=0}^{n-1} \rho^{(i+2)(i+3)/2} + \rho^{(n+1)(n+2)/2},
 \end{aligned} \tag{3.27}$$

where, again, a_1, a_2, b_1 , and b_2 are determined by eq. (3.21), (3.22), (3.24) and (3.25). The above is an exact equation of the response curve, although, unfortunately, it is not possible to calculate the double sums in a closed form. Nevertheless, for a given n , we can calculate and plot $P_n(1)$ versus ρ , providing that n is not too large. This has been done for $n = 12$, as shown in Figure 3.3. Again, there is excellent agreement between the theoretical curve representing infinite lattice and computer simulations done on a finite lattice. It is also possible to compute $P_n(1)$ for a special case of $\rho = 1/2$. In this case $P_0 \left(\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix} \right) = 1/4$ for any a_1, a_2 , hence

$$P_n^{(s)}(1) = 2 \sum_{i=0}^{n-1} \frac{1}{4} 2^{-(i+1)(i+2)/2} - \sum_{i=0}^{n-1} 2^{-(i+2)(i+3)/2} + 2^{-(n+1)(n+2)/2},$$

which simplifies to

$$P_n^{(s)}(1) = \frac{1}{2} - \frac{1}{4} \sum_{i=0}^{n-1} 2^{-i(i+3)/2}.$$

The asymptotic density is then given by

$$P^{(s)}(1) = \lim_{n \rightarrow \infty} P_n^{(s)}(1) = \frac{1}{2} - \frac{1}{4} \sum_{i=0}^{\infty} 2^{-i(i+3)/2},$$

where, again, it is not possible to compute the sum in a closed form. We can, however, approximate the infinite sum by taking the finite number of term and use the integral bounds for the remainder,

$$\int_{k+1}^{\infty} 2^{-x(x+3)/2} dx \leq \sum_{i=k}^{\infty} 2^{-i(i+3)/2} \leq \int_k^{\infty} 2^{-x(x+3)/2} dx.$$

The above integrals can be expressed in terms of the error function, hence for any positive integer k we obtain

$$\sum_{i=0}^{k-1} 2^{-i(i+3)/2} + \frac{\sqrt{\pi} 2^{1/8}}{a} (1 - A) \leq \sum_{i=0}^{\infty} 2^{-i(i+3)/2} \leq \sum_{i=0}^{k-1} 2^{-i(i+3)/2} + \frac{\sqrt{\pi} 2^{1/8}}{a} (1 - A),$$

where $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$, $A = \operatorname{erf}(ak + \frac{5}{2}a)$ and $a = \frac{\sqrt{2 \ln 2}}{2}$. The sum converges very fast and using the above inequalities for $k = 5$ we obtain

$$0.1791839087 \dots \leq P^{(s)}(1) \leq 0.1791839597 \dots,$$

that is, $P^{(s)}(1)$ with accuracy of seven digits after the decimal point.

We once again complete our analysis by using our cardinalities of preimage sequences to consider the asymptotic behaviour of two-dimensional Rule 130.

Proposition 3.16. *The two-dimensional version of Rule 130 asymptotically emulates Rule 170 (the right shift rule).*

Proof. Using the results obtained in Section 3.3.2 and considering the basic blocks on which Rules 130 and 170 differ, we conclude that the set $A_0 = \left\{ \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} 1, \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} 1 \right\}$. Using the data in Table 3.5, we may use Proposition 3.7 and conclude that

$$d(\mathbf{g}_{130}^{n+1}, g_{170} \circ \mathbf{g}_{130}^n) = \frac{\operatorname{card}[\mathbf{g}_{130}^{-n} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} 1 \right)] + \operatorname{card}[\mathbf{f}_{130}^{-n} \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix} 1 \right)]}{2^{(n^2+5n+6)/2}} = 2^{-n^2-4n-5}, \quad (3.28)$$

which clearly goes to zero as n becomes large. Therefore, we can say that the one-dimensional version of Rule 130 asymptotically emulates the right shift rule. We also note that the rate of convergence is much faster than that of the one-dimensional case. \square

Surjective Cellular Automata

During our investigation into response sequences of two-dimensional cellular automata with L-neighbourhood, we discovered that many rules have so-called balanced preimage sets - that is, all blocks of the same size and shape have the same number of preimages. This is a result of the Balance Theorem, originally published in [27] for rules with Moore neighbourhood. We present a detailed proof of this theorem for both L-neighbourhood and the von Neumann neighbourhood.

In general, it is undecidable whether a given cellular automaton is surjective [23]. We wondered however, if there would be a way to find a comprehensive list of all surjective rules for a specific case of a “small” neighbourhood. In the case of rules with L-neighbourhood, we were able to find a list of all surjective rules. We used a computerized search to find rules which violate the balance condition and found that all remaining rules are those which are permutive and surjective ¹.

In the case of rules with von Neumann neighbourhood of radius 1, we were able to classify rules according to surjectivity for all but 4 of 270,754,432 minimal rules. These rules are interesting in that they seem to behave like surjective rules, but are not permutive with respect to any local site, nor do they fail the balance condition for any small sized block. Further study of these rules is clearly required.

4.1 Definitions and Prior Results

We first present the definition of a surjective cellular automaton in two dimensions. A global rule G is surjective if, for any $s \in \mathcal{G}^{\mathbb{Z}^2}$, there exists $t \in \mathcal{G}^{\mathbb{Z}^2}$ such that $G(t) = s$. There are many difficulties in proving properties regarding infinite configurations, so many papers ([1],[5],[20],[24],[27],[28]) have been written on measurable properties of cellular automata that imply surjectivity.

If an element $q \in \mathcal{G}$ exists such that $g(\begin{smallmatrix} q \\ q \end{smallmatrix}) = q$, then we call q a *quiescent state*. A configuration $s \in \mathcal{G}^{\mathbb{Z}^d}$ will be said to be a *finite configuration* if all but finitely many entries are in the quiescent state.

¹A version of this chapter (and portions of Chapter 3) has been accepted for publication. H. Fuks and A. Skelton. Response Curves and Preimage Sequences of Two-Dimensional Cellular Automata. *CA-CSC proc.*, 2011.

If a 2D CA does not have a quiescent state, there must exist some element $q \in \mathcal{G}$, such that $g(\begin{smallmatrix} q \\ q \end{smallmatrix}) = p \neq q$. We define a local function $g^* : \mathcal{T}_1 \rightarrow \mathcal{T}_1$ (with corresponding global function $G^* : \mathcal{G}^{\mathbb{Z}^2} \rightarrow \mathcal{G}^{\mathbb{Z}^2}$) that exchanges instances of states q and p , such that $g^*(q) = p$, $g^*(p) = q$, and $g^*(a) = a$ if $a \neq p, a \neq q$. We need the following lemma before proceeding.

Lemma 4.1. *$F^* \circ F$ is surjective iff F is surjective.*

Proof. (forward direction) If $F^* \circ F$ is surjective, then for any $t \in \mathcal{G}^{\mathbb{Z}^2}$, there exists $s \in \mathcal{G}^{\mathbb{Z}^2}$, such that $(F^* \circ F)(s) = t$. Now consider configuration $t_{p \leftrightarrow q}$, derived from t such that all instances of p and q are exchanged. Clearly, the sets $\{t\} = \{t_{p \leftrightarrow q}\} = \mathcal{G}^{\mathbb{Z}^2}$. Now we consider

$$\begin{aligned} (F^* \circ F)(s) &= t \\ (F^* \circ F^* \circ F)(s) &= F^*(t) \\ F(s) &= t_{p \leftrightarrow q}. \end{aligned}$$

Therefore, for any $t_{p \leftrightarrow q} \in \mathcal{G}^{\mathbb{Z}^2}$, there exists $s \in \mathcal{G}^{\mathbb{Z}^2}$ such that $F(s) = t_{p \leftrightarrow q}$, so F is surjective. A similar argument proves the other direction. \square

Therefore, if our CA does not have a quiescent state, we continue the proof with the composition function $F^* \circ F$. We will also require the following classical result in our next section.

Theorem 4.1. [28] *A global function F is surjective iff F is injective when restricted to finite configurations.* \square

4.2 Surjectivity with L-neighbourhood

We now present our version of the *Balance Theorem*, tailored to L-shaped neighborhoods. Proof of this theorem for Moore neighbourhoods has been originally published in [27] and we adapt a portion of the original methodology and use some ideas borrowed from [20, 24, 30].

Theorem 4.2. *A 2D CA with L-neighbourhood is surjective iff for all $n \geq 1$ and all $b \in \mathcal{T}_n$,*

$$\text{card}[\mathbf{f}^{-1}(b)] = N^{n+1}. \quad (4.1)$$

Proof. We will first prove that the balance condition (4.1) is sufficient for surjectivity. Assume that for all $n \geq 1$ and all $b \in \mathcal{T}_n$, $\text{card}[\mathbf{f}^{-1}(b)] = N^{n+1}$. Consider any infinite

configuration, $t \in \mathcal{G}^{\mathbb{Z}^2}$. Define for all $n \geq 1$, the set,

$$S_n = \{s \in \mathcal{G}^{\mathbb{Z}^2} : f(s_{[n+1]}) = t_{[n]}\}.$$

Our assumption guarantees that all S_n are non-empty for $n \geq 1$. We also know that $S_{n+1} \subseteq S_n$. We show that S_n is a clopen set, by considering the complement of S_n :

$$\overline{S_n} = \{s \in \mathcal{G}^{\mathbb{Z}^2} : f(s_{[n+1]}) \neq t_{[n]}\}.$$

We show that $\overline{S_n}$ is open. Let $s \in \overline{S_n} \subset \{0, 1\}^{\mathbb{Z}^2}$ be an arbitrary configuration. For all $\epsilon > 0$, we choose $k \in \mathbb{Z}$, where $k > n$, such that $\frac{1}{k+1} < \epsilon$. We may now pick an infinite configuration $s' \in \{0, 1\}^{\mathbb{Z}^2}$ such that $d(s, s') = \frac{1}{k^*+1}$ (2.2), where $k^* > k$. Since $s \in \overline{S_n}$, we know that $s' \in \overline{S_n}$, and

$$d(s, s') = \frac{1}{k^*+1} < \frac{1}{k+1} < \epsilon.$$

Thus, $\overline{S_n}$ is open, and similar analysis shows that S_n must also be open. We conclude that S_n is a clopen set. By the Nested Set Theorem [26], there must exist an $s \in \mathcal{G}^{\mathbb{Z}^2}$, such that $F(s) = t$, which completes the proof of sufficiency.

In order to show that the balance condition is necessary for surjectivity, we will use induction with respect to n . To make the proof easier to follow, we consider the base case $n = 1$ in Lemma 4.2, while the inductive step is subject to Lemma 4.3.

Lemma 4.2. *If a CA with L-neighbourhood is surjective then $\forall b_0 \in \mathcal{G}$, $\text{card}[f^{-1}(b_0)] = N^2$.*

Proof. Assume that for some symbol $b_0 \in \mathcal{G}$, $\text{card}[f^{-1}(b_0)] \geq N^2 + 1$. Given an integer $k \geq 4$ (to be chosen later), we construct a block b , in which there are k^2 occurrences of b_0 and all other entries are arbitrary (see Figure 4.1a). We denote the set of all possible constructions of block b by \mathcal{B}^* , and note that $\text{card}[\mathcal{B}^*] = N^{2(k^2-k-(k-1))} = N^{2k^2-4k+2}$.

We now consider the preimage of the set \mathcal{B}^* (see Figure 4.1b). We know that

$$\text{card}[f^{-1}(b) : b \in \mathcal{B}^*] = (\text{card}[f^{-1}(b_0)])^{k^2} \geq (N^2 + 1)^{k^2}.$$

Let us now define the *border* of a block, similarly as done in [27]. An element of a block is considered part of the border if there exists an L-neighbourhood that contains both that element and an element which does not belong to the block. This is illustrated in Figure 4.1c, where the border sites are shown as shaded. If we now consider blocks which have the shape of preimages of b (as shown in Figure 4.1b),

then there are $N^{2k+2k+2(k-1)+2(k-2)+1} = N^{8k-5}$ possible borders for blocks of this shape.

If the number of preimage blocks, equal to $(N^2+1)^{k^2}$, is larger than the number of possible borders N^{8k-5} (guaranteed for $k \geq 4$), then there must exist some preimage blocks which share a common border. In fact, there will be a border which is shared by at least $\left\lceil \frac{(N^2+1)^{k^2}}{N^{8k-5}} \right\rceil$ preimage blocks.

We will now demonstrate that for sufficiently large k the above expression exceeds the number of blocks in \mathcal{B}^* , that is,

$$\frac{(N^2+1)^{k^2}}{N^{8k-5}} > N^{2k^2-4k+2}. \quad (4.2)$$

By taking a natural logarithm of both sides and solving the resulting quadratic inequality for k , one finds that the inequality (4.2) is satisfied if if we choose

$$k > \left\lceil \frac{2 \ln N + \sqrt{\ln N [10 \ln N - 3 \ln(N^2+1)]}}{\ln(N^2+1) - 2 \ln N} \right\rceil. \quad (4.3)$$

It can be verified that the right hand side of (4.3) is greater than 4 for all $N \geq 2$.

Therefore, if we choose k satisfying (4.3), then there exists some block b , constructed as shown in Figure 4.1a, which has at least two preimages sharing a common border. Let two of these preimages be c_1 and c_2 , and let their image be b .

Let us define $s_{\{b\}}$ to be a configuration which is constructed by placing the block b on the plane such that its right-angle corner is at the origin, and making all sites not belonging to b to be in the quiescent state. Since $c_1 \neq c_2$, we also have $s_{\{c_1\}} \neq s_{\{c_2\}}$, and, moreover, $F(s_{\{c_1\}}) = F(s_{\{c_2\}})$. Therefore, when restricted to finite configurations, F is not injective, thus by Theorem 4.1, F is not surjective, which is a contradiction. Therefore, for all $b_0 \in \mathcal{G}$, $\text{card}[f^{-1}(b_0)] = N^2$. \square

Lemma 4.3. *If a 2D CA with L -neighbourhood is surjective and if for some $n > 0$ we have $\text{card}[f^{-1}(b)] = N^{n+1}$ for all $b \in \mathcal{T}_n$, then for all $b' \in \mathcal{T}_{n+1}$, $\text{card}[f^{-1}(b')] = N^{n+2}$.*

Proof. Assume that F is surjective. For any $n \geq 1$, if $b \in \mathcal{T}_n$, there must exist $t \in \mathcal{G}^{\mathbb{Z}^2}$ such that b is contained in t . Since F is surjective, there exists $s \in \mathcal{G}^{\mathbb{Z}^2}$, such that $G(s) = t$ and thus $f^{-1}(b)$ is contained in s . Therefore, for all $b \in \mathcal{T}_n$, $\text{card}[f^{-1}(b)] > 0$, and we define

$$m_n = \min (\text{card} [f^{-1}(b)] : b \in \mathcal{T}_n) > 0. \quad (4.4)$$

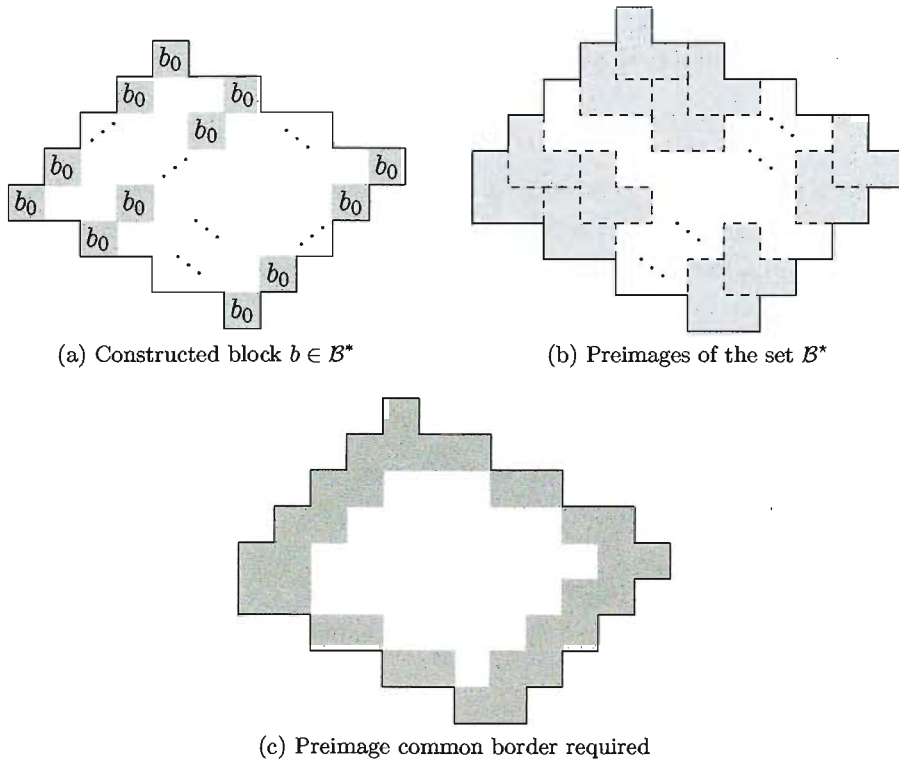


Figure 4.1: Constructions used in $n = 1$ case of the proof of Lemma 4.2

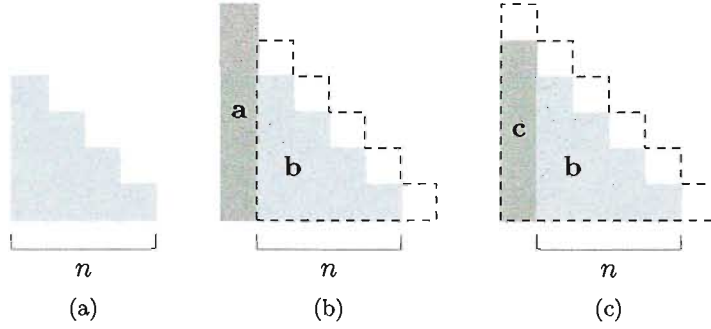


Figure 4.2: Constructions used in the proof of Lemma 4.3, where dashed lines denote preimages of shaded blocks. (a) An element of \mathcal{T}_n . (b) Element of the set $\{a f^{-1}(b)\}$. (c) Element of the set $\{f^{-1}(cb)\}$.

Suppose now that b is a triangular block in \mathcal{T}_n , as shown in Figure 4.2a. We use the notation (cb) to refer to the augmentation of block $b \in \mathcal{T}_n$ with column c of height $(n+1)$, as shown in Figure 4.2b. Comparing Figures 4.2a and 4.2b we see that

$$\sum_a \text{card}[a f^{-1}(b)] = \sum_c \text{card}[f^{-1}(cb)], \quad (4.5)$$

where the sum on the left hand side runs over all possible columns of height $n+2$, and the sum on the right hand side over all possible columns of height $n+1$. By the hypothesis of our Lemma,

$$\sum_a \text{card}[a f^{-1}(b)] = \underbrace{N^{n+2}}_{\# \text{ of choices for } a} \cdot \underbrace{N^{n+1}}_{\text{card}[f^{-1}(b)]} = N^{2n+3}. \quad (4.6)$$

Using (4.4), we obtain

$$\sum_c \text{card}[f^{-1}(cb)] \geq \underbrace{N^{n+1}}_{\# \text{ of choices for } c} \cdot \underbrace{m_{n+1}}_{\text{from (4.4)}} = N^{n+1} m_{n+1}. \quad (4.7)$$

From equations (4.5), (4.6) and (4.7) we can infer that $m_{n+1} \leq N^{n+2}$. Since $\text{card}[\mathcal{T}_{n+1}] = N^{\frac{n^2+3n+2}{2}}$ and the number of all possible preimages of elements of \mathcal{T}_{n+1} is equal to $\text{card}[\mathcal{T}_{n+2}] = N^{\frac{n^2+5n+6}{2}}$, we know that the average number of preimages per element of \mathcal{T}_{n+1} is $\frac{N^{(n^2+5n+6)/2}}{N^{(n^2+3n+2)/2}} = N^{n+2}$. Since the minimum number of preimages is equal to the average number of preimages, we conclude that for all $b' \in \mathcal{T}_{n+1}$, $\text{card}[f^{-1}(b') : b' \in \mathcal{T}_{n+1}] = N^{n+2}$. \square Combining Lemmas 4.2 and 4.3, we can carry out the induction over n , thus completing the proof that for all $n \geq 1$ and all $b \in \mathcal{T}_n$, $\text{card}[f^{-1}(b)] = N^{n+1}$. \square

4.2.1 Non-surjective binary 2D CA

In the remainder of this section we consider only the case when $\mathcal{G} = \{0, 1\}$. There are 256 possible binary CA with L-shaped neighbourhood. We checked each rule for a possible violation of the balance condition (4.1) by finding preimages of all blocks in \mathcal{T}_n , for consecutively increasing n values from $n = 1$. If a block of size n is found which violates the balance condition, then the rule is clearly non surjective and can be eliminated from the list. We performed this procedure for n values from 1 to 6, and identified 217 rules which are non surjective. For these rules we found not only blocks violating balance conditions, but also blocks which have no preimages at all, that is, Garden of Eden blocks.

We say that a rule f has a Garden of Eden at level n , if n is the minimal number such that there exists a block $b \in \mathcal{T}_n$ for which $\text{card}[f^{-1}(b)] = 0$. We now list all 217 aforementioned non-surjective rules ordered by the level of their Garden of Eden using their Wolfram codes (2.6)

Level 1: 0, 255

Level 2: 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 32, 34, 40, 42, 48, 50, 64, 68, 72, 76, 80, 84, 96, 112, 159, 171, 175, 179, 183, 187, 191, 205, 207, 213, 215, 221, 223, 235, 237, 239, 241, 243, 245, 247, 249, 251, 253

Level 3: 1, 3, 5, 7, 9, 11, 13, 17, 19, 21, 24, 27, 29, 31, 33, 35, 36, 39, 47, 49, 53, 55, 59, 63, 65, 66, 69, 71, 79, 81, 83, 87, 93, 95, 111, 115, 117, 119, 123, 125, 126, 127, 128, 129, 130, 132, 136, 138, 140, 143, 144, 160, 162, 168, 172, 174, 176, 184, 186, 189, 190, 192, 196, 200, 202, 206, 208, 216, 219, 220, 222, 224, 226, 228, 231, 234, 236, 238, 242, 246, 248, 250, 252, 254

Level 4: 23, 26, 28, 38, 41, 43, 44, 46, 52, 56, 58, 62, 70, 73, 74, 77, 78, 82, 88, 92, 94, 97, 98, 100, 107, 109, 110, 113, 114, 116, 118, 121, 122, 124, 131, 133, 134, 137, 139, 141, 142, 145, 146, 148, 155, 157, 158, 161, 163, 167, 173, 177, 178, 181, 182, 185, 193, 197, 199, 203, 209, 211, 212, 214, 217, 227, 229, 232

Level 5: 22, 25, 37, 61, 67, 91, 103, 104, 151, 152, 164, 188, 194, 218, 230, 233

We found that all the remaining rules, i.e., those not listed above, share a common property known as permutivity, to be discussed in the next subsection.

4.2.2 Permutivity and surjectivity

Sites belonging to the L-shaped neighbourhood ($\begin{smallmatrix} a_{0,1} \\ a_{0,0} \end{smallmatrix} a_{1,0}$) will be identified by their indices as $(0, 1)$, $(0, 0)$, and $(1, 0)$. Similarly as done in [20], a local function f will be called *permutive* with respect to the $(0, 1)$ site if for any choice of $y, z \in \mathcal{G}$ the

function $x \rightarrow f(\begin{smallmatrix} x \\ y \ z \end{smallmatrix})$ is one-to-one. Permutivity with respect to the central site $(0, 0)$ or the right neighbour $(0, 1)$ is defined similarly. All rules which were not eliminated by the procedure described in the previous section turned out to be permutive with respect to one of these three sites.

In one dimension, it is known that rules permutive with respect to one of the variables located at the left or the right end of the neighbourhood are surjective, as proved in [20]. Recently, this results has been generalized to two dimensions by Dennunzio and Formenti. They demonstrated in [5] that any rule with Moore neighbourhood (of any radius) which is permutive with respect of one of the corner sites is also surjective.

In what follows we show how one can prove a similar result specifically for the L-shaped neighbourhood.

Proposition 4.1. *A binary 2D CA with L-shaped neighbourhood is surjective if its local mapping is permutive with respect to one of the three neighbourhood sites.*

Proof. The proof will assume that the rule is permutive with respect to the central site $(0, 0)$. Proofs for permutivity with respect to $(0, 1)$ or $(1, 0)$ are similar.

Let us first note that if a rule is permutive with respect to $(0, 0)$, then there must exist numbers $x_0, \dots, x_3 \in \{0, 1\}$ such that the local function takes the form

$$f(\begin{smallmatrix} a_0 \\ a_1 \ a_2 \end{smallmatrix}) = \begin{cases} 0 & \text{if } (\begin{smallmatrix} a_0 \\ a_1 \ a_2 \end{smallmatrix}) \in \{ \begin{smallmatrix} 0 \\ x_0 \ 0 \end{smallmatrix}, \begin{smallmatrix} 0 \\ x_1 \ 1 \end{smallmatrix}, \begin{smallmatrix} 1 \\ x_2 \ 0 \end{smallmatrix}, \begin{smallmatrix} 1 \\ x_3 \ 1 \end{smallmatrix} \} \\ 1 & \text{if } (\begin{smallmatrix} a_0 \\ a_1 \ a_2 \end{smallmatrix}) \in \{ \begin{smallmatrix} 0 \\ \bar{x}_0 \ 0 \end{smallmatrix}, \begin{smallmatrix} 0 \\ \bar{x}_1 \ 1 \end{smallmatrix}, \begin{smallmatrix} 1 \\ \bar{x}_2 \ 0 \end{smallmatrix}, \begin{smallmatrix} 1 \\ \bar{x}_3 \ 1 \end{smallmatrix} \} \end{cases}, \quad (4.8)$$

where \bar{x}_i denotes $1 - x_i$. Assuming the above form of f , let us consider an arbitrary block $b \in \mathcal{T}_n$. We will now show how to construct all preimages of b under f . First of all, we claim that blocks $c \in \mathcal{T}_{n+1}$ of the form

$$\begin{array}{ccccccc} & & & & & & \alpha_1 \\ & & & & & & \\ & & & & & & \\ c_{1,n} & & \ddots & & & & \\ \vdots & & \ddots & & \ddots & & \\ & & & & & & \\ c_{1,1} & \dots & c_{n,1} & \alpha_{n+1} & & & \end{array},$$

are the only preimages of b , where each α_i ($1 \leq i \leq n+1$) is an arbitrary value in $\{0, 1\}$, and values of $c_{i,j} \in \{0, 1\}$ can be determined by an iterative algorithm.

To see that this is indeed true, we now present an algorithm with which we can construct all possible preimages. It consists of the following steps:

1. Starting from $b_{1,n}$, we wish to find all neighbourhoods $\begin{smallmatrix} a_0 \\ a_1 \end{smallmatrix} a_2$ such that

$$f \left(\begin{smallmatrix} a_0 \\ a_1 \end{smallmatrix} a_2 \right) = b_{1,n}.$$

The structure of the local mapping gives us four possible such neighbourhoods $\begin{smallmatrix} a_0 \\ a_1 \end{smallmatrix} a_2 = \begin{smallmatrix} \alpha_0 \\ c_{1,n} \end{smallmatrix} \alpha_1$, where

$$c_{1,n} = (1 - b_{1,n})x_{2\alpha_1+\alpha_2} + b_{1,n}(1 - x_{2\alpha_1+\alpha_2}),$$

and the values of α_1 and α_2 are arbitrarily selected.

Now, for all values of $i \in \{2, \dots, n\}$, we repeat step 2 as follows.

2. Since $b_{i,n-i+1}$ is given and α_i has been freely chosen in the previous iteration, we wish to know all neighbourhoods $\begin{smallmatrix} a_0 \\ a_1 \end{smallmatrix} a_2$, such that

$$g_1 \left(\begin{smallmatrix} a_0 \\ a_1 \end{smallmatrix} a_2 \right) = b_{i,n-i+1}.$$

The structure of the local mapping gives us two possible neighbourhoods $\begin{smallmatrix} a_0 \\ a_1 \end{smallmatrix} a_2 = \begin{smallmatrix} \alpha_i \\ c_{i,n-i+1} \end{smallmatrix} \alpha_{i+1}$, where

$$c_{i,n-i+1} = (1 - b_{i,n-i+1})x_{2\alpha_i+\alpha_{i+1}} + b_{i,n-i+1}(1 - x_{2\alpha_i+\alpha_{i+1}}),$$

and α_{i+1} is another arbitrarily selected value.

After the above step is repeated the required number of times, we will have determined all possible configurations for the top two diagonals. The top diagonal, indeed, will consist of $n + 1$ arbitrary values $\alpha_1, \dots, \alpha_{n+1}$.

We now construct the rest of the preimage and show that all other values are uniquely determined based on each choice of the α values in the top diagonal. For all values of $j \in \{1, \dots, n-i\}$ and then for all $i \in \{1, \dots, n-j\}$, we repeat step 3 as follows.

3. Since $b_{i,n-i-j+1}$ is fixed, we wish to know all neighbourhoods $\begin{smallmatrix} a_0 \\ a_1 \end{smallmatrix} a_2$, such that

$$g_1 \left(\begin{smallmatrix} a_0 \\ a_1 \end{smallmatrix} a_2 \right) = b_{i,n-i-j+1}.$$

Since $c_{i,n-i-j+2}$ and $c_{i+1,n-i-j+1}$ were fixed in a previous iteration, the structure of the local mapping tells us that our neighbourhood must have the form $\begin{smallmatrix} a_0 \\ a_1 \end{smallmatrix} a_2 = \begin{smallmatrix} c_{i,n-i-j+2} \\ c_{i,n-i-j+1} \end{smallmatrix} c_{i+1,n-i-j+1}$, where

$$c_{i,n-i-j+1} = (1 - b_{i,n-i-j+1})x_{i'} + b_{i,n-i-j+1}(1 - x_{i'}),$$

and $i' = 2c_{i,n-i-j+2} + c_{i+1,n-i-j+1}$. Note that no new arbitrary parameter appears here, thus the neighbourhood is determined uniquely.

The only arbitrary values in the preimage are the $(n+1)$ values of α_i on the main diagonal. Therefore, we know that there are exactly 2^{n+1} preimages for a given $b \in \mathcal{T}_n$. We may now apply Theorem 4.2 and conclude that any CA whose local mapping is of the form of eq. (4.8) is surjective. \square

We tested for permutivity all rules for which we were not able to find the Garden of Eden, and they all turned out to be permutive, and therefore also surjective. These are rules with code numbers 15, 30, 45, 51, 54, 57, 60, 75, 85, 86, 89, 90, 99, 101, 102, 105, 106, 108, 120, 135, 147, 149, 150, 153, 154, 156, 165, 166, 169, 170, 180, 195, 198, 201, 204, 210, 225, and 240. Since they belong to equivalency classes, we choose only the minimal rule (2.7) and obtain our result.

Proposition 4.2. *Among two-dimensional binary rules with L-shaped neighbourhood, only the following minimal rules are surjective:*

15, 30, 45, 51, 54, 57, 60, 90, 105, 106, 108, 150, 154, 156, 170 and 204.

We remark in passing that this list includes code numbers of all elementary rules which are surjective in one dimension. Code numbers representing rules which are surjective in 2D but not in 1D include only 54, 57, 108, 156, and all these rules are permutive with respect to $(0,0)$.

An important consequence of the Balance Theorem is in relation to the response curves of surjective rules. All blocks have the same number of preimages as any other block of the same size. We can therefore conclude that, in the special case $\rho = 1/2$, all blocks are equally likely to occur. Therefore, the response curve of such rules will always pass through the point $(1/2, 1/2)$. This is confirmed in the experimental curves in Appendix A. We note, however, that for other values of ρ , the asymptotic density is not necessarily equal to $1/2$.

4.3 Surjectivity with von Neumann neighbourhood

We now prove a similar result for two-dimensional cellular automata with von Neumann neighbourhood. The balance theorem in von Neumann neighbourhood can be stated as follows.

Theorem 4.3. *A 2D CA with von Neumann neighbourhood, G , is surjective iff for all $n \geq 1$ and for all von Neumann blocks d of size n , $\text{card}[g^{-1}(d)] = N^{4n}$.*

Figure 4.3: An element of \mathcal{D}_n

Proof. Assume that for all $n \geq 1$ and all $d \in \mathcal{D}_n$, $\text{card}[\mathbf{g}^{-1}(d)] = N^{4n}$. Consider any infinite configuration, $t \in \mathcal{G}^{\mathbb{Z}^2}$. Define for all $n \geq 1$, the set,

$$S_n = \{s \in \mathcal{G}^{\mathbb{Z}^2} : \mathbf{g}(s_{[b]}) = t_{[b]}\},$$

such that $b \in \mathcal{D}_n$ and $b' \in \{\mathbf{g}^{-1}(b)\} \subset \mathcal{D}_{n+1}$. Our assumption guarantees that all S_n are non-empty for $n \geq 1$ and we also know that $S_{n+1} \subseteq S_n$. We now show that S_n is a clopen set. We consider the complement of S_n as follows:

$$\overline{S_n} = \{s \in \mathcal{G}^{\mathbb{Z}^2} : \mathbf{g}(s_{[b']}) \neq t_{[b]}\}.$$

We show that $\overline{S_n}$ is open. Let $s \in \overline{S_n} \subset \{0, 1\}^{\mathbb{Z}^2}$ be an arbitrary configuration. For all $\epsilon > 0$, we choose $k \in \mathbb{Z}$, where $k > n$, such that $\frac{1}{k+1} < \epsilon$. We may now pick an infinite configuration $s' \in \{0, 1\}^{\mathbb{Z}^2}$ such that $d(s, s') = \frac{1}{k^*+1}$, where $k^* > k$. Since $s \in \overline{S_n}$, we know that $s' \in \overline{S_n}$, and

$$d(s, s') = \frac{1}{k^*+1} < \frac{1}{k+1} < \epsilon.$$

Therefore, $\overline{S_n}$ is open. Similar analysis shows that S_n must also be open. Therefore, we conclude that S_n is a clopen set. We now know that, by the Nested Set Theorem [26], there must exist an $s \in \mathcal{G}^{\mathbb{Z}^2}$, such that $G(s) = t$, which completes the proof.

Now, assume that G is surjective. For any $n \geq 1$, if $d \in \mathcal{D}_n$, there must exist $t \in \mathcal{G}^{\mathbb{Z}^2}$ such that d is contained in t . Since G is surjective, there exists $s \in \mathcal{G}^{\mathbb{Z}^2}$, such that $G(s) = t$ and thus $\mathbf{g}^{-1}(d)$ is contained in s . Therefore, for all $d \in \mathcal{D}_n$, $\text{card}[\mathbf{g}^{-1}(d)] > 0$. So we set

$$m_n = \min (\text{card} [\mathbf{g}^{-1}(d)] : d \in \mathcal{D}_n) > 0. \quad (4.9)$$

We now use an induction proof, presented in the following two lemmas.

Lemma 4.4. *If a 2D CA with von neumann neighbourhood is surjective then for any $d_0 \in \mathcal{G}$, $\text{card}[\mathbf{g}^{-1}(d_0)] = N^4$.*

Proof. Assume that for some $d_0 \in \mathcal{G}$, $\text{card}[\mathbf{g}^{-1}(d_0)] \geq N^4 + 1$. Now for some $k \geq 2$ (to be chosen later), we construct a block a , in which there are k^2 occurrences of d_0 and all other entries are arbitrary (see Figure 4.4a). We denote as \mathcal{A}^* , the set of all possible constructions of block a , and note that $\text{card}[\mathcal{A}^*] = N^{4(k-1)^2} = N^{4k^2-8k+4}$.

We now consider the preimage of the set \mathcal{A}^* (see Figure 4.4b). We know that

$$\text{card}[\mathbf{g}^{-1}(a) : a \in \mathcal{A}^*] = (\text{card}[\mathbf{g}^{-1}(d_0)])^{k^2} \geq (N^4 + 1)^{k^2}.$$

There are $N^{8k+8(k-2)+4} = N^{16k-12}$ possible borders (shaded in Figure 4.4b) for the set of preimages of \mathcal{A}^* . An element is considered part of the border if there exists an von Neumann neighbourhood that contains both that element and an element in the quiescent state. There must exist a specific border for which

$$\text{card}[\mathbf{g}^{-1}(a) : a \in \mathcal{A}^*, \text{ sharing specific border}] \geq \frac{(N^4 + 1)^{k^2}}{N^{16k-12}}.$$

We now must find a specific value of k such that

$$\frac{(N^4 + 1)^{k^2}}{N^{16k-12}} > N^{4k^2-8k+4}. \quad (4.10)$$

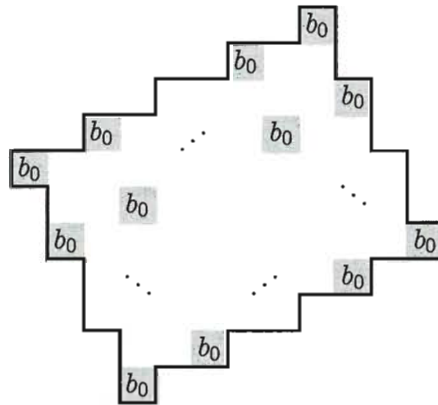
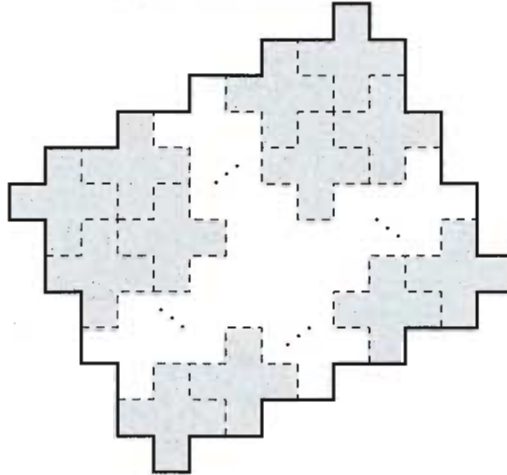
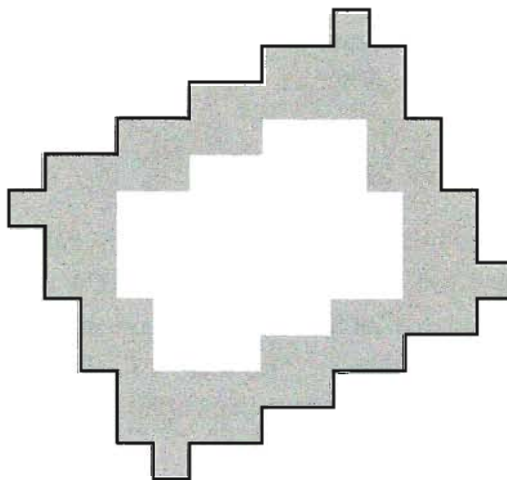
After rearranging and solving the resulting quadratic inequality for k , we discover that if we choose

$$k \gg \left\lceil \frac{4 \ln N + 2\sqrt{2 \ln N [6 \ln N - \ln(N^2 + 1)]}}{\ln(N^2 + 1) - 4 \ln N} \right\rceil, \quad (4.11)$$

(it can be verified that $k \geq 2$ for all $N \geq 2$), then

$$\text{card}[\mathbf{g}^{-1}(a) : a \in \mathcal{A}^*, \text{ sharing specific border}] \geq \text{card}[\mathcal{A}^*] + 1.$$

Therefore, there are at least two distinct preimages, which share a common border, that map to the same image. Let two of these preimages be c_1 and c_2 , let their image be d_1 , and let $s \in \mathcal{G}^{\mathbb{Z}^2}$ be comprised entirely of the quiescent state. We now create finite configurations $s_{[c_1]}, s_{[c_2]}, s_{[d_1]}$, such that $s_{[c_1]} \neq s_{[c_2]}$, and $G(s_{[c_1]}) = G(s_{[c_2]}) = s_{[d_1]}$. Therefore, when restricted to finite configurations, G is not injective. By Theorem 4.1, G is not surjective, which is a contradiction. Therefore, for all $d_0 \in \mathcal{G}$, $\text{card}[\mathbf{g}^{-1}(d_0)] = N^4$. \square

(a) Constructed block $b \in \mathcal{B}^*$ (b) Preimages of the set \mathcal{B}^* 

(c) Preimage common border required

Figure 4.4: Constructions used in $n = 1$ case of the proof of Lemma 4.4

Lemma 4.5. *If a 2D CA with von Neumann neighbourhood is surjective and for all $d \in \mathcal{D}_n$, $\text{card}[\mathbf{g}^{-1}(d)] = N^{4n}$ (for some $n \geq 1$), then for all $d' \in \mathcal{D}_{n+1}$, $\text{card}[\mathbf{g}^{-1}(d')] = N^{4n+4}$.*

Proof. We construct in Figures 4.5a,b, blocks in \mathcal{D}_{n+2} for which we may use a double counting argument. We use the notation (cd) to refer to the augmentation of block $d \in \mathcal{D}_n$ with diamond border c of height $(2n+1)$ as in Figure 4.5a. We know that

$$\sum_a \text{card}[a \mathbf{g}^{-1}(d)] = \sum_c \text{card}[\mathbf{g}^{-1}(cd)]. \quad (4.12)$$

By our induction assumption we know that

$$\sum_a \text{card}[a \mathbf{g}^{-1}(d)] = \underbrace{N^{2(2n+1)+2}}_{\# \text{ of choices for } a} \cdot \underbrace{N^{4n}}_{\text{card}[\mathbf{g}^{-1}(d)]} = N^{8n+4}. \quad (4.13)$$

From equation 4.9, we know that

$$\sum_c \text{card}[\mathbf{g}^{-1}(cd)] \geq \underbrace{N^{2(2n-1)+2}}_{\# \text{ of choices for } c} \cdot \underbrace{m_{n+1}}_{\text{from (4.9)}} = N^{4n} m_{n+1}. \quad (4.14)$$

From equations 4.12, 4.13 and 4.14, we can conclude that $m_{n+1} \leq N^{4n+4}$. Since $\text{card}[\mathcal{D}_{n+1}] = N^{2n^2+2n+1}$, and $\text{card}[\mathbf{g}^{-1}(d) : d \in \mathcal{D}_{n+1}] = N^{2n^2+6n+5}$, we know that the average number of preimages per element of \mathcal{D}_{n+1} is N^{4n+4} . Since the minimum number of preimages must be less than the average number of preimages, we conclude that for all $d' \in \mathcal{D}_{n+1}$, $\text{card}[\mathbf{g}^{-1}(d') : d' \in \mathcal{D}_{n+1}] = N^{4n+4}$. \square

Combining the results in Lemmas 4.4 and 4.5, we can conclude that for all $n \geq 1$ and all $d \in \mathcal{D}_n$, $\text{card}[\mathbf{g}^{-1}(d)] = N^{4n}$. \square

4.3.1 Surjective binary 2D CA

We attempted to determined the list of all surjective rules for the case of $\mathcal{G} = \{0, 1\}$. From the $2^{2^5} = 4,294,967,296$ possible binary CA rules with von Neumann neighbourhood, we eliminated all non-surjective ones by computerized search for violation of the balance condition of Theorem 4.3. This left 260766 candidates for surjectivity, and among those, 260668 ($\approx 99.99\%$) are permutive with respect to $(0, 1)$, $(0, -1)$, $(1, 0)$ or $(-1, 0)$.

Proposition 4.3. *A 2D CA with von Neumann neighbourhood is surjective if its local mapping is permutive with respect to $(0, 1)$, $(0, -1)$, $(1, 0)$ or $(-1, 0)$.*

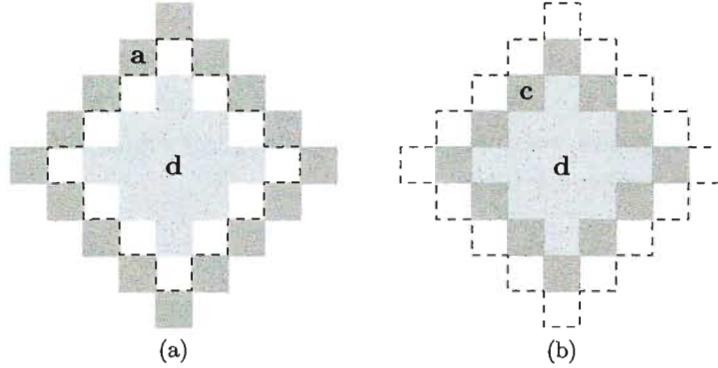


Figure 4.5: Constructions used in the induction step of the proof of Lemma 4.5, where dashed lines again denote preimages of shaded blocks. Figure 4.5a shows the set $\{a \mathbf{g}^{-1}(d)\}$, and Figure 4.5b shows the set $\{\mathbf{g}^{-1}(cd)\}$.

Proof. We will present an outline of the proof assuming permutivity with respect to $(0, -1)$. In this case, the local function has the form

$$g_1 \left(\begin{smallmatrix} a_1 \\ a_2 \ a_3 \ a_4 \\ a_5 \end{smallmatrix} \right) = \begin{cases} 1 & \text{if } \left(\begin{smallmatrix} a_1 \\ a_2 \ a_3 \ a_4 \\ a_5 \end{smallmatrix} \right) = \left\{ \begin{smallmatrix} * & * & * \\ * & x_i & * \end{smallmatrix} \right\}_{i=0}^{15} \\ 0 & \text{if } \left(\begin{smallmatrix} a_1 \\ a_2 \ a_3 \ a_4 \\ a_5 \end{smallmatrix} \right) = \left\{ \begin{smallmatrix} * & * & * \\ * & \bar{x}_i & * \end{smallmatrix} \right\}_{i=0}^{15} \end{cases}, \quad (4.15)$$

where $x_i \in \mathcal{G}$ is determined for each of the 16 choices of the four arbitrary values denoted by $*$. Now, consider a block $d \in \mathcal{D}_n$

$$\begin{array}{ccccc} & & d_{0,(n-1)} & & \\ & \ddots & \vdots & \ddots & \\ d_{-(n-1),0} & \cdots & d_{0,0} & \cdots & d_{(n-1),0} \\ & \ddots & \vdots & \ddots & \\ & & d_{0,-(n-1)} & & \end{array}$$

We will show how to construct the set of all blocks $c \in \mathcal{D}_{n+1}$, such that $\mathbf{g}^{-1}(d) = c$.

1. Since we know value of $d_{0,(n-1)}$, the symmetry of the local mapping allows us to conclude that the top of c is of the following form:

$$\begin{array}{ccccc} & & * & & \\ & * & & * & \\ c_{0,(n-2)} & & & & \end{array}$$

If $n = 1$, we are finished, so let's assume $n \geq 2$ and proceed. After each step, we will select one particular value for each arbitrary entry, and denote this fixed choice as \star . We note, however, that in order to obtain the set of all possible preimages, every subsequent step should be repeated for each possible choice of the arbitrary value.

2. Since we know the value of $d_{0,(n-2)}$, we know that top part of the preimage must be

$$\begin{array}{ccccc} & & \star & & \\ & \star & & \star & \\ \star & c_{0,(n-2)} & & \star & \\ & & c_{0,(n-3)} & & \end{array}$$

3. Since we know the value of $d_{\pm 1,(n-2)}$, we may determine more entries of the preimage as follows

$$\begin{array}{ccccccc} & & & \star & & & \\ & & \star & & \star & & \star \\ \star & & \star & & c_{0,(n-2)} & & \star \\ & c_{-1,(n-3)} & & c_{0,(n-3)} & & c_{1,(n-3)} & \end{array}$$

Now, if $n = 2$, we proceed to step 5 to complete the preimages. If $n \geq 3$, we repeat step 4 for each value of j from $(n - 3)$ to 0.

4. We know all values $d_{i,j}$, where $|i| \leq (n - j)$, and all values $c_{i',j-1}$, where $|i'| \leq (n - (j - 1))$, which have been determined in a previous iteration. By continuing the same process as in steps 2 and 3 (i.e., stepping down one place in the middle and then working horizontally toward the left and right edges), we can determine the values of row $j - 1$ in the preimage as follows

- (a) if $j \geq 1$, then we obtain

$$\begin{array}{ccccccccccccccc} & & \star & & \star & & c_{-(n-j-2),j} & \cdots & c_{0,j} & \cdots & c_{(n-j-2),j} & & \star & & \star \\ \star & & \star & c_{-(n-j-1),j-1} & & c_{-(n-j-2),j-1} & \cdots & c_{0,j-1} & \cdots & c_{(n-j-2),j-1} & & c_{(n-j-1),j-1} & \star & & \star \end{array}$$

- (b) if $j = 0$, then we obtain

$$\begin{array}{ccccccccccccccc} \star & & \star & & c_{-(n-j-2),j} & \cdots & c_{0,j} & \cdots & c_{(n-j-2),j} & & \star & & \star \\ & c_{-(n-j-1),j-1} & & c_{-(n-j-2),j-1} & \cdots & c_{0,j-1} & \cdots & c_{(n-j-2),j-1} & & c_{(n-j-1),j-1} & & & \end{array}$$

We note that, in this case, since the next row down is shorter in length, we did not introduce any arbitrary elements in the row $j = -1$.

5. We now complete the lower half of the preimage. We proceed in the same manner as before, by traveling down one place in the centre and then working horizontally to the left and right edges. We note that now, however, since the rows are becoming shorter as we progress (as in step 4b), all remaining elements of the preimage will be uniquely determined.

We may now count the number of possible preimages by counting the number of arbitrary sites in our preimage block c . For each value of j , such that $0 \leq j \leq (n-2)$, there are four arbitrary values in row j . By adding the four arbitrary values in the top two rows, we get, for all $n \geq 1$, $\text{card}[\mathbf{g}_1^{-1}(a)] = 2^{4(n-1)+4} = 2^{4n}$. Therefore, by Theorem 4.3, any CA whose local mapping is of the form described in (4.15) must be surjective. \square

By the virtue of Proposition 4.3, all 260668 aforementioned permutive rules are thus surjective. Another 34 are equivalent to *L-shaped* rules permutive with respect to $(0,0)$, and therefore surjective too. This leaves 64 unclassified rules. These, however, are not all truly different, as many of them are related to each other by a symmetry transformation (subsection 2.2.2). These transformations split the set of 64 rules mentioned above into 4 equivalence classes, whose four “minimal” representatives have code numbers 381926505, 636065370, 768833670, 1019451029. We performed some heuristic surjectivity tests on these four rules, and we are strongly convinced that they are surjective. However, since they are not permutive with respect to any site, we were not able to prove their surjectivity rigorously.

α -Asynchronous Cellular Automata

Any deterministic cellular automata model assumes that the local function will always be applied to every site at every time step. This is not, however, how many physical systems behave. Probabilistic and asynchronous cellular automata assume that some sites will be updated and some will remain constant. Probabilistic CA were originally applied to Ising spin models [6], but have been used in many applications, including disease spread models [2, 3].

There are many ways to apply an update rule. In some cases, a single site is chosen to be updated, while all others remain constant [12]. This fully asynchronous update scheme is applied to a lattice with periodic boundary conditions. A variety of other update methods are described in [4]. We will consider an update scheme in which each block may be independently updated with some probability α at any given time step. This technique allows us to consider a smooth transition from fully synchronous behaviour when $\alpha = 1$ to fully asynchronous behaviour as $\alpha \rightarrow 0$.

In [10], the authors considered the same class of probabilistic rules, defined on a periodic lattice of length $n = 100$. They present experimental response curves from a specific initial lattice. They describe the sensitivity to asynchronicity for each rule - that is, the extent to which the dynamics of the rule are dependent on the level of asynchronicity. Of our three examples, they present experimental curves for Rule 200A (which they denote E), and Rule 206A (denoted B), which is in the same equivalency class as Rule 140A.

We approach this problem using similar techniques as in Chapter 3. We consider sets of preimages and describe their structure to determine the density of ones after an arbitrary number of iterations. We provide an in depth analysis of 3 rules, those which differ by only a single basic block output from the identity rule. We also apply the Local Structure Theory to these rules and note that the Level 3 Local Structure equations provides an excellent approximation to the true dynamics of the rule.

5.1 Definitions

We define our probabilistic cellular automata in terms of a related deterministic cellular automata, using a random variable and transition probabilities.

If we denote by $s_i(t)$ the value of site i at time t , then we may write the value of site i at the next time step as a random variable X such that

$$s_i(t+1) = X_{i,s_{i-1}(t),s_i(t),s_{i+1}(t)}.$$

If f is a local mapping of a 1D deterministic CA as defined in Section 2.1, then the related 1D asynchronous CA is defined as the value of this random variable, so that

$$X_{i,s_{i-1}(t),s_i(t),s_{i+1}(t)} = \begin{cases} f(s_{i-1}(t), s_i(t), s_{i+1}(t)) & \text{with probability } \alpha \\ s_i(t) & \text{with probability } 1 - \alpha \end{cases}, \quad (5.1)$$

for each block $s_{i-1}(t)s_i(t)s_{i+1}(t) \in \mathcal{B}_3$. We define a *1-step transition probability*,

$$\omega(s_i(t+1)|s_{i-1}(t)s_i(t)s_{i+1}(t)), \quad (5.2)$$

as the conditional probability that a site $s_i(t)$ with nearest neighbours $s_{i-1}(t)$, $s_i(t)$, and $s_{i+1}(t)$ changes its state to $s_i(t+1)$ in a single time step.

We will refer to this asynchronous CA as Rule WA, where W is the Wolfram number of f as defined in (2.4).

We now define a *1-step block transition probability* ω so that, for any $b = b_0b_1 \dots b_rb_{r+1} \in \mathcal{B}_{r+2}$ and any $c = c_1c_2 \dots c_{r-1}c_r \in \mathcal{B}_r$,

$$\omega(c|b) = \prod_{i=1}^r \omega(c_i|b_{i-1}b_ib_{i+1}). \quad (5.3)$$

We also define a *n -step block transition probability* ω recursively, so that, when $n \geq 2$ and for any block $b \in \mathcal{B}_{r+2n}$ and any block $c \in \mathcal{B}_r$,

$$\omega^n(c|b) = \sum_{b' \in \mathcal{B}_{r+2n-2}} \omega(b'|b) \omega^{n-1}(c|b'), \quad (5.4)$$

which may be written explicitly as

$$\omega^n(c|b) = \sum_{\substack{b_{n-1} \in \mathcal{B}_{r+2(n-1)} \\ \vdots \\ b_1 \in \mathcal{B}_{r+2}}} \omega(c|b_2) \left(\prod_{i=1}^{n-2} \omega(b_i|b_{i+1}) \right) \omega(b_{n-1}|b). \quad (5.5)$$

5.2 Response Surface

We wish now to derive a formula for the probability of occurrence of block $b \in \mathcal{B}_r$ after application of the 1-step block operator. We must consider the possibility that block b was created from any of the 2^{r+2} blocks in \mathcal{B}_{r+2} . We assume that the elements of these preimage blocks were chosen independently with Bernoulli measure. Since this measure is translationally invariant, we can write the probability of occurrence of block b after application of the 1-step block operator as

$$P_1(b) = \sum_{a \in \mathcal{B}_{r+2}} P_0(a) \omega(b|a). \quad (5.6)$$

Similarly, we can also write the probability of occurrence of block b after an iteration of the n -step block operator as

$$P_n(b) = \sum_{a \in \mathcal{B}_{r+2n}} P_0(a) \omega^n(b|a). \quad (5.7)$$

where the transition probability is defined in (5.4) or (5.5).

Since many of these transition probabilities will be zero, we define, for any block $b \in \mathcal{T}_r$, the n -step preimage set $\mathbf{f}^{-n}(b)$, to be the set of blocks $a \in \mathcal{T}_{r+2n}$ such that $\omega^n(b|a) > 0$. That is, the set of blocks that can be mapped to b under the n -step block evolution operator with some non-zero probability. Then we can write (5.7) as

$$P_n(b) = \sum_{a \in \mathbf{f}^{-n}(b)} P_0(a) \omega^n(b|a). \quad (5.8)$$

We note that our initial probability depends on some parameter $\rho \in [0, 1]$, and our transition probability depends on some parameter $\alpha \in [0, 1]$. Therefore, we may consider (5.8) to be a function from $[0, 1]^2 \rightarrow [0, 1]$. We denote such a relation to be a *response surface*. When we consider the special case of $\rho = 1/2$, (5.8) is a function from $[0, 1] \rightarrow [0, 1]$. We denote such a relation to be a *symmetric response curve* and use the notation $P_n^{(s)}(b)$. We also note that taking the limit of (5.8) as $n \rightarrow \infty$ (if such a limit exists) provides us with a formula for the asymptotic density of ones, $P(b)$, that is, the probability of block b after a large number of iterations.

5.2.1 Theoretical Response Surfaces

We now consider the class of α -asynchronous rules in which $\omega(1|b) = 0, 1$ for all but one basic block. There are eight such rules, but due to symmetry, only 3 are

minimal. For these 3 examples, we now attempt to find explicit formulae for the response curves of both a single cell in state 1 and response curves for each of the eight basic blocks. In the first two cases, we were able to completely solve explicitly for all response curves. In the third case, however, we were unable to write the formulae for the transition probabilities in a closed form. We start with the rule with the simplest preimage set structure, Rule 200A.

Example: Rule 200A

Consider an α -asynchronous rule defined as

$$\omega(1|b) = \begin{cases} 0 & \forall b \in \{000, 001, 100, 101\} \\ 1 & \forall b \in \{011, 110, 111\} \\ 1 - \alpha & \forall b \in \{010\} \end{cases}. \quad (5.9)$$

Note that if $\alpha = 1$, then this rule is equivalent to deterministic Rule 200 (2.4). We will then refer to the asynchronous rule defined in (5.9) as Rule 200A. For ease of notation and future calculations, we will let $\beta = 1 - \alpha$.

We wish to find a response surface for Rule 200A and to apply (5.8), we begin by finding the set of all potential preimages blocks and their respective transition probabilities.

Proposition 5.1. *The set $\mathbf{f}^{-n}(1)$ consists of all blocks of the form*

$$\{ \underbrace{\star \cdots \star}_n 1 \underbrace{\star \cdots \star}_n \}.$$

Proof. From (5.9), we can see that an element in state 0 will always remain in state 0, so any block in the complement of our conjectured set will never be transformed to a single 1 under n iterations of Rule 200A. Similarly, a block in our set could transform to a single 1, with some non-zero probability. \square

We now define the following subset of $\mathbf{f}^{-n}(1)$, $B_n = \{ \underbrace{\star \cdots \star}_{n-1} 010 \underbrace{\star \cdots \star}_{n-1} \}.$

Proposition 5.2. *For any block $b \in \{\mathbf{f}^{-n}(1) \setminus B_n\}$, we have $\omega^n(1|b) = 1$.*

Proof. In the set $\{\mathbf{f}^{-n}(1) \setminus B_n\}$, we will always have a centre component block either 011, 110 or 111. From (5.9), we can see that these blocks will always be preserved under application of Rule 200A with probability 1. \square

Proposition 5.3. *For any block $b \in B_n$, we have $\omega^n(1|b) = \beta^n$.*

Proof. In each iteration, the 0s in the centre block will be preserved with probability 1, so we need only consider the transition $010 \rightarrow 1$, which occurs in each iteration of Rule 200A with probability β . \square

We may now use (5.8) and considering the sets and transition probabilities described in Propositions 5.2 and 5.3, we conclude that

$$\begin{aligned}
 P_n(1) &= \sum_{b^* \in f^{-n}(1) \setminus B_n} P_0(b^*) \omega^n(1|b^*) + \sum_{b^* \in B_n} P_0(b^*) \omega^n(1|b^*) \\
 &= 1 \cdot (2\rho^2(1-\rho) + \rho^3) + \beta^n \cdot \rho(1-\rho)^2 \\
 &= \rho^2(2-\rho) + \beta^n \rho(1-\rho)^2.
 \end{aligned} \tag{5.10}$$

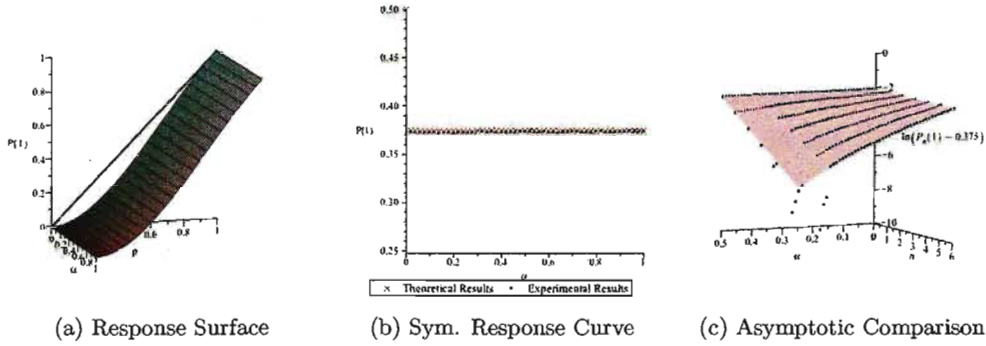


Figure 5.1: Rule 200A - Graphs

Therefore, the asymptotic density of ones, (Figure 5.1a), is given by

$$P(1) = \lim_{n \rightarrow \infty} P_n(1) = \begin{cases} \rho & \text{if } \alpha = 0 \\ \rho^2(2-\rho) & \text{if } \alpha \in (0, 1] \end{cases}.$$

When $\rho = 1/2$, the symmetric response curve is given by

$$P_n^{(s)}(1) = \frac{3}{8} + \frac{1}{8}\beta^n. \tag{5.11}$$

The theoretical symmetric response curve is plotted in Figure 5.1b together with experimental results obtained with an array of length 20000, iterated 100000/ α times with $\alpha > 0.1$ and 1000000 times, with periodic boundary conditions, averaged

over 100 runs. There is a close agreement between the theoretical and experimental results. We can confirm the asymptotic behaviour of (5.11) as follows

$$P_n(1) - \frac{3}{8} = \frac{1}{8}\beta^n$$

$$\ln\left(P_n(1) - \frac{3}{8}\right) = \ln\left(\frac{1}{8}\right) + n \ln(\beta). \quad (5.12)$$

In Figure 5.1c, we plot $\ln(P_n(1) - \frac{3}{8})$, both experimentally with an array of 200000 elements iterated step-by-step to $n = 6$, and theoretically from (5.12). We notice excellent agreement for values of $\alpha \leq 0.4$. When $\alpha > 0.5$, we also have excellent agreement for $n = 1$. Otherwise, it appears that the speed of convergence for large values of α prevents accuracy of the logarithm plot.

Basic Blocks For Rule 200A, we were also able to find explicit formulae for the response surfaces for each of the eight basic blocks. We once again use (5.8). In Table 5.1, the set of all n -step preimage blocks of each basic block are shown, with their corresponding initial probabilities and respective transition probabilities.

Table 5.1 can be used to find formulae of basic block response surfaces, such as

$$P_n(000) = \rho^2(1 - \rho)^2(1 + \rho) + \rho(1 - \rho)^2(1 - 2\rho^2)\beta^n - \rho^2(1 - \rho)^3\beta^{2n}.$$

In the special case when $\rho = 1/2$, we have the following n -step and asymptotic probabilities, assuming that $\alpha \neq 0$,

$P_n^{(s)}(000) = \frac{13}{32} - \frac{5}{16}\beta^n + \frac{1}{32}\beta^{2n},$	$P^{(s)}(000) = 13/32,$
$P_n^{(s)}(001) = \frac{3}{32} + \frac{1}{16}\beta^n - \frac{1}{32}\beta^{2n},$	$P^{(s)}(001) = 3/32,$
$P_n^{(s)}(010) = \frac{1}{8}\beta^n,$	$P^{(s)}(010) = 0,$
$P_n^{(s)}(011) = \frac{1}{8},$	$P^{(s)}(011) = 1/8,$
$P_n^{(s)}(100) = \frac{3}{32} + \frac{1}{16}\beta^n - \frac{1}{32}\beta^{2n},$	$P^{(s)}(100) = 3/32,$
$P_n^{(s)}(101) = \frac{1}{32} + \frac{1}{16}\beta^n + \frac{1}{32}\beta^{2n},$	$P^{(s)}(101) = 1/32,$
$P_n^{(s)}(110) = \frac{1}{8},$	$P^{(s)}(110) = 1/8,$
$P_n^{(s)}(111) = \frac{1}{8},$	$P^{(s)}(111) = 1/8.$

Table 5.1: Rule 200A - Initial and Transition Probabilities of Basic Blocks

$b \in \mathcal{B}_3$	$b^* \in \mathbf{f}^{-n}(b)$	$\omega^n(b b^*)$	$P_0(b^*)$
000	*...*01010*...*	$1 - 2\beta^n + \beta^{2n}$	$\rho^2(1 - \rho)^3$
	*...*0010*...*	$1 - \beta^n$	$\rho(1 - \rho)^3$
	*...*010*...*	$1 - \beta^n$	$\rho(1 - \rho)^2$
	*...*0100*...*	$1 - \beta^n$	$\rho(1 - \rho)^3$
	*...*000*...*	1	$(1 - \rho)^3$
001	*...*0011*...*	1	$\rho^2(1 - \rho)^2$
	*...*01011*...*	$1 - \beta^n$	$\rho^3(1 - \rho)^2$
	*...*0010*...*	β^n	$\rho(1 - \rho)^3$
	*...*01010*...*	$\beta^n - \beta^{2n}$	$\rho^2(1 - \rho)^3$
010	*...*010*...*	β^n	$\rho(1 - \rho)^2$
011	*...*011*...*	1	$\rho^2(1 - \rho)$
100	*...*1100*...*	1	$\rho^2(1 - \rho)^2$
	*...*11010*...*	$1 - \beta^n$	$\rho^3(1 - \rho)^2$
	*...*0100*...*	β^n	$\rho(1 - \rho)^3$
	*...*01010*...*	$\beta^n - \beta^{2n}$	$\rho^2(1 - \rho)^3$
101	*...*01010*...*	β^{2n}	$\rho^2(1 - \rho)^3$
	*...*11010*...*	β^n	$\rho^3(1 - \rho)^2$
	*...*01011*...*	β^n	$\rho^3(1 - \rho)^2$
	*...*11011*...*	1	$\rho^4(1 - \rho)$
110	*...*011*...*	1	$\rho^2(1 - \rho)$
111	*...*111*...*	1	ρ^3

Example: Rule 140A

Consider an α -asynchronous rule defined as

$$\omega(1|b) = \begin{cases} 0 & \forall b \in \{000, 001, 100, 101\} \\ 1 & \forall b \in \{010, 011, 111\} \\ 1 - \alpha & \forall b \in \{110\} \end{cases} \quad (5.13)$$

Note that if $\alpha = 1$, then this rule is equivalent to deterministic Rule 140 (2.4). We will then refer to the asynchronous rule defined in (5.13) as Rule 140A.

We now find the set of all potential preimage blocks and their respective transition probabilities.

Proposition 5.4. *The set $\mathbf{f}^{-n}(1)$ consists of all blocks of the form*

$$\{ \underbrace{\star \cdots \star}_n 1 \underbrace{\star \cdots \star}_n \}.$$

Proof. From (5.13), we can see that an element in state 0 will always remain in state 0, so that for any block $b' \in \mathcal{B}_{2n+1} \setminus \mathbf{f}^{-n}(1)$, we have $\omega^n(1|b') = 0$. under Rule 140A. A block in $\mathbf{f}^{-n}(1)$ however, could transform to a single 1 with some non-zero probability. \square

To determine our transition probabilities, we must consider subsets of the set of potential preimage blocks. We start by defining $C_n^k \subset \mathbf{f}^{-n}(1)$ to be the set of blocks of the form

$$\{ \underbrace{\star \cdots \star}_{n-1} 1 \quad \underline{1} \quad \underbrace{1 \cdots 1}_{k-1} 0 \underbrace{\star \cdots \star}_{n-k} \},$$

where $1 \leq k \leq n+1$. Intuitively, the value of k refer to the place (as counted to the right of the underlined centre 1) in which the first occurrence of 0 is located. We also define the set

$$C_n = \bigcup_{k=0}^{n+1} C_n^k = \{ \underbrace{\star \cdots \star}_{n-1} 1 \underline{1} \underbrace{\star \cdots \star}_n \},$$

and note that the compliment of C_n within the set of potential preimages is given by

$$\mathbf{f}^{-n}(1) \setminus C_n = \{ \underbrace{\star \cdots \star}_{n-1} 0 \underline{1} \underbrace{\star \cdots \star}_n \}.$$

Proposition 5.5. *For any block $c^* \in \mathbf{f}^{-n}(1) \setminus C_n$ we have $\omega^n(1|c^*) = 1$.*

Proof. From (5.13), the centre block $0\underline{1}$ will be preserved for the first $(n-1)$ -steps with probability 1. Finally, any block $0\underline{1}\star$ will be transformed to a single 1 with probability 1. \square

Proposition 5.6. *For any block $c \in C_n^k$, we have*

$$\omega^n(1|c) = \begin{cases} \beta^n & \text{if } k = 1 \\ \beta^n \left(\frac{\alpha}{\beta} \right)^{k-1} \binom{n-1}{k-1} + \beta^{n-k+1} \sum_{j=0}^{k-2} \binom{n-k+j}{j} \alpha^j & \text{if } 2 \leq k \leq n. \\ 1 & \text{if } k = n+1 \end{cases}$$

Proof. For ease of calculations, we will, for any block $c \in C_n^k$, use the notation $\gamma_n^k = \omega^n(1|c)$. To calculate this transition probability, we will first write a formula

for n -step transition probability recursively in terms of possible $(n-1)$ -step transition probabilities. We do so by cases on the value of k

1. When $k = 1$, consider the following transition

$$\begin{array}{ccccccc|ccc} \star & \cdot & \cdot & \star & 1 & 1 & 0 & \star & \cdot & \cdot & \star \\ ? & \cdot & ? & 1 & 1 & 0 & ? & \cdot & ? & & \end{array}$$

The shaded transition will occur with probability β .

2. When $2 \leq k \leq n$, consider the following transition

$$\begin{array}{ccccccc|ccc} \star & \cdot & \cdot & \star & 1 & 1 & \cdot & \cdot & 1 & 1 & 0 & \star & \cdot & \cdot & \star \\ ? & \cdot & ? & 1 & 1 & 1 & \cdot & \cdot & 1 & x & 0 & ? & \cdot & ? & \end{array}$$

We know that $x = 1$ with probability β , resulting in a block in C_{n-1}^k , and $x = 0$ with probability α , resulting in a block in C_{n-1}^{k-1} .

3. When $k = n + 1$, consider the following transition

$$\begin{array}{cccc|c|ccc} \star & \cdot & \cdot & \star & 0 & 1 & 1 & \cdot & \cdot & \cdot & 1 \\ ? & \cdot & ? & 0 & 1 & 1 & \cdot & \cdot & 1 & & \end{array}$$

which will occur with probability 1.

Combining these cases, we obtain the following recursive formula

$$\gamma_n^k = \begin{cases} \beta\gamma_{n-1}^1 & \text{if } k = 1 \\ \alpha\gamma_{n-1}^{k-1} + \beta\gamma_{n-1}^k & \text{if } 2 \leq k \leq n \\ 1 & \text{if } k = n + 1 \end{cases} \quad (5.14)$$

We may then solve this recursive formula to obtain our desired formula. When $k = 1$ or $k = n + 1$, our formula follows trivially from Proposition 5.14. When $2 \leq k \leq n$, our formula can be proved by induction on n . When $n = 2$, we only have the case when $k = 2$, where

$$\gamma_2^2 = \beta^1\alpha^1\binom{1}{1} + \beta^1\sum_{j=0}^0\binom{j}{j}\alpha^j = \beta\alpha + \beta = 1 - \alpha^2.$$

Now, we consider the following inductive step when $3 \leq k \leq n$,

$$\begin{aligned}
\gamma_n^k &= \alpha \gamma_{n-1}^{k-1} + \beta \gamma_{n-1}^k \\
&= \alpha \left[\beta^{n-1} \left(\frac{\alpha}{\beta} \right)^{k-2} \binom{n-2}{k-2} + \beta^{n-k+1} \sum_{j=0}^{k-3} \binom{n-k+j}{j} \alpha^j \right] + \\
&\quad + \beta \left[\beta^{n-1} \left(\frac{\alpha}{\beta} \right)^{k-1} \binom{n-2}{k-1} + \beta^{n-k} \sum_{j=0}^{k-2} \binom{n-k+j-1}{j} \alpha^j \right] \\
&= \alpha \beta^{n-1} \left(\frac{\alpha}{\beta} \right)^{k-2} \binom{n-2}{k-2} + \beta \beta^{n-1} \left(\frac{\alpha}{\beta} \right)^{k-1} \binom{n-2}{k-1} \\
&\quad + \alpha \beta^{n-k+1} \sum_{j=0}^{k-3} \binom{n-k+j}{j} \alpha^j + \beta \beta^{n-k} \sum_{j=0}^{k-2} \binom{n-k+j-1}{j} \alpha^j \\
&= \beta^n \left(\frac{\alpha}{\beta} \right)^{k-1} \binom{n-1}{k-1} + \beta^{n-k+1} \left[\sum_{j=1}^{k-2} \binom{n-k+j-1}{j-1} \alpha^j \right. \\
&\quad \left. + \beta + \beta \sum_{j=1}^{k-2} \binom{n-k+j-1}{j} \alpha^j \right] \\
&= \beta^n \left(\frac{\alpha}{\beta} \right)^{k-1} \binom{n-1}{k-1} + \beta^{n-k+1} \sum_{j=0}^{k-2} \binom{n-k+j}{j} \alpha^j.
\end{aligned}$$

A similar procedure is used to prove the formula when $k = 2$, thus completing the proof. \square

We may now use (5.8) and considering the sets and transition probabilities described in Propositions 5.5 and 5.6, we conclude that

$$\begin{aligned}
P_n(1) &= \sum_{c^* \in \mathbf{f}^{-n}(1) \setminus C_n} P_0(c^*) \omega^n(1|c^*) + \sum_{c^* \in C_n} P_0(c^*) \omega^n(1|c^*) \\
&= \rho(1-\rho) + \rho^2(1-\rho)\beta^n + \rho^2(1-\rho) \sum_{k=2}^n \rho^{k-1} \gamma_n^k + \rho^{n+2}, \tag{5.15}
\end{aligned}$$

where

$$\sum_{k=2}^n \rho^{k-1} \gamma_n^k = \beta^n \sum_{k=2}^n \binom{n-1}{k-1} \left(\frac{\rho\alpha}{\beta} \right)^{k-1} + \sum_{k=2}^n \sum_{j=0}^{k-2} \binom{n-k+j}{j} \alpha^j \left(\frac{\rho}{\beta} \right)^{k-1}. \tag{5.16}$$

To simplify (5.15) and (5.16), we may now use the following two summation identities.

Lemma 5.1.

$$\sum_{k=2}^n \binom{n-1}{k-1} \left(\frac{\alpha\rho}{\beta} \right)^{k-1} = -1 + \left(1 + \frac{\alpha\rho}{\beta} \right)^{n-1}.$$

Proof. We use the binomial identity as follows

$$-1 + \left(1 + \frac{\alpha\rho}{\beta} \right)^{n-1} = -1 + \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{\alpha\rho}{\beta} \right)^k = \sum_{k=2}^n \binom{n-1}{k-1} \left(\frac{\alpha\rho}{\beta} \right)^{k-1}.$$

□

Lemma 5.2. When $\rho \neq 1$ and $\alpha \neq 0$,

$$\beta^n \sum_{k=2}^n \sum_{j=0}^{k-2} \binom{n-k+j}{j} \alpha^j \left(\frac{\rho}{\beta} \right)^{k-1} = \frac{\rho}{1-\rho} [(\beta + \rho\alpha)^{n-1} - \rho^{n-1}]. \quad (5.17)$$

Proof. We prove this identity by induction. When $n = 2$ both sides of the identity equal $\rho\beta$. If we denote by $h(n)$ the LHS of (5.17), then $h(n+1)$ is given by

$$\begin{aligned} & \beta^{n+1} \sum_{k=2}^{n+1} \sum_{j=0}^{k-2} \binom{n+1-k+j}{j} \alpha^j \left(\frac{\rho}{\beta} \right)^{k-1} \\ &= \rho\beta^n \sum_{m=1}^n \sum_{j=0}^{m-1} \binom{n-m+j}{j} \alpha^j \left(\frac{\rho}{\beta} \right)^{m-1} \quad \text{where } m = k-1 \\ &= \rho\beta^n \left(\sum_{m=2}^n \sum_{j=0}^{m-1} \binom{n-m+j}{j} \alpha^j \left(\frac{\rho}{\beta} \right)^{m-1} + 1 \right) \\ &= \rho\beta^n \left(\sum_{m=2}^n \sum_{j=0}^{m-2} \binom{n-m+j}{j} \alpha^j \left(\frac{\rho}{\beta} \right)^{m-1} + \sum_{m=2}^n \binom{n-1}{m-1} \left(\frac{\alpha\rho}{\beta} \right)^{m-1} + 1 \right) \\ &= \rho\beta^n \sum_{k=2}^n \sum_{j=0}^{k-2} \binom{n-k+j}{j} \alpha^j \left(\frac{\rho}{\beta} \right)^{k-1} + \rho\beta^n \left(\sum_{k=2}^n \binom{n-1}{k-1} \left(\frac{\alpha\rho}{\beta} \right)^{k-1} + 1 \right). \end{aligned}$$

Now, using the inductive hypothesis (5.17) and Lemma 5.1, we simplify to

$$\begin{aligned} h(n+1) &= \rho \frac{\rho}{1-\rho} [(\beta + \rho\alpha)^{n-1} - \rho^{n-1}] + \rho\beta^n \left(-1 + \left(1 + \frac{\alpha\rho}{\beta} \right)^{n-1} + 1 \right) \\ &= \frac{\rho}{1-\rho} [(\beta + \rho\alpha)^n - \rho^n]. \end{aligned}$$

□

Using Lemmas 5.1 and 5.2, we can now simplify (5.15) and (5.16) to give

$$P_n(1) = \rho(1 - \rho) + \rho^2 (1 - (1 - \rho)\alpha)^n. \quad (5.18)$$

and thus the asymptotic density (plotted in Figure 5.2a) is given by

$$P(1) = \lim_{n \rightarrow \infty} P_n(1) = \begin{cases} 1 & \text{if } \rho = 1 \\ \rho & \text{if } \alpha = 0 \\ \rho(1 - \rho) & \text{otherwise} \end{cases}. \quad (5.19)$$

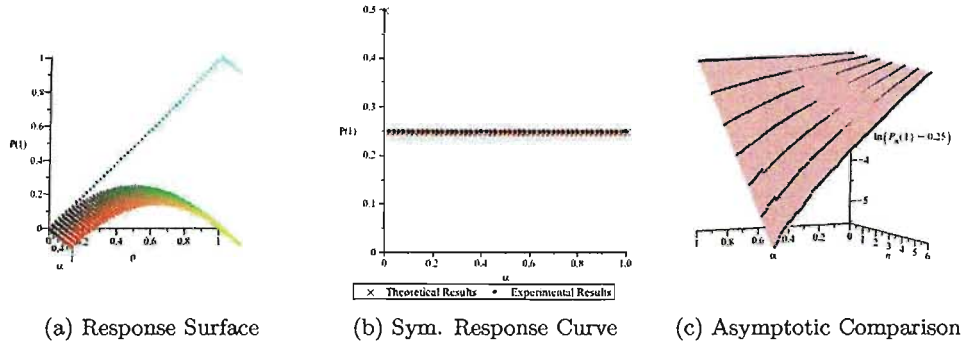


Figure 5.2: Rule 140A - Graphs

In the special case when $\rho = 1/2$, we obtain

$$P_n^{(s)}(1) = \frac{1}{4} + \frac{1}{4} \left(1 - \frac{\alpha}{2}\right)^n. \quad (5.20)$$

In Figure 5.2b, the theoretical formula (5.20) is plotted with direct simulated iteration of Rule 140 in which an array of length 20000 was iterated $100000/\alpha$ times with $\alpha > 0.1$ and 1000000 times with $\alpha \leq 0.1$, with periodic boundary conditions, averaged over 100 runs. We check the asymptotic convergence of (5.20) by considering

$$\begin{aligned} P_n(1) - \frac{1}{4} &= \frac{1}{4} \left(1 - \frac{\alpha}{2}\right)^n \\ \ln \left(P_n(1) - \frac{1}{4}\right) &= \ln \left(\frac{1}{4}\right) + n \ln \left(1 - \frac{\alpha}{2}\right). \end{aligned} \quad (5.21)$$

In Figure 5.2c, we plot $\ln(P_n(1) - \frac{1}{4})$, both experimentally with an array of 200000 elements iterated step-by-step to $n = 6$, and theoretically from (5.21). We notice excellent agreement for all values of α .

Basic Blocks For Rule 140A, we were also able to find explicit formulae for the response surfaces for each of the eight basic blocks. We once again use (5.8). In Table 5.2, the set of all n -step preimage blocks of four of the eight basic block are shown, with their corresponding initial probabilities and respective transition probabilities.

Table 5.2: Rule 140A - Initial and Transition Probabilities of Basic Blocks

$b \in \mathcal{B}_3$	$b^* \in \mathbf{f}^{-n}(b)$	$\omega^n(b b^*)$	$P_0(b^*)$
001	$\underbrace{\star \cdots \star}_n \underline{001} \underbrace{\star \cdots \star}_n$	1	$\rho(1-\rho)^2$
	$\underbrace{\star \cdots \star}_{n-1} 1 \underline{101} \underbrace{\star \cdots \star}_n$	$1 - \beta^n$	$\rho^3(1-\rho)$
011	$\underbrace{\star \cdots \star}_n \underline{011} \underbrace{1 \cdots 1}_{k-1} 0 \underbrace{\star \cdots \star}_{n-k}$ where $1 \leq k \leq n+1$	see 5.14	see 5.14
101	$\underbrace{\star \cdots \star}_n 1 \underline{101} \underbrace{\star \cdots \star}_n$	β^n	$\rho^3(1-\rho)$
	$\underbrace{\star \cdots \star}_{n-1} 0 \underline{101} \underbrace{\star \cdots \star}_n$	1	$\rho^2(1-\rho)^2$
111	$\underbrace{\star \cdots \star}_n \underline{111} \underbrace{1 \cdots 1}_{k-1} 0 \underbrace{\star \cdots \star}_{n-k}$ where $1 \leq k \leq n+1$	see 5.14	see 5.14

We can use the results in Table 5.2 and consistency conditions (2.3) to find formulae for all eight basic block symmetric response curves. We summarize those results as follows, where we assume that $\alpha \neq 0$.

$$P_n^{(s)}(000) = \frac{5}{16} - \frac{1}{4} \left(1 - \frac{\alpha}{2}\right)^n + \frac{1}{16} \beta^n, \quad P^{(s)}(000) = 5/16, \quad (5.22)$$

$$P_n^{(s)}(001) = \frac{1}{16} + \frac{1}{16} \beta^n, \quad P^{(s)}(001) = 1/16, \quad (5.23)$$

$$P_n^{(s)}(010) = \frac{1}{4} - \frac{1}{8} \left(1 - \frac{\alpha}{2}\right)^n, \quad P^{(s)}(010) = 1/4, \quad (5.24)$$

$$P_n^{(s)}(011) = \frac{1}{8} \left(1 - \frac{\alpha}{2}\right)^n, \quad P^{(s)}(011) = 0, \quad (5.25)$$

$$P_n^{(s)}(100) = \frac{3}{16} - \frac{1}{16} \beta^n, \quad P^{(s)}(100) = 3/16, \quad (5.26)$$

$$P_n^{(s)}(101) = \frac{1}{16} + \frac{1}{16} \beta^n, \quad P^{(s)}(101) = 1/16, \quad (5.27)$$

$$P_n^{(s)}(110) = \frac{1}{8} \left(1 - \frac{\alpha}{2}\right)^n, \quad P^{(s)}(110) = 0, \quad (5.28)$$

$$P_n^{(s)}(111) = \frac{1}{8} \left(1 - \frac{\alpha}{2}\right)^n, \quad P^{(s)}(111) = 0. \quad (5.29)$$

Example: Rule 76A

Consider an α -asynchronous rule defined as

$$\omega(1|b) = \begin{cases} 0 & \forall b \in \{000, 001, 100, 101\} \\ 1 & \forall b \in \{010, 011, 110\} \\ 1 - \alpha & \forall b \in \{111\} \end{cases} . \quad (5.30)$$

Note that if $\alpha = 1$, then this rule is equivalent to deterministic Rule 76 (2.4). We will then refer to the asynchronous rule defined in (5.13) as Rule 76A. In this section, we will often have need of the Kroenecker delta function, which we define as

$$\delta_{(x,y)} = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y. \end{cases}$$

We now find the set of all potential preimage blocks and their respective transition probabilities. We start by defining $E_n^{k_1, k_2}$ to be the set of blocks of the form

$$\{\underbrace{\star \cdots \star}_{n-k_1-1} 0 \underbrace{1 \cdots 1}_{k_1} \underline{1} \underbrace{1 \cdots 1}_{k_2} 0 \underbrace{\star \cdots \star}_{n-k_2-1}\}.$$

where $1 \leq k_1, k_2 \leq n$. Intuitively, the values of k_1, k_2 refer to the number of 1's to the left and right, respectively, of the centre $\underline{1}$ before the first occurrence of a 0.

Proposition 5.7. *The set $\mathbf{f}^{-n}(1)$ consists of all blocks in*

$$E_n = \bigcup_{k_1, k_2=1}^n E_n^{k_1, k_2} = \{ \underbrace{\star \cdots \star}_n 1 \underbrace{\star \cdots \star}_n \}.$$

Proof. From (5.30), we can see that an element in state 0 will always remain in state 0, so that for any block $e' \in \mathbf{f}^{-n}(1) \setminus E_n$, we have $\omega^n(1|e') = 0$. under Rule 76A. A block in $\mathbf{f}^{-n}(1)$ however, could transform to a single 1 with some non-zero probability. \square

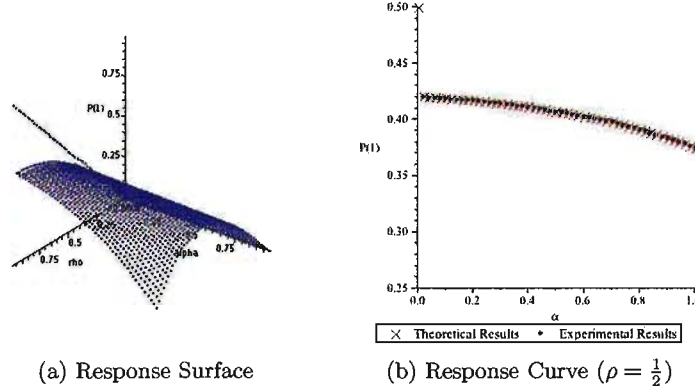


Figure 5.3: Rule 76A - Graphs

Proposition 5.8. *For any block $e \in E_n^{k_1, k_2}$, we have*

$$\omega^n(1|e) = \begin{cases} \omega^{n-1}(1|e') & \text{if } k_1 = 0, k_2 = 0 \\ \sum_{i=0}^{k_2-1} \alpha^{1-\delta_{(i, k_2-1)}} \beta^i \omega^{n-1}(1|e') & \text{if } k_1 = 0, 1 \leq k_2 \leq n \\ \sum_{j=0}^{k_1-1} \alpha^{1-\delta_{(j, k_1-1)}} \beta^j \omega^{n-1}(1|e') & \text{if } 1 \leq k_1 \leq n, k_2 = 0 \\ \sum_{j=0}^{k_1-1} \sum_{i=0}^{k_2-1} \alpha^{2-\delta_{(j, k_1-1)}-\delta_{(i, k_2-1)}} \beta^{j+i+1} \omega^{n-1}(1|e') & \text{if } 1 \leq k_1, k_2 \leq n \end{cases},$$

where $e' \in E_{n-1}^{0,0}, E_{n-1}^{0,i'}, E_{n-1}^{j',0}$ and $E_{n-1}^{j',i'}$, respectively, and $j' = j + \delta_{(j, k_1-1)} - \delta_{(j, n-1)}$, $i' = i + \delta_{(i, k_2-1)} - \delta_{(i, n-1)}$, $\varepsilon_1^{1,1} = \varepsilon_1^{1,2} = \varepsilon_1^{2,1} = 1$, and $\varepsilon_1^{2,2} = \beta$.

Proof. The derivation of this proposition is long and tedious and similar in structure to the derivation of (5.14). Since we were unable to derive a closed-form equation for the transition probabilities, we omit the details but note, however, that the full four-page proof is available on request. \square

If we consider (5.8) and the result of Proposition 5.8 we conclude that for all $e \in E_n^{j,i}$,

$$P_n(1) = \sum_{j=0}^n \sum_{i=0}^n \rho^{j+i+1} (1-\rho)^{2-\delta_{(j,n)}-\delta_{(i,n)}} \omega^n(1|e). \quad (5.31)$$

To plot the response surface (see Figure 5.3a), we iterated (5.31) to $n = 15/\alpha$ for $\alpha > 0.1$ and $n = 150$ when $\alpha \leq 0.1$.

Our symmetric response curve is given by

$$P_n^{(s)}(1) = \sum_{j=0}^n \sum_{i=0}^n 2^{-i-j-3+\delta_{(j,n)}+\delta_{(i,n)}} \omega^n(1|e). \quad (5.32)$$

In Figure 5.3b, (5.32) is plotted with direct simulated iteration of Rule 76. For the theoretical plot, the formula was iterated to $n = 15/\alpha$ for $\alpha > 0.1$ and $n = 150$ when $\alpha \leq 0.1$. For the simulated plot, an array of length 20000 was iterated $100000/\alpha$ times with $\alpha > 0.1$ and 1000000 times with $\alpha \leq 0.1$, with periodic boundary conditions, averaged over 100 runs. We can see that there is a close agreement between the theoretical and experimental results.

5.2.2 Experimental Response Surfaces

For all other minimal rules, we constructed experimental response surfaces. In each case an array of length 20000 was iterated $100000/\alpha$ times with $\alpha > 0.1$ and 1000000 times with $\alpha \leq 0.1$, with periodic boundary conditions, averaged over 100 runs. The results of these iterations are found in Appendix B.

We note that many of these response surfaces possess discontinuities. In particular, most rules have a discontinuity at $\alpha = 0$ the asynchronicity value at which the rule is equivalent to the Identity Rule. Some rules also possess a discontinuity at $\alpha = 1$, when the rule is strictly deterministic. All rules, however, do appear to have smooth transitions from the limiting case $\alpha \rightarrow 1$ to the limiting case $\alpha \rightarrow 0$.

Some rules have dramatically differing dynamics at the values $\alpha = 0, 1$ than the intermediate values. Consider, for example, the experimental response surface for Rule 154 B.5p. When $\alpha = 0, 1$, the asymptotic density of ones is $P(1) < 1$. For all other values of α , however, the rule stabilizes at a configuration entirely consisting of cells in state 1.

5.3 Local Structure Theory

In [19], the authors introduce the Local Structure Theory for cellular automata, in which they derive a method by which we can write the probability of a block of length m , as a map entirely in terms of probabilities of other blocks of length m . They show that fixed points of this map can be used to approximate the asymptotic behaviour of the cellular automata itself. The basis of this method is determining the one-step preimages of a given block and using Bayesian extensions to derive the required map. We now show the procedure for an elementary cellular automata

local function, f , using blocks of length 2, with the assumption that the same steps can be performed for blocks on any length.

If we consider a block, $c = c_1c_2 \in \mathcal{B}_2$, we can find the set of preimages of c under f to be

$$f^{-1}(c) = \{b_1, \dots, b_N\},$$

where, for each $1 \leq i \leq N$, block $b_i = b_{i1}b_{i2}b_{i3}b_{i4} \in \mathcal{B}_4$.

Therefore, the $(n+1)$ -step probability of occurrence of block c is given by

$$P_{n+1}(c_1c_2) = P_n(b_{11}b_{12}b_{13}b_{14}) + \dots + P_n(b_{N1}b_{N2}b_{N3}b_{N4}).$$

Using Bayesian extensions, we obtain

$$\begin{aligned} P_{n+1}(c_1c_2) \approx & \frac{P_n(b_{11}b_{12})P_n(b_{12}b_{13})P_n(b_{13}b_{14})}{P_n(b_{12})P_n(b_{13})} + \dots + \\ & + \frac{P_n(b_{N1}b_{N2})P_n(b_{N2}b_{N3})P_n(b_{N3}b_{N4})}{P_n(b_{N2})P_n(b_{N3})}. \end{aligned}$$

Using Kolmogorov consistency conditions when $k = 0$ (2.1), we obtain

$$\begin{aligned} P_{n+1}(c_1c_2) \approx & \frac{P_n(b_{11}b_{12})P_n(b_{12}b_{13})P_n(b_{13}b_{14})}{(P_n(b_{12}0) + P_n(b_{12}1))(P_n(b_{13}0) + P_n(b_{13}1))} + \dots + \\ & + \dots + \frac{P_n(b_{N1}b_{N2})P_n(b_{N2}b_{N3})P_n(b_{N3}b_{N4})}{(P_n(b_{N2}0) + P_n(b_{N2}1))(P_n(b_{N3}0) + P_n(b_{N3}1))}. \end{aligned} \quad (5.33)$$

Therefore, we have written the level- $(n+1)$ probabilities of a block of length 2 entirely in terms of level- n probabilities of other blocks of length-2. Using consistency condition identities (2.1), we can obtain a system of two independent equations of the form (5.33), each entirely in terms of probabilities of two of the four blocks of length 2.

We can either iterate this system from initial probabilities ($n = 0$) to obtain approximations for $P_n(c)$, or find fixed points of this system to obtain an approximation for the asymptotic density $P(c)$. It was shown in [19] that as the size of the block used for local structure approximations increases, so does the accuracy of such an approximation. We now derive Level 2 and Level 3 Local Structure Theory approximations for Rules 76A, 140A and 200A (Figure 5.4).

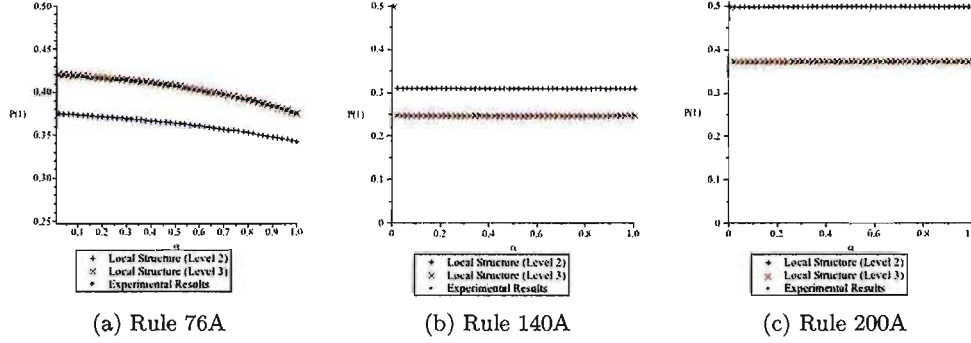


Figure 5.4: Plots of $P^{(s)}(1)$ for Rules 76A, 140A and 200A with experimental results plotted with Local Structure Theory approximations at both Level 2 and Level 3. Note that in each case, the Level 3 LST approximation appears to be exact, while the Level 2 LST approximation clearly differs from the actual results.

5.3.1 Level Two Approximations

Example: Rule 200A

Considering the sets of potential preimages of blocks 01 and 11, we obtain

$$\begin{aligned}
 P_n(00) &= P_{n-1}(0000) + P_{n-1}(0001) + P_{n-1}(1000) + P_{n-1}(1001) + \\
 &\quad + \alpha [P_{n-1}(0010) + P_{n-1}(0100) + P_{n-1}(0101) + P_{n-1}(1010)], \\
 P_n(01) &= P_{n-1}(0011) + P_{n-1}(1011) + \beta [P_{n-1}(0010) + P_{n-1}(1010)].
 \end{aligned}$$

Using the procedure outlined in Section 5.3, and letting $a_n := P_n(00)$ and $b_n := P_n(01)$ we obtain a system of two independent equations, which we can write as

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \Phi(a_n, b_n) = \begin{pmatrix} a_n + \frac{\alpha b_n^2}{1 - a_n - b_n} \\ b_n - \frac{\alpha b_n^2}{1 - a_n - b_n} \end{pmatrix}. \quad (5.34)$$

It is easy to see that a fixed point solution of (5.34) is given by $b = 0$ from which we determine that, for some parameter a which depends on α and the initial densities of blocks 00 and 01,

$$P(00) = a, \quad P(01) = P(10) = 0, \quad P(11) = 1 - a.$$

Unfortunately, this is not an explicit solution and tells us only the asymptotic relationship between densities of all blocks of length 2. Instead, we can iterate (5.34) starting with initial conditions $a_0 = b_0 = 1/4$. We plot $P_n(1) = 1 - P_n(00) - P_n(01)$

versus α and the results of iteration to $n = 100$ is shown in Figure (5.4c). It is evident that iteration of Local Structure Level Two does not provide an accurate approximation of density.

Example: Rule 140A

Using the same procedure, and letting $a_n := P_n(00)$ and $d_n := P_n(11)$, we obtain

$$\begin{pmatrix} a_{n+1} \\ d_{n+1} \end{pmatrix} = \Phi(a_n, d_n) = \begin{pmatrix} a_n + \frac{\alpha d_n(1-a_n-d_n)}{1-a_n+d_n} \\ d_n - \frac{\alpha d_n(1-a_n-d_n)}{1-a_n+d_n} \end{pmatrix}.$$

Results of the iteration to $n = 100$ is shown in Figure (5.4b) and shows that Local Structure Level Two iterations do not accurately approximate true density of ones.

Example: Rule 76A

Finally, if we let $b_n := P_n(01)$ and $d_n := P_n(11)$, we obtain a map

$$\begin{pmatrix} b_{n+1} \\ d_{n+1} \end{pmatrix} = \Phi(b_n, d_n) = \begin{pmatrix} b_n + \frac{\alpha d_n^2(b_n + \beta d_n)}{(b_n + d_n)^2} \\ \frac{d_n(b_n + \beta d_n)^2}{(b_n + d_n)^2} \end{pmatrix}.$$

Results of the iteration to $n = 100$ is shown in Figure (5.4a) and shows that Local Structure Level Two iterations do not accurately approximate true density of ones.

5.3.2 Level Three Approximations

Example: Rule 200A

Considering sets of potential preimages of blocks 000, 001, 010 and 101, we obtain

$$\begin{aligned} P_n(000) &= P_{n-1}(00000) + P_{n-1}(00001) + P_{n-1}(10000) + P_{n-1}(10001) + \\ &\quad + \alpha [P_{n-1}(00010) + P_{n-1}(10010) + P_{n-1}(00100) + P_{n-1}(00101) + \\ &\quad + P_{n-1}(10100) + P_{n-1}(10101) + P_{n-1}(01000) + P_{n-1}(01001)] + \\ &\quad + \alpha^2 P_{n-1}(01010), \\ P_n(001) &= P_{n-1}(00011) + P_{n-1}(10111) + \beta [P_{n-1}(00010) + P_{n-1}(10010)] + \\ &\quad + \alpha P_{n-1}(01011) + \alpha \beta P_{n-1}(01010), \\ P_n(010) &= \beta [P_{n-1}(00100) + P_{n-1}(00101) + P_{n-1}(10100) + P_{n-1}(10101)], \\ P_n(101) &= P_{n-1}(11011) + \beta [P_{n-1}(01011) + P_{n-1}(11010)] + \beta^2 P_{n-1}(01010). \end{aligned}$$

Using the procedure outlined in Section 5.3 and letting $a_n := P_n(000)$, $b_n := P_n(001)$, $c_n := P_n(010)$ and $f_n := P_n(101)$, we obtain a set of the following four independent equations, which we write as a iterative map as

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \\ f_{n+1} \end{pmatrix} = \Phi(a_n, b_n, c_n, f_n) = \begin{pmatrix} a_n + \alpha c_n \left(1 + \frac{2b_n}{b_n + f_n}\right) + \alpha^2 \frac{c_n^2 f_n}{(b_n + f_n)^2} \\ \left(1 - \frac{\alpha c_n}{b_n + f_n}\right) \left(b_n + \frac{\alpha c_n f_n}{b_n + f_n}\right) \\ \beta c_n \\ f_n \left(1 - \frac{\alpha c_n}{b_n + f_n}\right)^2 \end{pmatrix}. \quad (5.35)$$

From (5.35), we obtain the fixed point solution $c = 0$, where upon parameterization, we again fail to obtain an explicit solution of our asymptotic densities.

In this case, however, it turns out that it is possible to solve this system of discrete equations directly to compare to our direct results.

We start by solving the third component of (5.35) with an initial condition $c_0 = 1/8$ to obtain the solution

$$c_n = \frac{1}{8} \beta^n. \quad (5.36)$$

Using the second and fourth components of (5.35), we can obtain the following two equations

$$\begin{aligned} b_{n+1} + f_{n+1} &= b_n + f_n - \alpha c_n, \\ b_{n+1} - f_{n+1} &= \left(1 - \frac{\alpha c_n}{b_n + f_n}\right) \left(b_n - f_n + \frac{2\alpha c_n f_n}{b_n + f_n}\right). \end{aligned}$$

If we let $x_n := b_n + f_n$ and $y_n := b_n - f_n$, our system is transformed to

$$x_{n+1} = x_n - \alpha c_n, \quad (5.37)$$

$$y_{n+1} = \left(1 - \frac{\alpha c_n}{x_n}\right)^2 y_n + \alpha c_n \left(1 - \frac{\alpha c_n}{x_n}\right). \quad (5.38)$$

To solve equations (5.37) and (5.38), with initial conditions $x_0 = 1/4$, $y_0 = 0$, we must use the following Lemma.

Lemma 5.3. [9] *A difference equation of the form $y_{n+1} = A_n y_n + g_n$ has solution*

$$y_n = \left(\prod_{i=0}^{n-1} A_i\right) y_0 + \sum_{r=0}^{n-1} \left(\prod_{i=r+1}^{n-1} A_i\right) g_r.$$

Using Lemma 5.3, we solve (5.37) to obtain

$$x_n = \frac{1}{4} - \frac{\alpha}{8} \sum_{r=0}^{n-1} \beta^r = \frac{1}{4} - \frac{\alpha}{8} \left(\frac{\beta^n - 1}{\beta - 1} \right) = \frac{1}{8} (1 + \beta^n). \quad (5.39)$$

Using our solutions for c_n and x_n ((5.36) and (5.39)), we can rewrite (5.38) as

$$y_{n+1} = \frac{(1 + \beta^{n+1})^2}{(1 + \beta^n)^2} y_n + \frac{\alpha}{8} \beta^n \frac{(1 + \beta^{n+1})}{(1 + \beta^n)}. \quad (5.40)$$

Again, using Lemma 5.3, we can solve (5.40) to obtain

$$\begin{aligned} y_n &= \frac{1}{8} \alpha \sum_{r=0}^{n-1} \left(\prod_{i=r+1}^{n-1} \left(\frac{1 + \beta^{i+1}}{1 + \beta^i} \right)^2 \right) \cdot \beta^r \frac{1 + \beta^{r+1}}{1 + \beta^r} \\ &= \frac{1}{8} \alpha \sum_{r=0}^{n-1} \left(\frac{(1 + \beta^n)^2}{(1 + \beta^{r+1})^2} \right) \cdot \beta^r \frac{1 + \beta^{r+1}}{1 + \beta^r} \\ &= \frac{1}{8} \alpha (1 + \beta^n)^2 \sum_{r=0}^{n-1} \frac{\beta^r}{(1 + \beta^r)(1 + \beta^{r+1})}. \end{aligned} \quad (5.41)$$

To simplify, we need the following summation identity.

Lemma 5.4.

$$\sum_{r=0}^{n-1} \frac{\beta^r}{(1 + \beta^r)(1 + \beta^{r+1})} = \frac{\beta^n - 1}{2(\beta - 1)(1 + \beta^n)}. \quad (5.42)$$

Proof. By induction. The $n = 1$ case is trivial. We complete the proof as follows

$$\begin{aligned} \sum_{r=0}^{n-1} \frac{\beta^r}{(1 + \beta^r)(1 + \beta^{r+1})} &= \frac{\beta^{n-1}}{(1 + \beta^{n-1})(1 + \beta^n)} + \sum_{r=0}^{n-2} \frac{\beta^r}{(1 + \beta^r)(1 + \beta^{r+1})} \\ &= \frac{\beta^{n-1}}{(1 + \beta^{n-1})(1 + \beta^n)} + \frac{\beta^{n-1} - 1}{2(\beta - 1)(1 + \beta^{n-1})} \\ &= \frac{2\beta^{n-1}(\beta - 1) + (\beta^{n-1} - 1)(1 + \beta^n)}{2(\beta - 1)(1 + \beta^{n-1})(1 + \beta^n)} \\ &= \frac{\beta^n - 1}{2(\beta - 1)(1 + \beta^n)}. \end{aligned}$$

□

We may now simplify (5.41) to give

$$y_n = \frac{1}{8} \alpha (1 + \beta^n)^2 \frac{\beta^n - 1}{2(\beta - 1)(1 + \beta^n)} = \frac{1}{16} (1 - \beta^{2n}). \quad (5.43)$$

We may now work backwards, and obtain the solution

$$b_n = \frac{1}{2}(x_n + y_n) = \frac{3}{32} - \frac{1}{32}\beta^{2n} + \frac{1}{16}\beta^n, \quad (5.44)$$

$$f_n = \frac{1}{2}(x_n - y_n) = \frac{1}{32} + \frac{1}{32}\beta^{2n} + \frac{1}{16}\beta^n. \quad (5.45)$$

We may now rewrite the first component of (5.35) to give

$$a_{n+1} = a_n + \frac{5\alpha}{16}\beta^n - \frac{\alpha(2-\alpha)}{32}\beta^{2n}. \quad (5.46)$$

Once again, using Lemma 5.3 and the initial condition $a_0 = 1/8$, we obtain

$$\begin{aligned} a_n &= \frac{1}{8} + \frac{5\alpha}{16} \sum_{r=0}^{n-1} \beta^r - \frac{\alpha(2-\alpha)}{32} \sum_{r=0}^{n-1} \beta^{2r} \\ &= \frac{1}{8} + \frac{5\alpha}{16} \cdot \frac{1-\beta^n}{1-\beta} - \frac{\alpha(2-\alpha)}{32} \cdot \frac{1-\beta^{2n}}{1-\beta^2} \\ &= \frac{13}{32} - \frac{5}{16}\beta^n + \frac{1}{32}\beta^{2n}. \end{aligned} \quad (5.47)$$

We were surprised to note that the results obtained by solving the system (5.35) agree with the direct solutions obtained in Subsection 5.2.1. Therefore, the Local Structure Theory Level 3 approximations provide exact solutions for both the n -step density of basic blocks and the asymptotic densities.

Example: Rule 140A

Using the same procedure, and letting $a_n := P_n(000)$, $b_n := P_n(001)$, $c_n := P_n(010)$ and $d_n := P_n(011)$, we obtain the following map

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \\ d_{n+1} \end{pmatrix} = \Phi(a_n, b_n, c_n, d_n) = \begin{pmatrix} a_n + \frac{\alpha d_n b_n}{c_n + d_n} \\ b_n + \frac{\alpha d_n (c_n + d_n - b_n)}{c_n + d_n} \\ c_n + \frac{\alpha d_n^2}{1 - a_n - b_n - 2c_n - 2d_n} \\ d_n - \frac{\alpha d_n^2}{1 - a_n - b_n - 2c_n - 2d_n} \end{pmatrix}. \quad (5.48)$$

We were unable to solve explicitly this system, but iterations of the map showed that the solution is not an exact solution of the n -step density of basic blocks given in Table 5.2. The asymptotic solution, however, appears to be a very close approximation to the actual density of ones (Figure (5.4b)).

Example: Rule 76A

Finally, letting $b_n := P_n(001)$, $c_n := P_n(010)$, $d_n := P_n(011)$ and $j_n := P_n(111)$, we obtain the following map

$$\begin{pmatrix} b_{n+1} \\ c_{n+1} \\ d_{n+1} \\ j_{n+1} \end{pmatrix} = \Phi(b_n, c_n, d_n, j_n) = \begin{pmatrix} b_n + \frac{\alpha^2 j_n^2 (d_n + \beta j_n)}{(j_n + d_n)^2} \\ c_n + \frac{\alpha j_n (2d_n (j_n + d_n) + \alpha \beta j_n^2)}{(j_n + d_n)^2} \\ \frac{(d_n + \alpha j_n)(d_n + \beta j_n)^2}{(d_n + j_n)^2} \\ \frac{\beta j_n (d_n + \beta j_n)^2}{(d_n + j_n)^2} \end{pmatrix}. \quad (5.49)$$

We iterate (5.49) starting with initial conditions $b_0 = c_0 = d_0 = j_0 = \frac{1}{8}$. We plot $P_n(1) = P_n(010) + P_n(011) + P_n(110) + P_n(111)$ versus α and the results of iteration to $n = 100$ is shown in Figure (5.4a). It once again appears that iteration of Local Structure Level Three provides a very accurate approximation of the density of ones.

Conclusion and Further Work

In Chapter 3, we considered response curves for two-dimensional cellular automata with L-neighbourhood. We were able to derive explicit formulae for a class of shift or identity emulating rules. These rules fell into two classes: those with constant density and those whose density decayed to some fixed value. For many rules, we were also able to find response curves for all eight basic blocks. We believe that there are more rules for which a response curve can be derived and further research should be spent determining these formulae. We also believe it will be possible to prove our conjecture for L-shaped Rule 184 in which the basic block density formulae are dependent on Catalan numbers.

In Chapter 4, we presented a proof of the Balance Theorem adapted to the L-neighbourhood and then to the von Neumann neighbourhood. We used the property of permutivity and derived an algorithm by which a test of surjectivity can be performed. We found all surjective rules with L-neighbourhood and found them to be the class of permutive rules. When we consider the von Neumann neighbourhood, we confirmed that all permutive rules were surjective. We were not, however, able to state that these are the only surjective rules. Further study should be performed on the set of four “hard” rules for which surjectivity could not be determined. The local mappings of these rules display enough regularity that we believe their properties can be proved rigorously.

In Chapter 5, we derived response surfaces for a class of special rules in which the rule tables differ by only one basic block from the Identity Rule. In two of these cases, we were able to derive explicit response surface equations. In the third case, we found an approximate response surface. We were also able to determine experimental response surfaces for all minimal rules. We believe that it will be possible to determine explicit response surface formulae for other rules, but the dynamics of many other rules will make this very difficult. Further work in categorizing these rules is planned and there is much work to be done to better understand the dynamics of such CA. We also determined that the Local Structure Theory provides excellent approximations at Level 3 for a number of rules. We strongly believe that these approximations are exact, but further research is needed to explain the reason for this phenomenon.

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Experimental Response Curves

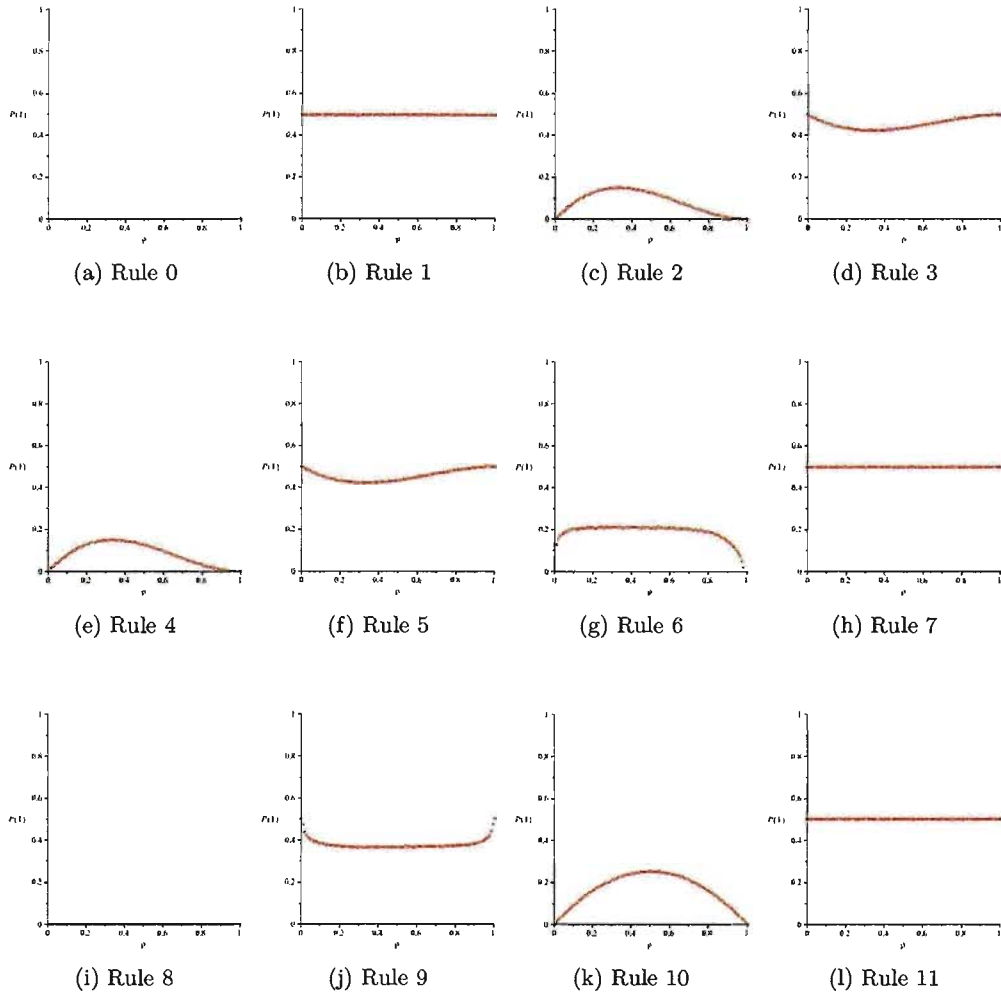


Figure A.1: Experimental Response Curves (Rules 0-11)

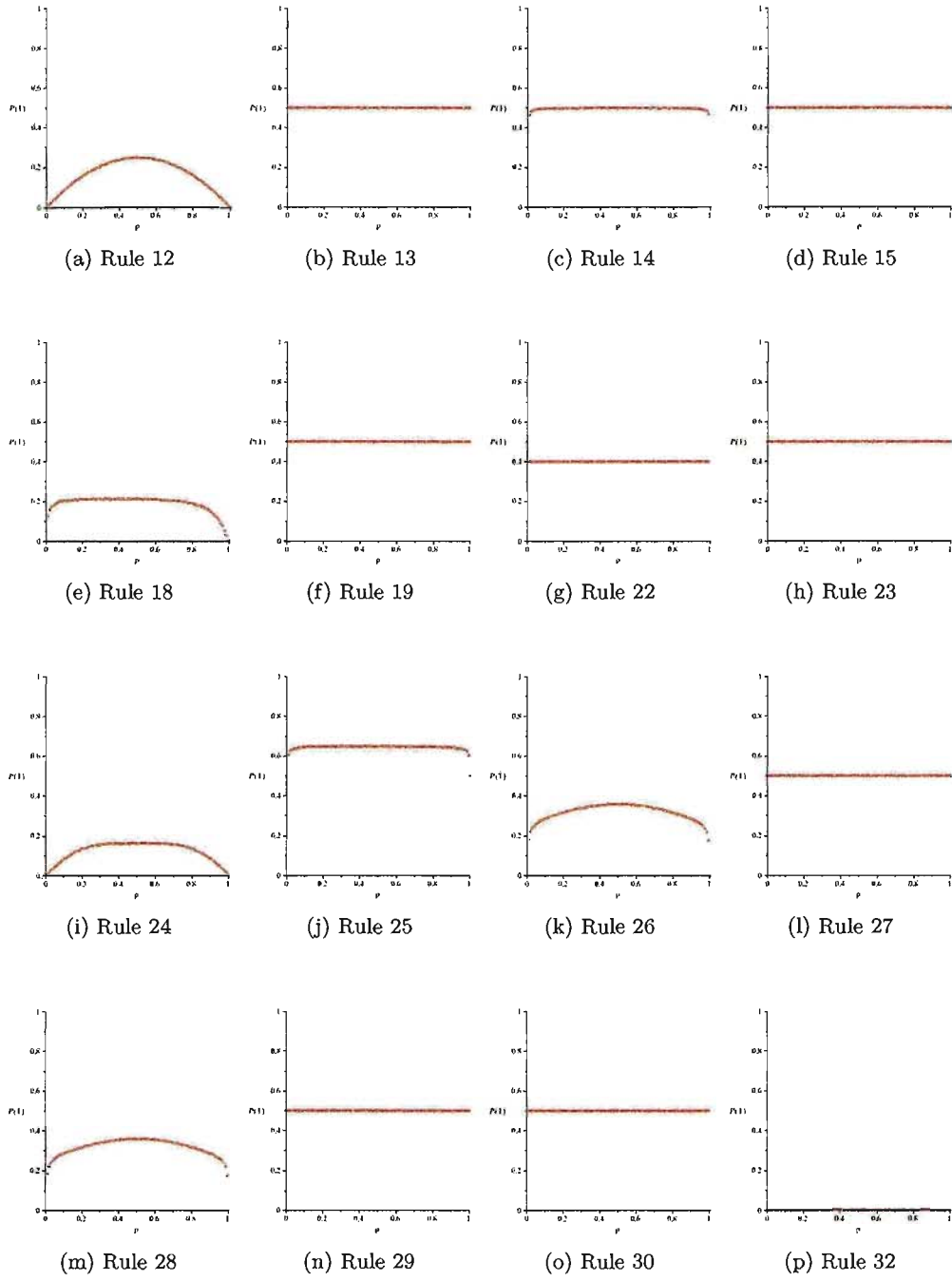


Figure A.2: Experimental Response Curves (Rules 12-32)

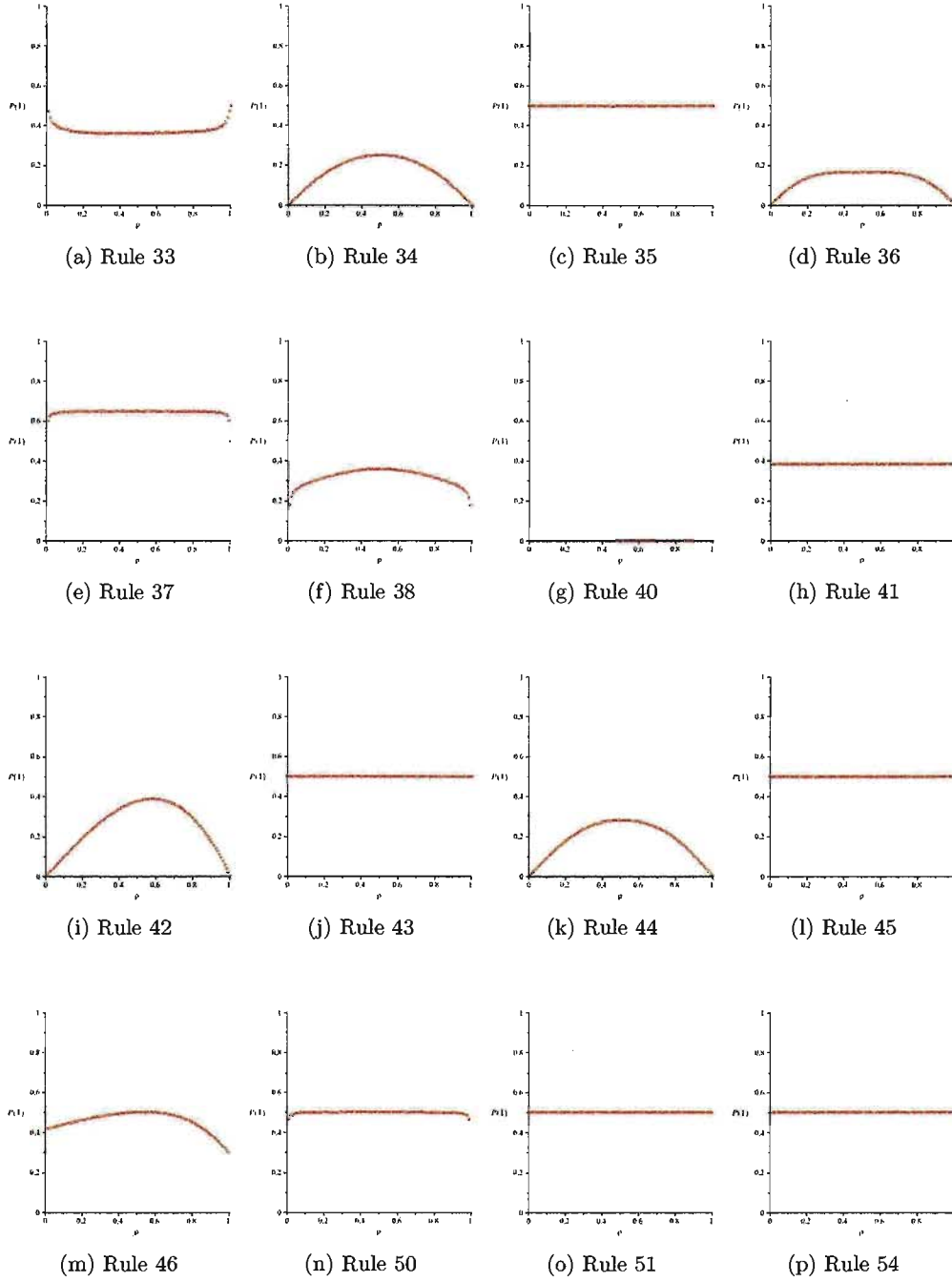


Figure A.3: Experimental Response Curves (Rules 33-54)

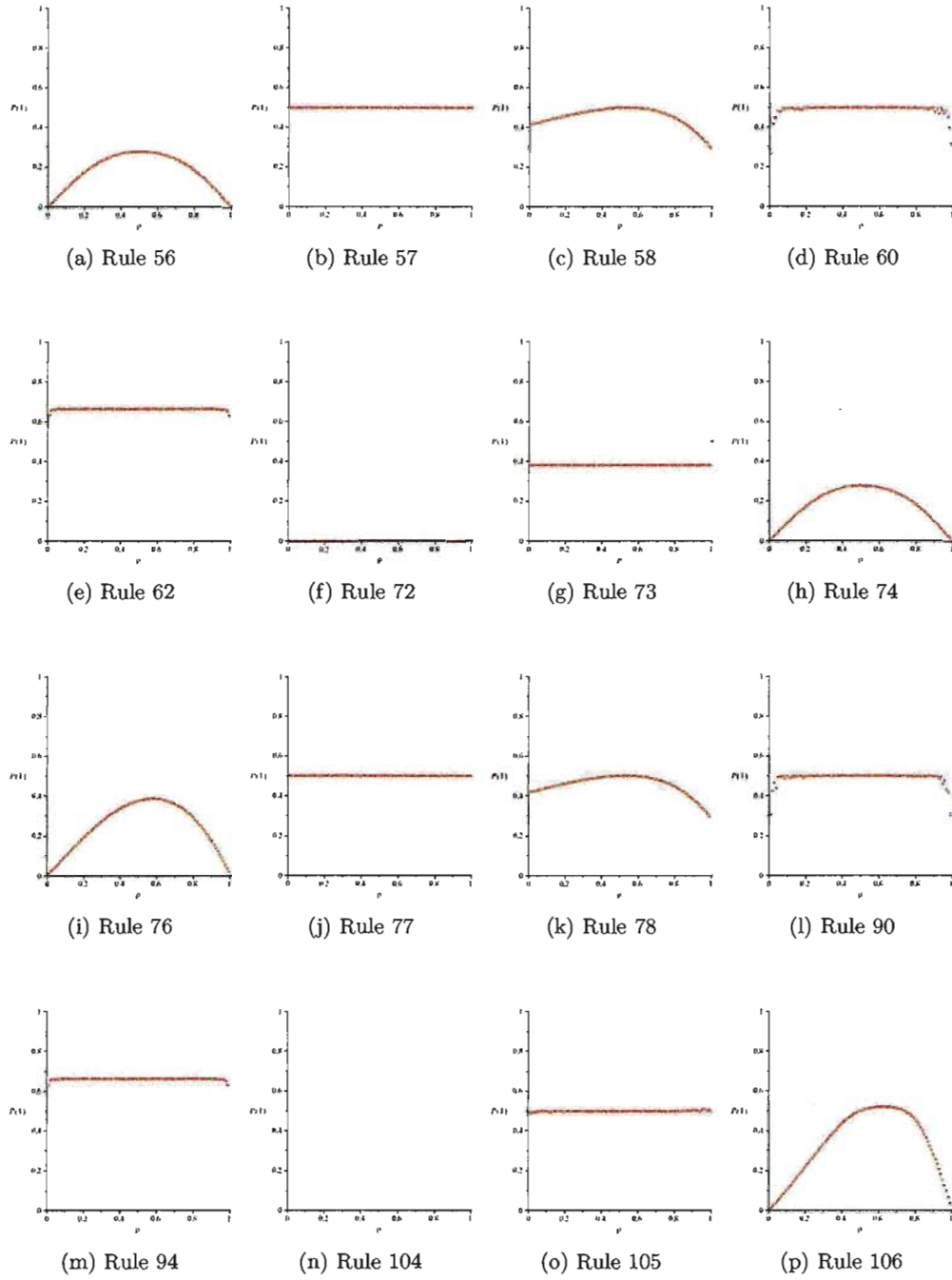


Figure A.4: Experimental Response Curves (Rules 56-106)

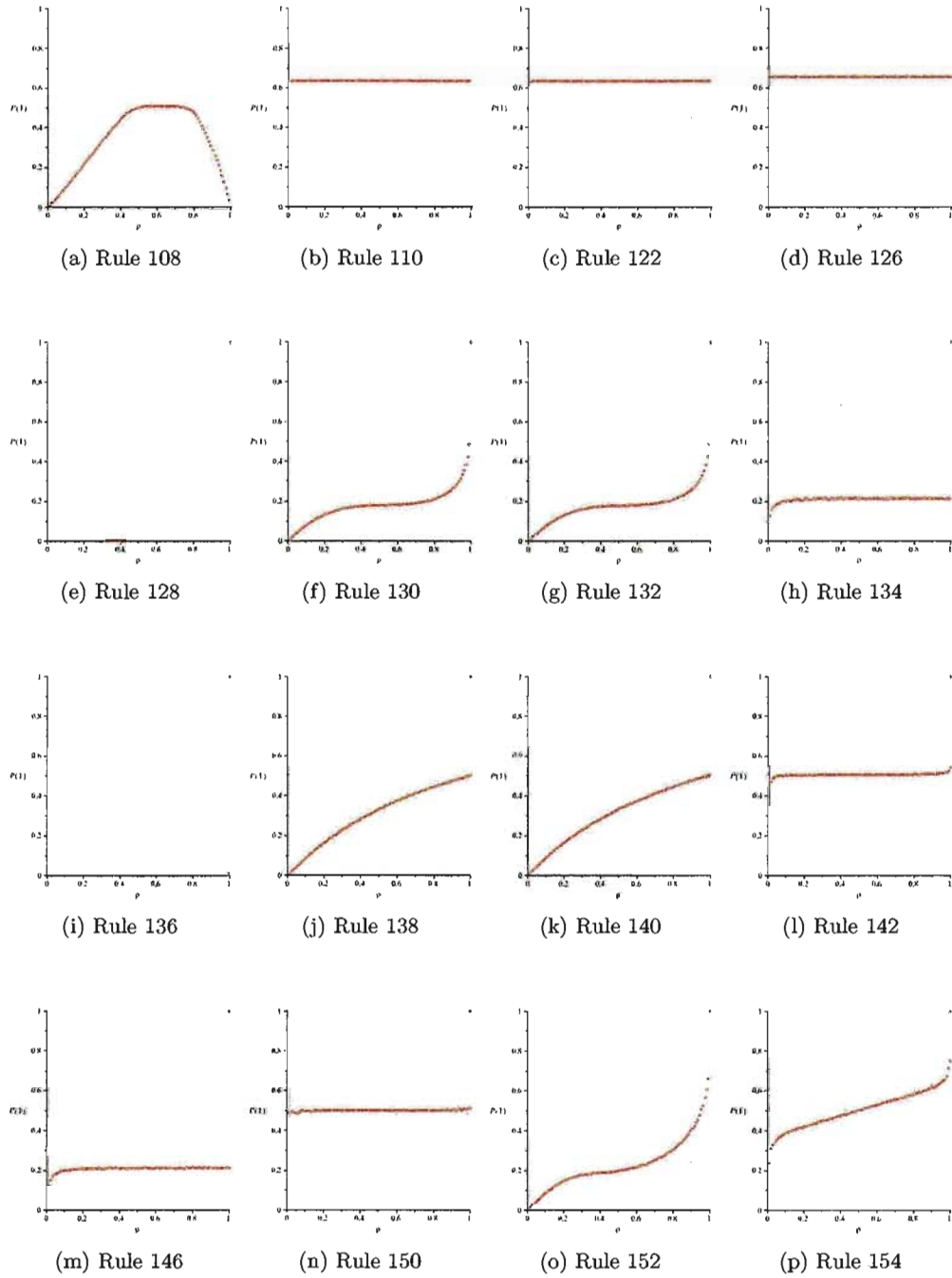


Figure A.5: Experimental Response Curves (Rules 108-154)

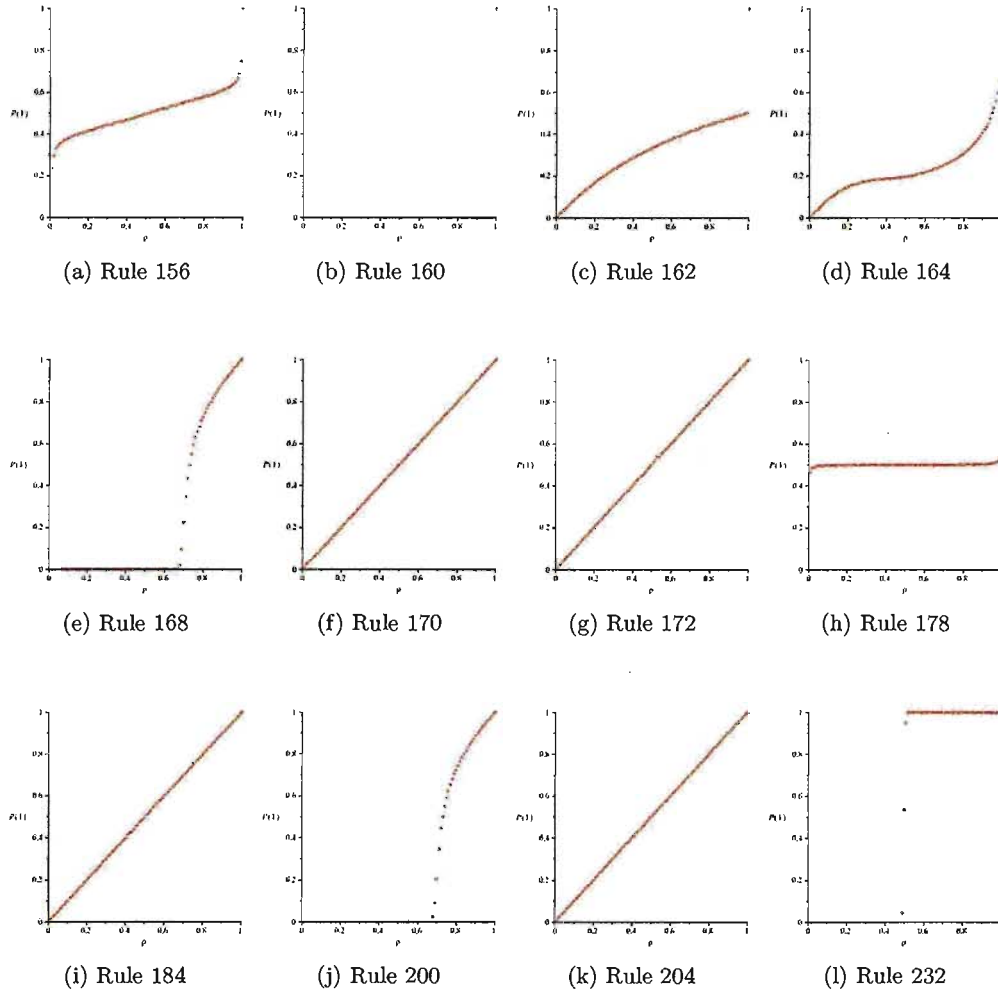


Figure A.6: Experimental Response Curves (Rules 156-232)

Experimental Response Surfaces

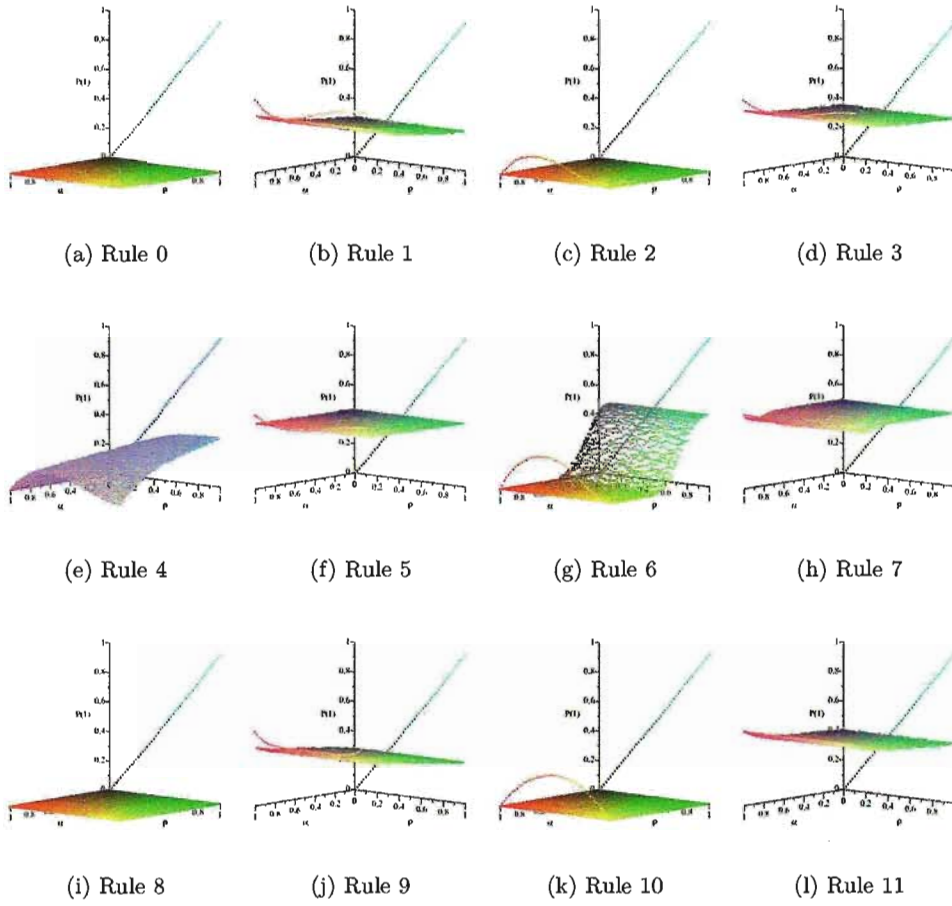
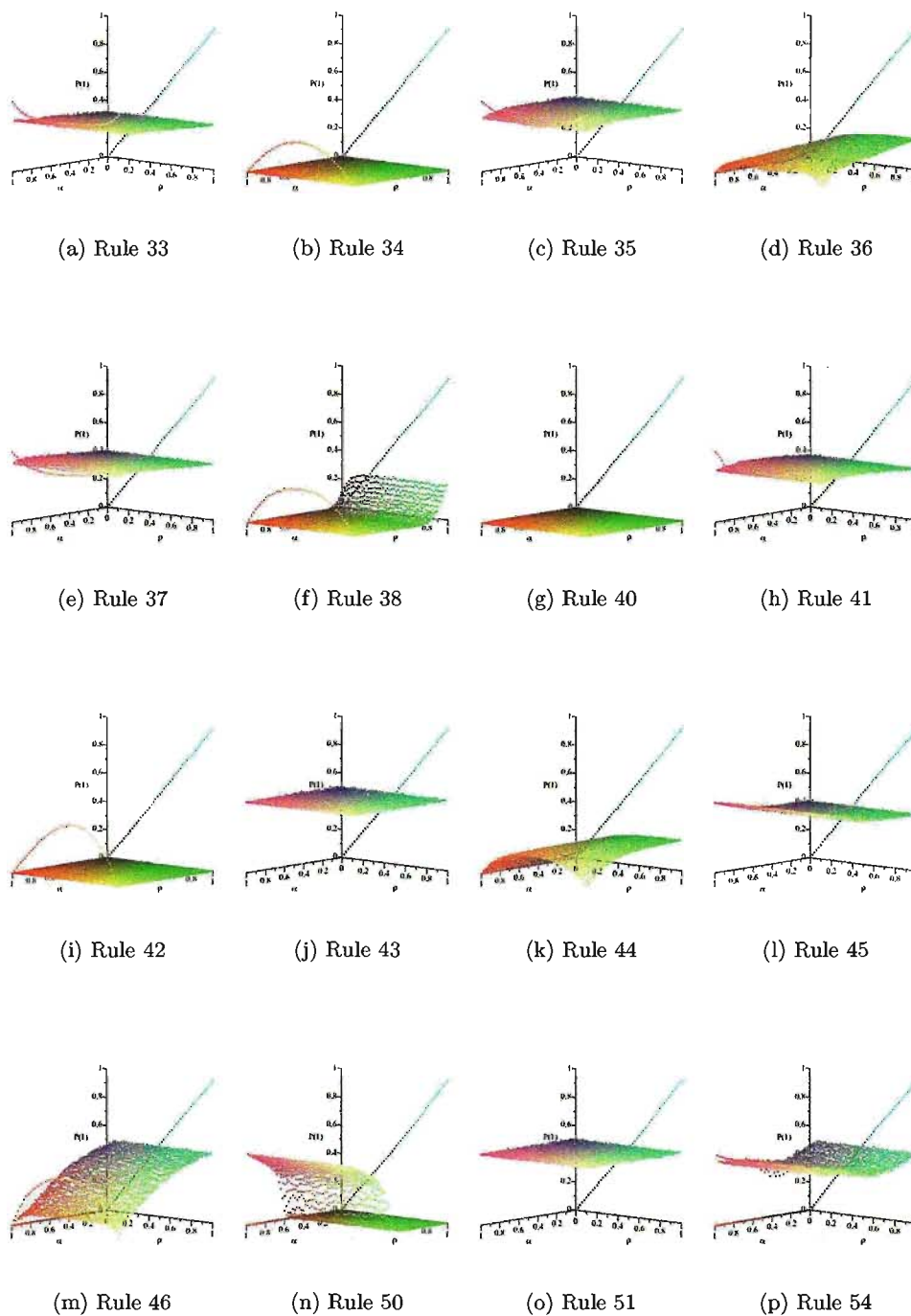


Figure B.1: Experimental Response Surfaces (Rules 0-11)





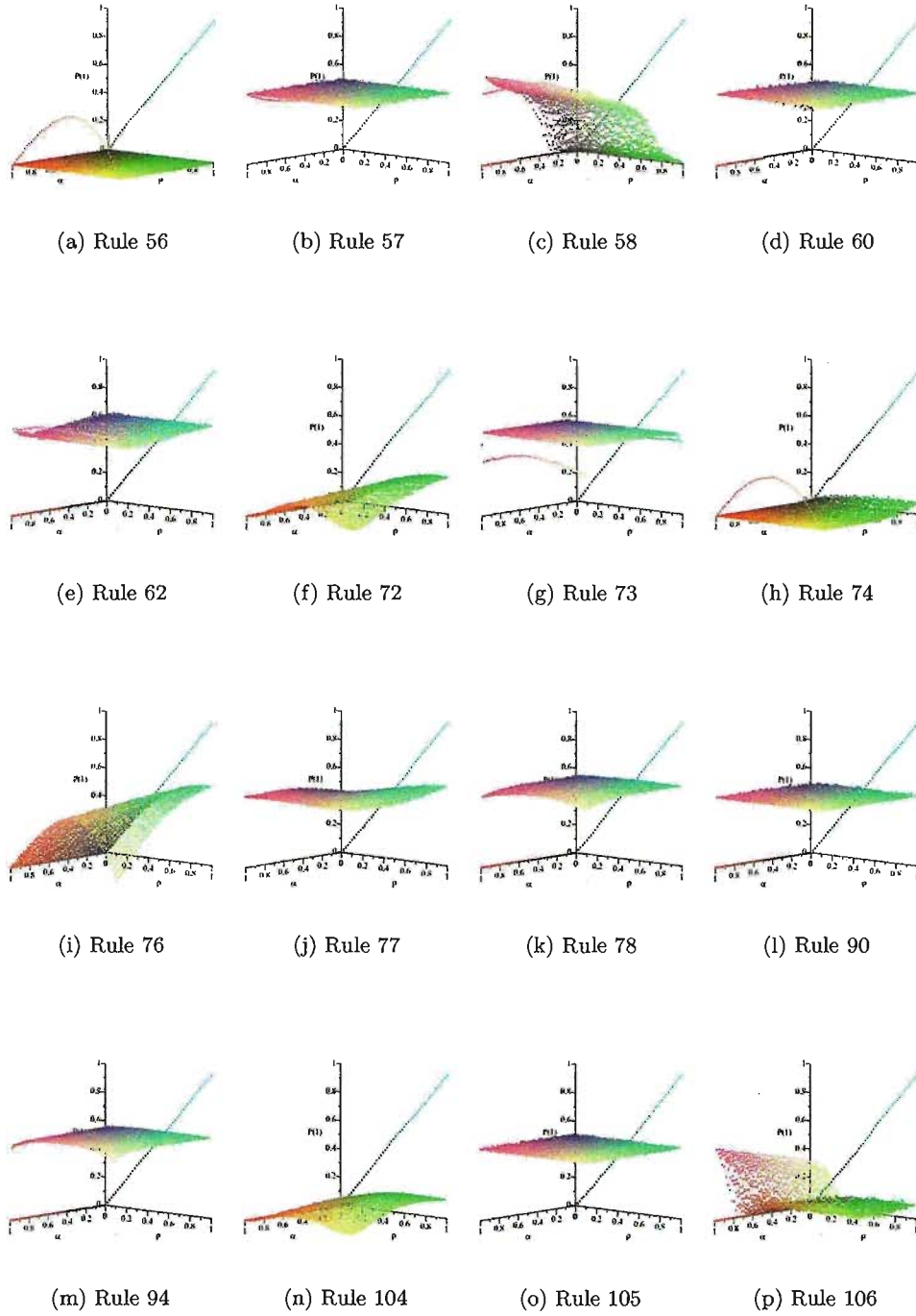


Figure B.4: Experimental Response Surfaces (Rules 56-106)

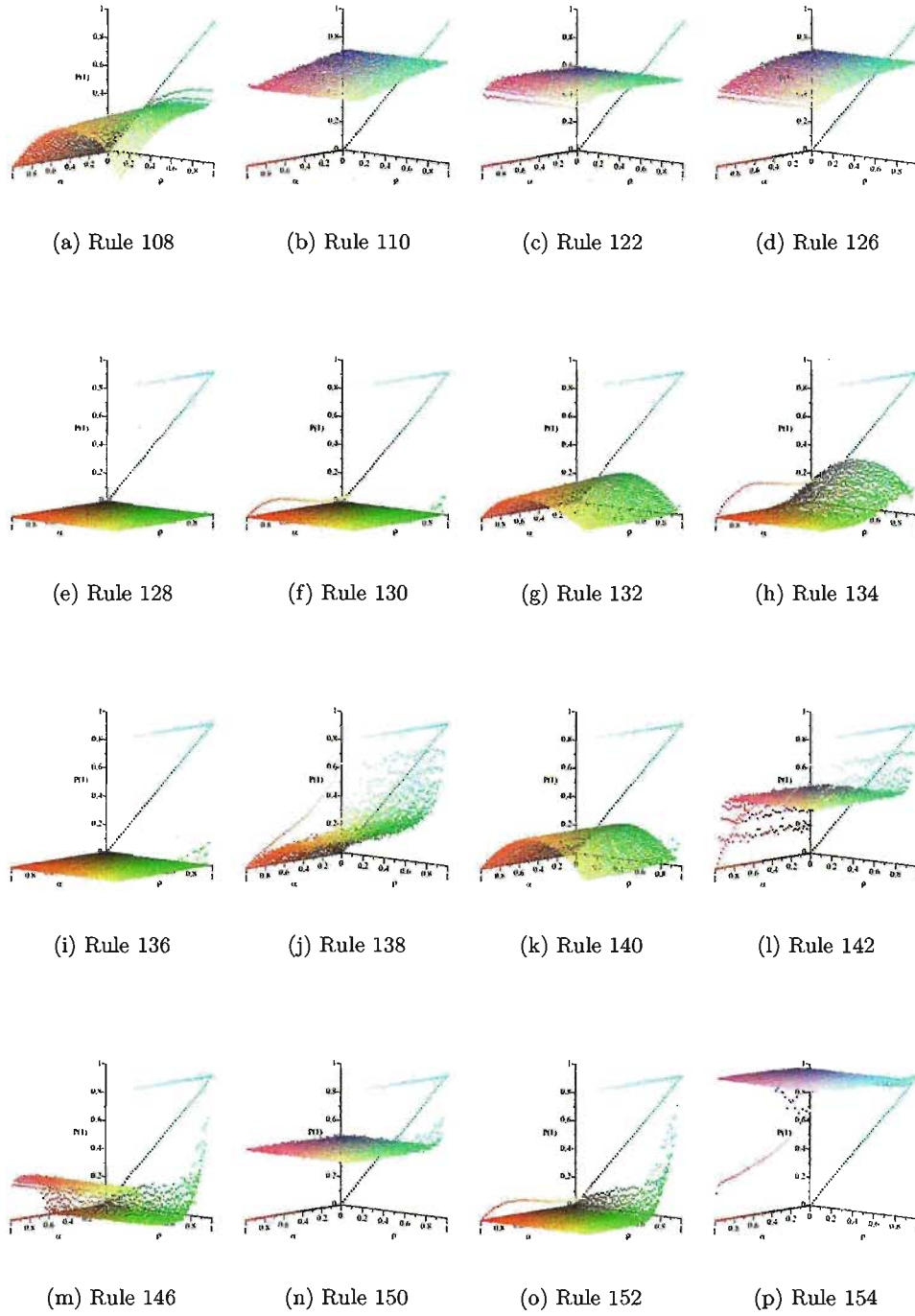


Figure B.5: Experimental Response Surfaces (Rules 108-154)

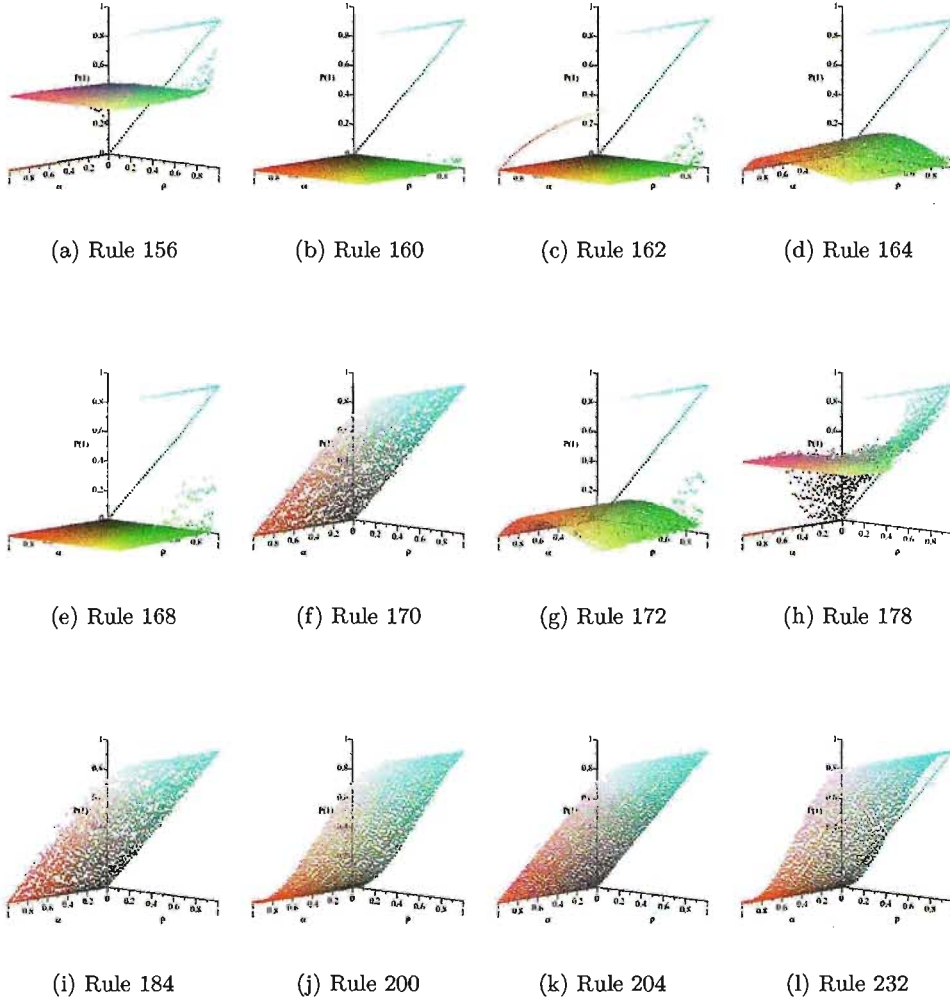


Figure B.6: Experimental Response Surfaces (Rules 156-232)


```

    for(b[5]=0; b[5]<=1; b[5]++){

        // iterate once and count all images
        block = 4*r[4*b[0]+2*b[1]+b[2]]+2*r[4*b[1]+2*b[3]+b[4]]+r
            [4*b[2]+2*b[4]+b[5]];
        imagecounter[block]++;

        // if image violates balance
        if(imagecounter[block] > 8){
            goto balancefailure;
        }
    }
}

// if no violation, print rule number
for(j=0; j<=length-1; j++){
    rulenum += pow(2,length-1-j)*myints[j];
}
cout << rulenum << "\n";
rulenum = 0;
}
balancefailure::

// reset the image counter
for(j=0; j<=length-1; j++){
    imagecounter[j] = 0;
}
} while ( next_permutation (myints,myints+length) );
}

// permutive check
int permutive(int r[8]){
    if(r[0]+r[4]==1 && r[1]+r[5]==1 && r[2]+r[6]==1 && r[3]+r[7]==1){
        return 0;
    }
    else if(r[0]+r[2]==1 && r[1]+r[3]==1 && r[4]+r[6]==1 && r[5]+r
        [7]==1){
        return 0;
    }
    else if(r[0]+r[1]==1 && r[2]+r[3]==1 && r[4]+r[5]==1 && r[6]+r
        [7]==1){
        return 0;
    }
    else{
        return 1;
    }
}
}

```

The following code checks for violations of the balance condition. First, we consider only those rules in which the local rule table contains an equal number of blocks mapping to 1 as to 0. Then, we check to see if the rule is permutive and if so, we ignore since we know it to be surjective. Finally, we construct all blocks in \mathcal{D}_3 , then \mathcal{D}_4 and apply the rule until we obtain a violation of the balance condition. The output of this program shows us that there are a small number of rules for which surjectivity can not be established in this way. Many of these rules are actually equivalent to known surjective rules with L-neighbourhood. We were unable to classify 16 outputted rules in terms of surjectivity.

[illegible]

```

    for(b[8]=0; b[8]<=1; b[8]++){
    for(b[9]=0; b[9]<=1; b[9]++){
    for(b[10]=0; b[10]<=1; b[10]++){
    for(b[11]=0; b[11]<=1; b[11]++){
    for(b[12]=0; b[12]<=1; b[12]++){

        // iterate once and count all images
        block = 16*r[16*b[0]+8*b[1]+4*b[2]+2*b[3]+b[6]]+8*r
            [16*b[1]+8*b[4]+4*b[5]+2*b[6]+b[9]]+4*r[16*b
            [2]+8*b[5]+4*b[6]+2*b[7]+b[10]]+2*r[16*b[3]+8*b
            [6]+4*b[7]+2*b[8]+b[11]]+r[16*b[6]+8*b[9]+4*b
            [10]+2*b[11]+b[12]];
        imagecounter[block]++;

        // if image violates balance
        if(imagecounter[block] > 256){
            goto balancefailure;
        }
    }}}}}}}}}}}

// no violation - set up all blocks in D4
for(b[0]=0; b[0]<=1; b[0]++){
for(b[1]=0; b[1]<=1; b[1]++){
for(b[2]=0; b[2]<=1; b[2]++){
for(b[3]=0; b[3]<=1; b[3]++){
for(b[4]=0; b[4]<=1; b[4]++){
for(b[5]=0; b[5]<=1; b[5]++){
for(b[6]=0; b[6]<=1; b[6]++){
for(b[7]=0; b[7]<=1; b[7]++){
for(b[8]=0; b[8]<=1; b[8]++){
for(b[9]=0; b[9]<=1; b[9]++){
for(b[10]=0; b[10]<=1; b[10]++){
for(b[11]=0; b[11]<=1; b[11]++){
for(b[12]=0; b[12]<=1; b[12]++){
for(b[13]=0; b[13]<=1; b[13]++){
for(b[14]=0; b[14]<=1; b[14]++){
for(b[15]=0; b[15]<=1; b[15]++){
for(b[16]=0; b[16]<=1; b[16]++){
for(b[17]=0; b[17]<=1; b[17]++){
for(b[18]=0; b[18]<=1; b[18]++){
for(b[19]=0; b[19]<=1; b[19]++){
for(b[20]=0; b[20]<=1; b[20]++){
for(b[21]=0; b[21]<=1; b[21]++){
for(b[22]=0; b[22]<=1; b[22]++){
for(b[23]=0; b[23]<=1; b[23]++){
for(b[24]=0; b[24]<=1; b[24]++){

```

```
// iterate once and count all images
block = 4096*r[16*b[0]+8*b[1]+4*b[2]+2*b[3]+b[6]] +
        2048*r[16*b[1]+8*b[4]+4*b[5]+2*b[6]+b[11]] +
        1024*r[16*b[2]+8*b[5]+4*b[6]+2*b[7]+b[12]] +
        512*r[16*b[3]+8*b[6]+4*b[7]+2*b[8]+b[13]] + 256*
r[16*b[4]+8*b[9]+4*b[10]+2*b[11]+b[16]] + 128*r
[16*b[5]+8*b[10]+4*b[11]+2*b[12]+b[17]] + 64*r
[16*b[6]+8*b[11]+4*b[12]+2*b[13]+b[18]] + 32*r
[16*b[7]+8*b[12]+4*b[13]+2*b[14]+b[19]] + 16*r
[16*b[8]+8*b[13]+4*b[14]+2*b[15]+b[20]] + 8*r
[16*b[11]+8*b[16]+4*b[17]+2*b[18]+b[21]] + 4*r
[16*b[12]+8*b[17]+4*b[18]+2*b[19]+b[22]] + 2*r
[16*b[13]+8*b[18]+4*b[19]+2*b[20]+b[23]] + r[16*
b[18]+8*b[21]+4*b[22]+2*b[23]+b[24]];
imagecounter2[block]++;

// if image violates balance
if(imagecounter2[block] > 4096){
    goto balancefailure;
}

}}}}}}}}}}}}}}}}}}}}}}}}}}}}

// if still no violation , print rule number
rulenumber += 1073741824*myints[0]+1073741824*myints[0];
for(j=1; j<=length-1; j++){
    rulenumber += pow(2,length-1-j)*myints[j];
}
cout << rulenumber << "\n";
rulenumber = 0;
}

balancefailure;;

// reset the image counters
for(j=0; j<=31; j++){
    imagecounter[j] = 0;
}
for(j=0; j<=8191; j++){
    imagecounter2[j] = 0;
}

} while ( next_permutation (myints, myints+length) );
}
```

```

// permutive check
int permutive(int r[32]){

    if(r[0]+r[16]==1 && r[1]+r[17]==1 && r[2]+r[18]==1 && r[3]+r
        [19]==1 && r[4]+r[20]==1 && r[5]+r[21]==1 && r[6]+r[22]==1 && r
        [7]+r[23]==1 && r[8]+r[24]==1 && r[9]+r[25]==1 && r[10]+r
        [26]==1 && r[11]+r[27]==1 && r[12]+r[28]==1 && r[13]+r[29]==1
        && r[14]+r[30]==1 && r[15]+r[31]==1){
        return 0;
    }
    else if(r[0]+r[8]==1 && r[1]+r[9]==1 && r[2]+r[10]==1 && r[3]+r
        [11]==1 && r[4]+r[12]==1 && r[5]+r[13]==1 && r[6]+r[14]==1 && r
        [7]+r[15]==1 && r[16]+r[24]==1 && r[17]+r[25]==1 && r[18]+r
        [26]==1 && r[19]+r[27]==1 && r[20]+r[28]==1 && r[21]+r[29]==1
        && r[22]+r[30]==1 && r[23]+r[31]==1){
        return 0;
    }
    else if(r[0]+r[2]==1 && r[1]+r[3]==1 && r[4]+r[6]==1 && r[5]+r
        [7]==1 && r[8]+r[10]==1 && r[9]+r[11]==1 && r[12]+r[14]==1 && r
        [13]+r[15]==1 && r[16]+r[18]==1 && r[17]+r[19]==1 && r[20]+r
        [22]==1 && r[21]+r[23]==1 && r[24]+r[26]==1 && r[25]+r[27]==1
        && r[28]+r[30]==1 && r[29]+r[31]==1){
        return 0;
    }
    else if(r[0]+r[1]==1 && r[2]+r[3]==1 && r[4]+r[5]==1 && r[6]+r
        [7]==1 && r[8]+r[9]==1 && r[10]+r[11]==1 && r[12]+r[13]==1 && r
        [14]+r[15]==1 && r[16]+r[17]==1 && r[18]+r[19]==1 && r[20]+r
        [21]==1 && r[22]+r[23]==1 && r[24]+r[25]==1 && r[26]+r[27]==1
        && r[28]+r[29]==1 && r[30]+r[31]==1){
        return 0;
    }
    else{
        return 1;
    }
}

```