Zero-Sum Problems in Finite Cyclic Groups

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Abstract

The purpose of this thesis is to investigate some open problems in the area of combinatorial number theory referred to as zero-sum theory. A zero-sequence in a finite cyclic group G is said to have the basic property if it is equivalent under group automorphism to one which has sum precisely $|G|$ when this sum is viewed as an integer. This thesis investigates two major problems, the first of which is referred to as the basic pair problem. This problem seeks to determine conditions for which every zero-sequence of a given length in a finite abelian group has the basic property. We resolve an open problem regarding basic pairs in cyclic groups by demonstrating that every sequence of length four in $\mathbb{Z}_p$ has the basic property, and we conjecture on the complete solution of this problem. The second problem is a 1988 conjecture of Kleitman and Lemke, part of which claims that every sequence of length n in $\mathbb{Z}_n$ has a subsequence with the basic property. If one considers the special case where n is an odd integer we believe this conjecture to hold true. We verify this is the case for all prime integers less than 40, and all odd integers less than 26. In addition, we resolve the Kleitman-Lemke conjecture for general n in the negative. That is, we demonstrate a sequence in any finite abelian group isomorphic to $\mathbb{Z}_{2p}$ (for $p \geq 11$ a prime) containing no subsequence with the basic property. These results, as well as the results found along the way, contribute to many other problems in zero-sum theory.

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Chapter 1

Introduction

Problems in combinatorial number theory typically involve number theoretic problems which involve combinatorial approaches in their formulations or solutions. The area of mathematics referred to as zero-sum theory is a class of problems set in combinatorial number theory. Typical zero-sum problems can occur naturally in many areas of mathematics, including but not limited to branches of combinatorics, number theory and geometry. There are also fundamental connections with graph theory, Ramsey theory and geometry.

The purpose of this thesis is to investigate some open problems in zero-sum theory. In particular, we investigate a 20 year old conjecture of Daniel Kleitman (MIT) and Mark Lemke (University of Minnesota) and approach some of the questions that arise as a result of these investigations. We also discuss in significant detail another open problem in zero-sums referred to as the basic pair problem. This question, as we will see, is intrinsically connected to the Kleitman-Lemke (KL) conjecture as well as many other problems in combinatorial number theory. We will summarize the nature of the basic pair problem and eventually solve an open conjecture which helps to resolve the overall problem. This result, which we refer to as the Four Sum Theorem, is the principal result in the first part of the thesis. The Four Sum Theorem helps to reduce the basic pair problem as it pertains to cyclic groups to one final conjecture we state in Chapter 2. The second part of the thesis is centered around the Kleitman-Lemke conjecture, which was originally presumed to hold true for all values of n. Chapter 3 deals with the conjecture only as it pertains to prime integers, in which case we believe that the original conjecture holds true. We demonstrate this assertion by verifying that it does indeed hold for all primes less than 40. Chapter 4 deals with the conjecture as it pertains to composite integers. In a surprising turn, we will prove that the conjecture is false for arbitrary values of n. While we still believe the conjecture may hold true for all odd integers, we demonstrate a class of even integers greater than or equal to 22 for which the conjecture fails to hold. We also verify that the conjecture does hold for all composite integers less than 22, and for the cases where n equals 24 or 25.

This thesis examines these and other questions, developing results along the way which contribute to other parts of zero-sum theory. Some previously known results are included in order to add merit and importance to the results found in this thesis, but the majority of the results are original in nature. Most of the proofs in this thesis take a combinatorial approach, and do not rely heavily on previously known non-trivial results. We begin by describing and introducing zero-sum theory, then by presenting some classical results, ultimately reaching the KL conjecture.

1.1 Preliminaries

In the case of zero-sum theory, algebraic methods are a powerful tool for solving many problems, and as such the problems are generally set in the framework of algebra. In general, we will assume G to be an additively written finite abelian group. Given a positive integer n, we will denote the cyclic group with n elements by $\mathbb{Z}_n$. In this thesis we typically only consider the case where $G = \mathbb{Z}_n$. A finite sequence $X = \{x_1, ..., x_k\}$ of elements of G (sometimes written in the literature as $x_1 * \cdots * x_k$), where the repetition of elements is allowed and their order is disregarded, is simply called a *sequence* over G. If $X = \{x_1, ..., x_k\}$ then k is called the *length* of X, sometimes written $|X| = k$. If X is a sequence with the property that $x_1 + ... + x_k = 0$ in G then X is called a *zero-sequence*. For example, consider the group $\mathbb{Z}_6$. Then $X = \{1, 3, 3, 3, 4, 4\}$ is a zero-sequence in $\mathbb{Z}_6$ of length 6 since $1+3+3+3+4+4 = 18 = 0 \ mod\ 6$. For longer sequences we may write a sequence $X = \{x_1^{l_1}, ..., x_k^{l_k}\}$ where the x_i's are distinct and have cardinality l_i in the sequence. (Note that a sequence written in this form has length $\sum_{i=1}^{k} l_i$, not k necessarily). So in the above example, we may write $X = \{1, 3^3, 4^2\}$. A *subsequence* Y of X in G is simply a sequence $Y = \{y_1, .., y_m\}$ that is obtained by deleting some of the elements of X. For example, $\{1, 3, 4, 4\}$ is a zero-subsequence of the above sequence X.

A typical zero-sum problem studies conditions which ensure that given sequences have non-empty zero-sum subsequences with prescribed properties. For example, the classic EGZ Theorem, which is generally considered to be the first non-trivial result in zero-sum theory, is an example of the type of problem generally encountered in zero-sums. In 1961 Erdös, Ginzburg and Ziv posed the problem to find the smallest integer k such that any sequence of elements of $\mathbb{Z}_n$ of size k necessarily has a zero-subsequence of length n. In [5] it is demonstrated that a sequence of size $2n-1$ integers always has such a subsequence. This question, along with the ongoing question of the determination of the Davenport Constant (which is discussed next) was the launching ground for research in zero-sum theory [See [9] and [1] for some survey papers on zero-sum theory]. The majority of this thesis is devoted to sequences of length n in $\mathbb{Z}_n$, and some of the properties incumbent upon such sequences. One of the most fundamental properties of such a sequence is related to the aforementioned Davenport constant, which we will briefly address now.

1.2 The Davenport Constant

A question originally posed by Baayen, Erdös and Davenport (see [13]) in 1967, the determination of an important constant (which has come to be called the Davenport constant) for the general finite abelian group has become the most famous problem in zero-sum theory. While Davenport's original motivation for solving the problem concerned prime ideal decompositions in algebraic number fields, the determination of the Davenport constant is now considered to be one of the most important unsolved problems concerning finite abelian groups. Determining the constant even for some very restricted families of groups has proved to be an interesting and difficult combinatorial problem.

Definition 1. *Let G be a finite abelian group (written additively). The Davenport constant of G, denoted $D(G)$, is the smallest integer k so that every sequence in G with length k has a nontrivial zero-sum subsequence.*

The determination of $D(\mathbb{Z}_n)$ is a fairly easy and well known result. Since much of this thesis is devoted to sequences of length n in $\mathbb{Z}_n$, this value is a useful and important observation.

Theorem 1.2.1. $D(\mathbb{Z}_n) = n$. *That is, any sequence of length n in $\mathbb{Z}_n$ contains a zero-subsequence.*

Proof. Note that the sequence $\{1^{n-1}\}$ is a sequence of length $n-1$ in $\mathbb{Z}_n$ which clearly has no zero-sum subsequence, hence $D(\mathbb{Z}_n) \geq n$. Then let $X = \{x_1, ..., x_n\}$ be any sequence of length n in $\mathbb{Z}_n$. Consider the sums $x_1, x_1 + x_2, \ldots, x_1 + x_2 + \cdots + x_n$. Note that there are n such sums, so if all of these sums are distinct then one must be equal to 0 *mod* n, and we have a zero-sum subsequence. So suppose that two sums are equal, and we can assume $x_1 + ... + x_i = x_1 + ... + x_j$ for some $i < j$. Then $x_{i+1} + ... + x_j = 0 \ mod \ n$ and we have a zero-sum subsequence. Hence $D(\mathbb{Z}_n) \leq n$. Therefore we have that $D(\mathbb{Z}_n) = n$. □

Therefore for cyclic groups we can always assume that a sequence of length n has a zero-sum subsequence. Expanding to general finite abelian groups proves very difficult. Recall by the fundamental theorem of finite abelian groups we can write $G = \mathbb{Z}_{n_1} \oplus ... \oplus \mathbb{Z}_{n_k}$ for any finite abelian group of order n. We note that the upper bound assertion of $D(G) \leq |G|$ in the above proof is still valid if we consider any finite abelian group G. We also have an easy lower bound if we consider the sequence with $n_i - 1$ copies of the element with a 1 in the i^{th} position and 0's elsewhere for each i. This sequence clearly has no subsequence which sums to zero, thus $D(G) \geq 1 + \sum_{i=1}^{k}(n_i - 1)$. However the situation gets much more difficult as soon we try to narrow these bounds any further. We mention two classical results of Olson, then proceed to the problem central to this thesis. For more information on the Davenport constant and its variants, the reader is encouraged to consult [12], [10], [3], [6], [8].

Theorem 1.2.2. *Olson[12]*

- *If* $a|b$ *then* $D(\mathbb{Z}_a \oplus \mathbb{Z}_b) = a + b - 1$.
- *If* p *is prime, then* $D(\mathbb{Z}_p^{l_1} \oplus \cdots \oplus Z_p^{l_k})) = 1 + \sum_{i=1}^{l}(p^{l_i} - 1)$

1.3 The Kleitman and Lemke Conjecture

In 1988 Mark Lemke and Daniel Kleitman [11] resolved a conjecture made by Paul Erdös and Mark Lemke few years earlier related to zero-sums. They showed that given some positive integers n and d with $d|n$, any set of d divisors of n has a subset with sum divisible by d and dividing n. The result and subsequent proof spanned several new questions centered in combinatorial number theory. In particular, they made the following conjecture, which has remained open for the past 20 years.

Conjecture 1.3.1. *Given positive integers n and d with $d|n$, and integers $x_1, ..., x_n$ there exists an integer m relatively prime to n and a subset S of $\{1, 2, ..., n\}$ with $|S| \leq d$ such that*

$$d | \sum_{i \in S} mod(mx_i, n)$$

and

$$\sum_{i \in S} mod(mx_i, n) | n$$

where mod (k, n) denotes the least positive residue of k modulo n.

They mentioned that they had verified this conjecture for all integers $n \leq 11$. We will ultimately show that this conjecture does not hold for general n. We consider the two separate cases when n is prime and composite, rewriting the above conjecture into the following two conjectures. The case where n is prime is discussed in Chapter 3 and the case where n is composite is discussed in Chapter 4. We start with the prime case.

Conjecture 1.3.2. *Given a prime p and integers $0 < x_1, ..., x_p < p$ there exists an integer m coprime to p and a subset S of $\{1, ..., p\}$ such that $\sum_{i \in S} mod(mx_i, p) = p$.*

Remark 1.3.3. *We exclude the case where some $x_i = p$. In this situation the conjecture is trivially true (take $S = \{i\}$ and $m = 1$). Also note the $d = 1$ is trivial. Take any integer x and there exists an m such that mod $(mx, p) = 1$, satisfying the conjecture.*

Note the statement becomes simpler in the case of a prime integer p. In particular we only need to consider the case where $d = p$, and m can be any integer satisfying $1 \leq m < p$. We take a moment to clarify what the conjecture says. We can consider the integers $0 < x_1, ..., x_p < p$ to be a sequence of non-zero elements in $\mathbb{Z}_p$. Then given a sequence X in $\mathbb{Z}_p$ we are looking for a subsequence with the property that the elements of X (as integers) sum to precisely p after multiplication *mod* p by some integer m coprime to p. For example, the sequence $X = \{3, 3, 3, 4, 4\}$ in $\mathbb{Z}_5$ has length 5. Then if we consider the subsequence $Y = \{3, 3, 4\}$ and the integer $m = 2$, note that $mod(2 \cdot 3, 5) = 1$ and $mod(2 \cdot 4, 5) = 3$ so that the sequence $2Y = \{1, 1, 3\}$ has element sum precisely p. Therefore we are trying to show that for any sequence X of length p, that a subsequence Y and m value exist with not only the property that Y is a zero-subsequence (and therefore the elements of Y will sum to a multiple of p, this always exists since $D(\mathbb{Z}_n) = n$) but that the elements of mY will sum to precisely p. Such sequences will be formally defined in the next chapter. We believe that Conjecture 1.3.2 is true for all p, and have found no counterexample to show otherwise. The next conjecture, which makes up the other part of Conjecture 1.3.1, will be proved false for arbitrary n, but it may hold true if we replace "composite integer" with "odd integer". We have no counterexample to prove the contrary.

Conjecture 1.3.4. *Given a positive composite integer n and d a divisor of n, and integers $x_1, ..., x_n$ not divisible by d, there exists an m relatively prime to n and a subset S of $\{1, 2, ..., n\}$ such that $d | \sum_{i \in S} mod(mx_i, n)$ and $\sum_{i \in S} mod(mx_i, n) | n$ with $|S| \leq d$.*

We now proceed to investigate these and other zero-sum problems. As stated, the remainder of the thesis is generally divided into two parts, the basic pair problem in Chapter 2 and the KL conjecture in Chapters 3 and 4. Chapter 2 is theoretical in nature, and introduces several new terms that will simplify the KL conjecture and related problems. The most significant result in this part is the Four Sum Theorem (Section 2.3), which solves a previously open conjecture regarding shorter sequences in $\mathbb{Z}_p$ in the affirmative. The first part of Chapter 3 (Sections 3.1-3.2) is also theoretical in nature. We introduce some new concepts in zero-sums and develop some results which clarify the KL conjecture and are applicable in other areas of zero-sum theory.

The second part of this chapter (3.3) offers some results directly aimed at simplifying the KL conjecture, then offers computational evidence that the conjecture holds for all $p < 40$. The first part of Chapter 4 (Sections 4.1-4.4) is again theoretical in nature and introduces more new concepts in zero-sums. The most significant result in this chapter is section 4.5 which resolves the KL conjecture in the negative. This section opens new questions in zero-sums and increases the significance of many earlier results contained in this thesis. The conclusion of the thesis (Section 4.6) is again computational in nature, offering evidence of when the KL conjecture does in fact hold for certain composite values of $n < 26$.

Chapter 2

Minimal Zero-Sequences and the Basic Pair Problem.

2.1 Introduction

In Chapter 1, we discussed the zero-sum problem, and noted some classical results involving zero-sums in finite abelian groups. We also introduced a conjecture of Lemke and Kleitman involving integer sums in $\mathbb{Z}_n$. In this chapter, we will introduce some zero-sequence concepts closely related to this conjecture. We introduce the notions of minimal zero-sequences and equivalent sequences. In particular, we will investigate the circumstances in which every minimal zero-sequence in $\mathbb{Z}_n$ of length k can be transformed under group automorphism into a sequence with sum precisely n. Such sequences are important in a number of frequently asked zero-sum questions in cyclic groups. If a sequence $X = \{x_1, ...x_k\}$ is a zero-sequence in G, but no proper subsequence of X is a zero-sequence, then we say X is a *minimal zero-sequence.* We will denote the set of all minimal zero-sequences in G by $MZS(G)$. Several important results in zero-sum theory are related to the set of potential sums of elements of a sequence. Define the *sumset* of a sequence to be the set of all $y \in G$ representable by a nonempty subsequence sum. We denote $\Sigma(X)$ as the sumset of X. If $0 \notin \Sigma(X)$ then X is called *zero-free* and if $0 \in \Sigma(X)$ then X has a zero-subsequence.

We say that two zero-sequences X and Y are *equivalent* if $X = \phi(Y)$ for some group automorphism ϕ. Notice that this relation is an equivalence relation on the set of all (minimal) zero-sum sequences. So $Aut(G)$ acts on $MZS(G)$, and we let $\vartheta(X) = \{\phi(X)|\phi \in Aut(G)\}$ be the set of all minimal zero-sequences equivalent to X. Since any automorphism in $\mathbb{Z}_n$ is equivalent to multiplication by an integer relatively prime to n, we always assume that $\phi(x) = mx$ for some m with $gcd(m,n) = 1$ and $1 \leq m < n$.

When $G = \mathbb{Z}_n$, take $\sigma(X) = \sum x_i$ where $1 \leq i \leq k$ to denote the sum of the elements of X as an integer, called the *sum* of X (for example, if $G = \mathbb{Z}_3$, and $X = \{2,2,2\}$, then $\sigma(X) = 6$). Then, using the notation of [14], we say that a zero-sequence X in $\mathbb{Z}_n$ is *basic* if it is equivalent to a zero-sequence whose sum is exactly n. If every minimal zero-sequence of length k from $\mathbb{Z}_n$ is basic, then $(\mathbb{Z}_n, k)$ is called a basic pair. It is an open question as to which $(\mathbb{Z}_n, k)$ are basic pairs. In this chapter we describe the basic pair problem, and eventually solve a conjecture involving relatively short sequences in $\mathbb{Z}_n$. This new result in turn reduces the open basic pair problem to the resolution of a new conjecture made in section 2.3.

2.2 Basic Pairs in $\mathbb{Z}_n$

Assume that $X = \{x_1, ..., x_n\}$ is a sequence of length n in $\mathbb{Z}_n$. Recall from Chapter 1 that $\mathbb{Z}_n$ has Davenport constant $D(\mathbb{Z}_n) = n$, and thus any sequence of length n has at least one non-empty zero-subsequence. Note that the existence of such a subsequence does not imply the existence of a basic subsequence. Also notice that if

$(\mathbb{Z}_n, k)$ is a basic pair, and X has a minimal zero-subsequence of length k, then we have verified Conjecture 1.3.1 in the case where $d = n$.

We will now introduce several important concepts in zero-sum problems. Let $v_X(x_i)$ denote the multiplicity of the element x_i in X. If the choice of sequence X is clear, we simply write $v(x_i)$. For now, let the height of the sequence X (denoted $H(X)$) to be the largest occurring multiplicity of any element in X. For example, if $X = \{1, 1, 4, 4, 4, 5, 7\}$ in $\mathbb{Z}_{11}$, then $H(X) = 3$ and $v(1) = 2$.

Another concept in zero-sum theory that is the focus of much research is the index of a sequence. If we take $X \in MZS(\mathbb{Z}_n)$ such that $\sigma(X) = kn$, Chapman, Freeze and Smith defined the *Index* of X denoted $Ind(X)$ to be the smallest possible $\bar{k}$ such that $\sigma(Y) = \bar{k}n$ for some $Y \in \vartheta(X)$. Note that using this definition, if $Ind(X) = 1$ we have immediately that X is a basic sequence. If $Ind(X) = 1$ for every sequence of length k in $MZS(G)$ then (G, k) is a basic pair. Chapman, Freeze and Smith proved in [2] that:

Theorem 2.2.1. *$Ind(X) = 1$ for every $X \in MZS(\mathbb{Z}_n)$ if and only if $n = 2, 3, 4, 5$ or 7.*

This implies that for these values of n that every minimal zero-sequence in $\mathbb{Z}_n$ is basic. Therefore for these n, given any value of k, $(\mathbb{Z}_n, k)$ is a basic pair. Also note that since any sequence of length n has some minimal zero-subsequence, we obtain that Conjecture 1.3.1 must hold for these values of n when $n = d$. The Index of a sequence has since been generalized and can take on a more complicated definition. Many interesting questions are centered around considering the smallest integer k such that a sequence (not necessarily a zero-sequence) is equivalent to one with sum k. For more information on the Index of sequences in finite abelian groups, consult [15], [7].

2.3 Short Minimal Zero-Sequences in $\mathbb{Z}_n$

In this section, we examine the structure of zero-sequences of short length in $\mathbb{Z}_n$ and the part of the basic pair problem dealing with short length sequences. We demonstrate that as k grows we can always find non-basic sequences in $\mathbb{Z}_n$, so determining for precisely which n and k all short minimal zero-sequences will be basic is a useful result. In particular, we prove the previously open problem that $(\mathbb{Z}_p, 4)$ is a basic pair for any prime p. We begin with some easy observations, in particular, the following example will show that $(\mathbb{Z}_n, 1)$, $(\mathbb{Z}_n, 2)$ and $(\mathbb{Z}_n, 3)$ are basic pairs for any n.

Example 2.3.1. *Note since the only element in $MZS(\mathbb{Z}_n)$ of length 1 is $\{0\}$, it is clear that $(\mathbb{Z}_n, 1)$ is a basic pair. To see that $(\mathbb{Z}_n, 2)$ is a basic pair, note that if $X = \{x_1, x_2\} \in MZS(\mathbb{Z}_n)$, then we can assume that $\sigma(X) < 2n$. Since X is a zero-sequence it must have sum n so X is basic. To see $(\mathbb{Z}_n, 3)$ is a basic pair, note if $X = \{x_1, x_2, x_3\} \in MZS(\mathbb{Z}_n)$, we can assume that $x_1 \leq x_2 \leq x_3$. Once again, we have that $\sigma(X) < 3n$. If $\sigma(X) = n$ we are done, so assume that $\sigma(X) = 2n$ and note that we must have that $x_2 \geq \frac{n}{2}$. Then let $\phi \in Aut(\mathbb{Z}_n)$ be the automorphism such that $\phi(x) = (n-1)x$. Then $\phi(x_3) \leq \phi(x_2) \leq \frac{n}{2}$. Thus $\phi(x_2) + \phi(x_3) \leq n$ and so $\sigma(\phi(X)) \leq n + (n-1) < 2n$ and so X is basic and $(\mathbb{Z}_n, 3)$ is a basic pair.*

The case for $(\mathbb{Z}_n, 4)$ is not so simple. It is not always possible to find an automorphism for every minimal zero-sequence that reduces its sum to n. For example, in $\mathbb{Z}_6$, $X = \{1, 3, 4, 4\}$ is only equivalent to the sequences X and $5X = \{5, 3, 2, 2\}$, each of which has sum 12. Hence X is a non-basic sequence. In fact, it is known that if $(n, 6) \neq 1$, there always exists a non-basic minimal zero sequence in $\mathbb{Z}_n$ of length 4. Ponomarenko [14] demonstrates that if n is even $\{1, \frac{n}{2}, \frac{n}{2}+1, n-2\}$ is always non-basic, and if $3|n$ that $\{1, \frac{n}{3}+1, \frac{2n}{3}+1, n-3\}$ is non-basic. We summarize with the following theorem.

Theorem 2.3.2. *$(\mathbb{Z}_n, k)$ is a basic pair for $k = 1, 2, 3$. If $(n, 6) \neq 1$ then $(\mathbb{Z}_n, 4)$ is a* *non-basic pair.*

This leaves the question of precisely when every minimal zero-sequence of length four in $\mathbb{Z}_n$ must be basic. We offer a partial solution to this problem by proving that $(\mathbb{Z}_p, 4)$ is a basic pair for any prime p. It is actually more convenient to prove the following equivalent theorem, which we do by first reducing its equivalency to that of Proposition 2.3.5.

Theorem 2.3.3 (Four Sum Theorem). *Let p be a prime and x_1, x_2, x_3, x_4 integers such that $\sum x_i = kp$ for some integer k and no proper sum is divisible by p. Then there exists an integer m with $(m,p) = 1$ such that $\sum mod(mx_i, p) = p$*

Remark 2.3.4. *The sequence $X = \{x_1, x_2, x_3, x_4\}$ is minimal, and any automorphism in $\mathbb{Z}_p$ is equivalent to multiplication by an m with $(m,p) = 1$. Thus, Theorem 2.3.3 implies that any minimal zero-sequence of length 4 is basic, and therefore $(\mathbb{Z}_p, 4)$ is a basic pair.*

In proving this theorem, we can assume that $1 \leq x_i \leq p-1$. Then we note that if $k = 1$ we can take $m = 1$. In fact, we must have $k = 2$ or $k = 3$ since $\sum x_i < 4p$. In addition, we may assume that $x_1 \neq x_2 \neq x_3 \neq x_4$. To see this, note that if we have at least two equal integers, say $X = \{x_1, x_1, x_2, x_3\}$ with $\sigma(X) = kp$, then there exists an integer m with $(m,p) = 1$, such that $mod(mx_1, p) = 1$. Then if $\bar{X} = mX = \{1, 1, \bar{x}_2, \bar{x}_3\}$, we have $\sigma(\bar{X}) = 1 + 1 + \bar{x}_2 + \bar{x}_3 = \bar{k}p$ where $1 \leq \bar{x}_2, \bar{x}_3 < p$, thus $\bar{k} < 2p$ unless $\bar{x}_2 = \bar{x}_3 = p - 1$, which is impossible since $\bar{X}$ is minimal. So we must have that $\bar{k} = 1$, and Theorem 2.3.3 is proved.

Also, by multiplying by a suitable integer m we can always assume that $x_4 = 1$. Thus we can assume that $X = \{x_1, x_2, x_3, 1\}$ where $1 < x_1 < x_2 < x_3 < p$ and $\sigma(X) = kp$. Since $\sigma(X) < 3p$, we can assume that $k = 2$.

We now note that if all $x_1, x_2, x_3 > \frac{p}{2}$ and $1+x_1+x_2+x_3 = 2p$, we let $x = p-x_2$ and $y = p-x_3$. Then $x_2 = p-x$ and $x_3 = p-y$ and so $1+x_1 = x+y$. Then $1 < y < x < x_1$, and note that $x < \frac{p}{2}$. Then we obtain $2 \times 1 = 2$, $2x_2 = 2(p-x) \equiv p - 2x$ $(mod\ p)$, $2x_3 = 2(p-y) \equiv p - 2y$ and $2x_1 \equiv 2x_1 - p$ since $p < 2x_1 < 2p$. Therefore $mod(2,p) + mod(2x_1,p) + mod(2x_2,p) + mod(2x_3,p) = 2 + 2x_1 - p + p - 2x + p - 2y = 2(1 + x_1 - x - y) + p = p$ and we are done.

So we may assume that not all $x_1, x_2, x_3 > \frac{p}{2}$. Note that if $x_1, x_2 < \frac{p}{2}$, then $\sigma(X) < 2p$. So we may assume that $x_1 < \frac{p}{2}$ and $x_2, x_3 > \frac{p}{2}$. We handle such situations as follows. Let $c = x_1$, $b = p - x_2$, and $a = p - x_3$ so that $a < b$. Then note that $1 - a - b + c = 0$, and therefore, $1 + c = a + b$, so we must have that $c > b$. Then $\sigma(X) = 1 + c + (p-a) + (p-b) = 2p$, and Theorem 2.3.3 will follow from the following proposition.

Proposition 2.3.5. *Assume that* $1 + c + (p-a) + (p-b) = 2p$ *where* $1 < a < b < c < \frac{p}{2}$. *Let* k *be the smallest positive integer such that* $\lceil \frac{kp}{c} \rceil = m < \frac{kp}{b}$. *Then* $mod(m,p) + mod(mc,p) + mod(m(p-a),p) + mod(m(p-b),p) = p$.

Proof. Note that such an integer k always exists. Since $\frac{b}{c}(p) < p - 1$, we have $\lceil \frac{b}{c}p \rceil = m \leq p-1 < p = \frac{b}{b}(p)$. Thus such a k exists and $k \leq b$. By using the minimality of k and the fact that $\frac{kp}{c}$ is not an integer, we can show that $(k-1)p < mb < kp$ and $kp < mc < kp + p$, so $mod(mc,p) = mc - kp$ and $mod(m(p-b),p) = kp - mb$. If $ma < p(*)$, then we have that $mod(m,p) + mod(mc,p) + mod(m(p-a),p) + mod(m(p-b),p) = m + mc - kp + p - ma + kp - mb = p + m(1 + c - a + b) = p$, and we are done. We now show that $(*)$ always holds by using a case by case analysis.

Case 1: If $k = 1$, then $m < \frac{p}{b} < \frac{p}{a}$, and therefore $ma < p$ and we are done.

We remark that if $k \geq 2$, then by the minimality of k, $\lceil \frac{(k-1)p}{c} \rceil = \lceil \frac{(k-1)p}{b} \rceil$. Thus,

$$\frac{(k-1)p}{b} - \frac{(k-1)p}{c} = \frac{(c-b)(k-1)p}{cb} < 1. \tag{2.1}$$

Case 2: If $k = 2$, then by (2.1), we have $\frac{(c-b)p}{cb} < 1$.

Subcase 1: If $a \leq \frac{b}{k} = \frac{b}{2}$, then $am < \frac{b}{k}\frac{k}{b}p = p$ and we are done.

Subcase 2: If $a > \frac{b}{k} = \frac{b}{2}$ then if $b = 2l$, then $a > l$ and $a - 1 \geq l$. Now $\frac{(k-1)(c-b)p}{bc} = \frac{a-1}{bc}p \geq \frac{lp}{bc} > \frac{2l}{b} = 1$, which is a contradiction to (2.1). If $b = 2l+1$, as before, if $a - 1 \geq \frac{b}{2}$ we find a contradiction to (2.1). So assume that $\frac{b}{2} > a - 1 > \frac{b}{2} - 1$, and thus $a = l + 1$. Now $c = a + b - 1 = 3l + 1$. Thus $c - b = a - 1 = l$ and again we have $\frac{(k-1)(c-b)}{bc}p = \frac{lp}{(2l+1)(c)} \geq \frac{3l}{2l+1} \geq 1$ if $p \geq 3c$, which is a contradiction to (2.1). So assume that $p = 2c + l_0$, for some l_0 odd. If $l_0 = 1$ then $p = 2c + 1 = 2(3l + 1) + 1 = 6l + 3 = 3(2l + 1)$ a contradiction since p is not divisible by 3. If $l_0 = 3$ then $p = 2c + 3$ and thus $\lceil \frac{p}{c} \rceil = \lceil 2 + \frac{3}{c} \rceil = 3 < \frac{p}{b} = \frac{6l+5}{2l+1}$. This implies that $k = 1$ a contradiction. If $l_0 \geq 5$ then $p \geq 2c + 5 = 6l + 7$. Now $\frac{lp}{(2l+1)(3l+1)} \geq \frac{l(6l+7)}{6l^2+5l+1} > 1$ a contradiction to (2.1).

Case 3. If $k \geq 3$. As before, if $a \leq \frac{b}{k}$ then $am < a\frac{kp}{b} \leq p$ and we are done. If $c - b = a - 1 \geq \frac{b}{k}$ then $\frac{(k-1)p(c-b)}{bc} \geq \frac{(k-1)bp}{kbc} > \frac{2(k-1)}{k} > 1$, a contradiction to (2.1). Thus, assume that $\frac{b}{k} + 1 > a > \frac{b}{k}$. Assume also that $b = kl + k_0$ for some $1 \leq k_0 < k$ and $l \geq 1$. Note that if $k_0 = 0$ then $a > \frac{b}{k} = l$. Thus $a - 1 \geq l = \frac{b}{k}$, a contradiction. Then $a = l + 1$ and $c = a + b - 1 = (k + 1)l + k_0$ and also $c - b = a - 1 = l$. Now, (2.1) reduces to the following:

$$\frac{(c-b)(k-1)p}{cb} = \frac{(k-1)lp}{(kl+k_0)c} < 1. \tag{2.2}$$

If $l \geq 2$ then we have $\frac{(c-b)(k-1)p}{cb} \geq \frac{2(k-1)l}{kl+k-1} = \frac{2lk-2l}{k(l+1)-1} = 1 + \frac{k(l-1)-2l+1}{k(l+1)-1} \geq 1$ (the first inequality holds since $\frac{p}{c} > 2$, $k-1 \geq k_0$ and the second holds since $k(l-1)-2l+1 \geq 3(l-1)-2l+1 = l-2 \geq 0$), which is a contradiction to (2.2).

If, on the other hand, $l = 1$, if $\frac{p}{c} > 3$ then $\frac{(k-1)lp}{(kl+k_0)c} > \frac{3(k-1)l}{kl+k-1} = \frac{3k-3}{2k-1} = 1 + \frac{k-2}{2k-1} > 1$, again a contradiction to (2.2).

So $p = 2c+s_0$ for some $c > s_0 \geq 1$, where s_0 is odd. Recall that $b = lk+k_0 = k+k_0$, so $a = 2$ and $c = b+1 = k+k_0+1$.

If $s_0 \geq 3$, then $(k-1)\frac{s_0}{c} - 1 \geq \frac{3k-3}{2k} - 1 = \frac{2k+k-3}{2k} - 1 = k-3 \geq 0$. Note that $\frac{(k-1)(c-b)p}{bc} = \frac{(k-1)(2c+s_0)}{(k+k_0)(k+k_0+1)} \geq \frac{(k-1)(2+\frac{s_0}{c})}{2k-1} = 1 + \frac{(k-1)\frac{s_0}{c}-1}{2k-1} \geq 1$, a contradiction to (2.2). so we must have $s_0 \leq 2$. Since s_0 is odd, we must have $s_0 = 1$.

If $s_0 = 1$, then $\frac{p}{c} = 2 + \frac{1}{k+k_0+1} < \frac{p}{b} = 2 + \frac{3}{k+k_0}$. Now let $X = \lceil \frac{k+k_0}{3} \rceil$. Then $X < \frac{k+k_0}{3} + 1 = \frac{k+k_0+3}{3} < k < b$ ($k \geq 3$). We claim that $X\frac{p}{b}$ is not an integer. Otherwise, (since p is a prime, $b < p$) $\frac{X}{b}$ is an integer, which means $b|X$ but $X < b$, a contradiction.

Next consider $\lceil X\frac{p}{c} \rceil = 2X + \lceil X\frac{1}{k+k_0+1} \rceil = 2X+1$ (since $\frac{X}{k+k_0+1} = \frac{X}{c} < 1$) and $X\frac{p}{b} = 2X + \frac{3X}{k+k_0} \geq 2X + 3\frac{k+k_0}{3}\frac{1}{k+k_0} \geq 2X+1$. Since $X\frac{p}{b}$ is not an integer, we have that $X\frac{p}{b} > 2X+1 = \lceil X\frac{p}{c} \rceil$. By the minimality of k, $X \geq k$ contradictory to $X < k$. In all cases, we showed that $ma < p$ must hold and the proof is complete.

□

We remark that most parts of the above proof works for the general case when $(n, 6) = 1$, and we are not aware of any counterexample to this situation. We make the following conjecture:

Conjecture 2.3.6. $(\mathbb{Z}_n, 4)$ *is a basic pair whenever* $gcd(n, 6) = 1$.

This conjecture turns out to be an important result. The next results will demonstrate how the basic condition of the set of minimal zero sequences of lengths longer than 4 is determined. In fact, a proof of the above conjecture would entirely solve the basic pair problem. We can illustrate with the following example involving sequences of length 5 in $\mathbb{Z}_n$.

Example 2.3.7. $(\mathbb{Z}_n, 5)$ is a non-basic pair for every integer $n > 7$.

Proof. First consider the case where n is even. Consider $X = \{1, 1, \frac{n}{2}, \frac{n}{2} + 1, n - 3\}$. Since $\sigma(X) = 2n$ but no subsequence has sum n, we have $X \in MZS(\mathbb{Z}_n)$. Then $\phi(x) = mx$ for some positive odd m which is less than n. It follows that $\phi(X) = \{m, m, \frac{n}{2}, mod(m\frac{n}{2} + m, n), mod(mn - 3m, n)\}$. (Note that for every m coprime to n, $mod(m\frac{n}{2}, n) = \frac{n}{2}$). Thus if $m \geq \frac{n}{4}$ we have $\sigma(\phi(X)) > m + m + \frac{n}{2} > n$. If $m < \frac{n}{4}$ then $mod(m\frac{n}{2} + m, n) = \frac{n}{2} + m$ and $\sigma(\phi(X)) > \frac{n}{2} + \frac{n}{2} + m > n$, so X is non-basic and so $(\mathbb{Z}_n, 5)$ is a non-basic pair when n is even.

We now claim that the sequence $X = \{1, 1, \frac{n-1}{2}, \frac{n+3}{2}, n - 3\}$ is always non-basic if n is odd. Note that $\sigma(X) = 2n$ but no subsequence sums to n, hence $X \in MZS(\mathbb{Z}_n)$. Then for any $\phi \in Aut(\mathbb{Z}_n)$, we have $\phi(x) = mx$ for some m relatively prime to n, and $\phi(X) = \{m, m, mod(m\frac{n-1}{2}, n), mod(m\frac{n+3}{2}, n), mod(m(n - 3), n)\}$. We now show that $\sigma(\phi(X)) > n$ for any possible m.

Case 1: If $m > \frac{n}{2}$, then since $m + m > n$, $\sigma(\phi(X)) > n$ and therefore X is not basic.

Case 2: If $\frac{n}{3} < m < \frac{n}{2}$. Then note that $m + m > 2\frac{n}{3}$. We also have that $mod(m(n - 3), n) = 2n - 3m$ since $n < 3m < n + \frac{n}{2}$, and therefore $\frac{n}{2} < 2n - 3m < n$.

Therefore $\sigma(\phi(X)) > n$ and X is non-basic.

Case 3: If $m < \frac{n}{6}$, note that $mod(mn - 3m, n) = n - 3m > \frac{n}{2}$ since $3m < \frac{n}{2}$. Thus, if $\phi(X)$ has another element greater than $\frac{n}{2}$, then $\sigma(\phi(X)) > n$, and X must be non-basic.

Subcase 1: If m is even, let $m = 2k$ so that $k < \frac{n}{12}$. Then note that $\phi(\frac{n-1}{2}) = 2k\frac{n-1}{2} = kn - k \equiv n - k(modn)$ and $n - k > \frac{n}{2}$. Therefore, X is non-basic.

Subcase 2: If m is odd, let $m = 2k + 1$. Then $\phi(\frac{n+3}{2}) = (2k + 1)\frac{n+3}{2} = k(n+3) + \frac{n+3}{2} \equiv 3k + \frac{n+3}{2}(modn)$ since $m = 2k+1 < \frac{n}{6}$ implies that $3k < \frac{n-6}{4} < \frac{n-3}{2}$. Hence, $\phi(\frac{n+3}{2}) > \frac{n}{2}$ and we have again that X is non-basic.

Case 4: If $\frac{n}{6} < m < \frac{n}{3}(\star)$ then $\phi(1) + \phi(1) > \frac{n}{3}$. Hence if we can find an element in $\phi(X)$ greater than $\frac{2n}{3}$, $\sigma(X) > n$ and we are done. We consider two cases.

Subcase 1: If m is even, then let $m = 2k$ so that $\phi(\frac{n-1}{2}) = k(n-1) \equiv n-k(modn)$ since from $\star$ we have that $\frac{5n}{6}n - k < \frac{11n}{12}$. Thus $\sigma(X) > n$, and X is non-basic.

Subcase 2: If m is odd, let $m = 2k + 1$. Then $\phi(\frac{n+3}{2}) \equiv 3k + \frac{n+3}{2}(modn)$ since from $\star$ we can assume that $\frac{n-6}{4} < 3k < \frac{n-3}{2}$. Hence $\frac{3n}{4} < 3k + \frac{n+3}{2} < n$. Thus once again $\sigma(X) > n$, so that in all cases we have X is non-basic and therefore $(\mathbb{Z}_n, 5)$ is a non-basic pair. $\square$

So we have shown that $(\mathbb{Z}_n, 3)$ is always a basic pair, $(\mathbb{Z}_n, 4)$ is basic if n is prime and non-basic if divisible by 2 or 3, and $(\mathbb{Z}_n, 5)$ is always non-basic. Thus sequences of length 4 are the threshold on the lower bound of the basic pair condition. We now move on to explore when longer sequences will satisfy this basic pair condition.

2.4 Long Minimal Zero-Sequences in $\mathbb{Z}_n$

We will now comment on the nature of longer minimal zero-sequences in $\mathbb{Z}_n$. Note that if $k > n$ then since $D(\mathbb{Z}_n) = n$ any sequence of length k must have a non-trivial zero-sum subsequence, so any sequence of length greater than n is not minimal. The following recent result is very useful, and characterizes precisely all long sequences in $MZS(\mathbb{Z}_n)$. It was recently proved by Savchev and Chen in [4].

Theorem 2.4.1. *If $\frac{n}{2} + 1 < k \leq n$ then $(\mathbb{Z}_n, k)$ is a basic pair.*

This next result will finish the classification of basic pairs, up to Conjecture 2.3.6. Recall that we have shown that all sequences of length less than 3 or greater than $\frac{n}{2} + 1$ are basic, and conjectured on the nature of sequences of length 4. We now handle the remaining cases with the following theorem, which is the final result we have regarding the basic pair problem.

Theorem 2.4.2. *For any $n \geq 8$, if $5 \leq k \leq \frac{n}{2} + 1$ then $(\mathbb{Z}_n, k)$ is a non-basic pair.*

Proof. If n is even consider the sequence $X = \{1^{k-3}, \frac{n}{2}, \frac{n}{2} + 1, n - (k-2)\}$. Then note that $X \in MZS(\mathbb{Z}_n)$, and we will show that X is non-basic. Let $\phi(x) = mx$ for $(m, n) = 1$. Then if $m \geq \frac{n}{2}$ we have $\phi(1) = m$ and since $v(1) \geq 2$ it follows that $\sigma(\phi(X)) > n$. If $m < \frac{n}{2}$ then note that $mod(m\frac{n}{2}, n) = \frac{n}{2}$ and that $mod(m(\frac{n}{2} + 1), n) = \frac{n}{2} + m$ and so $\sigma(mX) > \frac{n}{2} + \frac{n}{2} + m > n$.

If n is odd consider the sequence $X = \{1^{k-3}, \frac{n-1}{2}, \frac{n+3}{2}, n - (k-2)\}$.

Case 1: If $m < \frac{n}{4}$ we will show that $mod(m(\frac{n-1}{2}), n) + mod(m(\frac{n+3}{2}), n) > n$. To see this note if m is even let $m = 2l$, then $l < \frac{n}{8}$. Thus $m\frac{n-1}{2} = ln - l$ and as such $mod(m(\frac{n-1}{2}), n) = n - l$. Similarly $m\frac{n+3}{2} = ln + 3l$ and so $mod(m(\frac{n+3}{2}), n) = 3l$. Therefore $mod(m(\frac{n-1}{2}), n) + mod(m(\frac{n+3}{2}), n) = n + 2l > n$.

If m is odd let $m = 2l+1$ and we still have $l < \frac{n}{8}$. Then $m\frac{n-1}{2} = ln + (\frac{n-1}{2} - l)$ and so $mod(m(\frac{n-1}{2}), n) = \frac{n-1}{2} - l$. Also $m\frac{n+3}{2} = ln + (\frac{n+3}{2} + 3l)$. Then since $(\frac{n+3}{2} + 3l) < n$ whenever $n > 12$ we have that $mod(m(\frac{n+3}{2}), n) = \frac{n+3}{2} + 3l$, and the result follows. (The condition $n > 12$ is satisfactory since the conditions that m is a positive odd integer greater than 1 and less than $\frac{n}{4}$ require $n > 12$).

Case 2: If $m > \frac{n}{4}$ then our sequence must have $v(1) = 2$ or 3 otherwise $\sigma(mX) > n$, and this implies $k = 5$ or 6 respectively. The case $k = 5$ was handled in the Example 2.3.7, so we will show the case $k = 6$. Note since $m < \frac{n}{2}$ that $m(n-(k-2)) = mn - 4m$ and $n < 4m < 2n$ so $mod(mn-4m, n) = 2n - 4m$. But then $2n - 4m + 3m = 2n - m > n$ and so X is non-basic. $\square$

Chapter 3

Sequences of Length p in $\mathbb{Z}_p$.

3.1 Introduction

In this chapter we more closely examine sequences of length p in $\mathbb{Z}_p$. Our ultimate goal is to verify Conjecture 1.3.2 by hand for relatively small values of p. We do so by first proving some general results regarding sequences of length p in $\mathbb{Z}_p$. We restate Conjecture 1.3.2 using some of the terminology from Chapter 2.

Conjecture 3.1.1. *Every sequence in $\mathbb{Z}_p$ of length p has a basic subsequence.*

Note that the condition of having a basic subsequence is equivalent to the one outlined in the KL conjecture when $d = n$ (in this case $n = p$), thus this statement is equivalent to Conjecture 1.3.2. The purpose of verifying the conjecture by hand rather attempting to solve it by other means (i.e. a computer program) is to attempt to gain an understanding of methods that may be used to ultimately find a generalized method for proving the conjecture true for any value of p. In order to demonstrate that the conjecture holds for a given value of p, we first prove some preliminary results which simplify this process. Note that in theory, we could be faced with the arduous task of checking whether each of the $(p)^p$ sequences of length p have some subsequence with sum p under each of the $p-1$ automorphisms in $\mathrm{Aut}(\mathbb{Z}_p)$.

Such a task quickly becomes unreasonable as p grows. In this chapter we demonstrate several classes of sequences which can be shown to necessarily contain basic subsequences. We show how sequences with certain repetition values will always contain basic zero-subsequences, and attempt to generalize this concept. We conclude by proving that for each p less than 40, that all sequences not handled by previous results must contain such a subsequence, and therefore the conjecture holds for small values of p.

We note that part of the benefit of dealing with sequences in $\mathbb{Z}_p$ is that every non-identity element in $\mathbb{Z}_p$ has order p. This guarantees the existence of an automorphism which can essentially turn any element of a sequence into a 1 (or any other non-zero integer). Since our ultimate goal is to find a zero-subsequence and an automorphism which sends it to a sequence with sum p, being able to turn specific elements as small as possible is very helpful. Note we do not have this luxury when we deal with composite integers in Chapter 4. In most occurrences however, the more times any given element occurs in a sequence, the easier it becomes to find a basic subsequence. We begin by revisiting an important concept in dealing with zero-sequences, the height of a sequence.

3.2 The Height of the Sequence

Let $X = \{x_1, ..., x_p\}$ be a sequence in $\mathbb{Z}_p$. Define the height of the sequence X in $\mathbb{Z}_p$ to be $H(X) = max\{v(x_i)\}$ for $1 \leq i \leq p$. For instance if $X = \{1, 2, 2, 2, 4, 7\}$ in $\mathbb{Z}_p$ then $H(X) = 3$. If the choice of X is clear we write $H(X) = H$. We wish to analyze if a sequence with a given height will necessarily contain a basic subsequence. We illustrate with an easy example.

Example 3.2.1. *Suppose $H(X) = p$ for some sequence X in $\mathbb{Z}_p$. Then we have*

$X = \{x^p\}$ for some $x \in \mathbb{Z}_p$. Then we can find an automorphism ϕ such that $\phi(x) = 1$. Then $\phi(X) = \{1^p\}$ and $\sigma(\phi(X)) = p$. Hence X is a basic sequence.

Knowing the height of a sequence can also provide us information regarding the structure of some subsequences. The following result is due to Gao and is applicable for any finite abelian group G. It is a useful result and is used several times in this thesis and also in the solutions of other zero-sum problems. It states that any sequence X of sufficient length necessarily has a zero-subsequence of length at most the height or the maximal order any element in X.

Theorem 3.2.2. *(Gao) Let X be a sequence of length $|X| \geq |\mathbb{G}|$. Then X has a non-empty zero-sum subsequence Y, such that $|Y| \leq min\{H(X), max\{ord(x)|x \in X\}\}$.*

Remark 3.2.3. *We will prove this theorem for the case where where $G = \mathbb{Z}_p$. Then, given a sequence X with $|X| = p$, we know that $ord(x) = p$ for every $x \neq p$ when $x \in X$. Since we can always assume that $v(p) = 0$ (otherwise $\{p\}$ is a basic subsequence) we know $min\{H(X), max\{ord(x)|x \in X\}\} = H(X)$.*

Proof. We first note that if A, B are two subsets of an abelian group, and if $A \cap (-B) = \{0\}$ then $|A+B| \geq |A| + |B| - 1$. We now use this fact to prove the above theorem.

Let $h = H(X)$. We can divide X into h disjoint subsets $A_1, ..., A_h$. Let $B_i = \{0\} \cup A_i$. For any two subsets Y and Z of G, define $Y \oplus Z = Y \cup Z \cup (Y + Z)$. Assume to the contrary that, $0 \notin \sum_{\leq h}(X)$. Then, $0 \notin A_1 \oplus A_2 \oplus \cdots \oplus A_i$ for every $i \in \{1, \cdots, h\}$. Now we have $B_1 \cap (-B_2) = \{0\}$ and by the above observation we have, $|B_1 + B_2| \geq |B_1| + |B_2| - 1 = |A_1| + |A_2| + 1$. Therefore, $|A_1 \oplus A_2| = |B_1 + B_2| - 1 \geq |A_1| + |A_2|$. Continue the same progress above we finally get, $|A_1 \oplus \cdots \oplus A_h| \geq |A_1| + \cdots + |A_h| = |X| = |G|$, a contradiction. $\square$

Therefore given a sequence of length p in $\mathbb{Z}_p$ we can always find a zero-subsequence of length at most H. We note of course that the existence of such a zero-subsequence does not necessarily imply that it is basic. This difference is a basis for many of the following results. We will now look more closely at when a sequence has very small repetition value.

3.2.1 Small Heights

As an application of Gao's Theorem, we note the following corollary which tells us that for sequences with very small heights, we can assume that Conjecture 1.3.2 holds.

Corollary 3.2.4. *If X is a sequence in $(\mathbb{Z}_p)$ with $|X| = p$ and $H(X) \leq 4$, then X has a basic subsequence.*

Proof. If $H(X) = 1, 2, 3$ or 4, by Gao's Theorem we can assume that X has a zero-subsequence Y of length $1, 2, 3$ or 4. We know from Theorem 2.2.1 that $(\mathbb{Z}_n, k)$ is a basic pair for all n if $k = 1, 2, 3$. Thus if $|Y| \leq 3$ then Y is a basic sequence. If $|Y| = 4$, then by the Four Sum Theorem Y is basic. Hence we have proved the corollary. □

This corollary demonstrates that if we are looking for a counterexample to Conjecture 1.3.2, we can always assume that it has height at least 5. We saw in Chapter 2 that $(\mathbb{Z}_p, 5)$ is a non-basic pair for any prime $p > 7$. Therefore, even if we know that a sequence of length p has a zero-subsequence of length at most five, we can not conclude that it is basic. There are certain conditions which allow us to make this conclusion, we mention one situation which occurs occasionally.

Theorem 3.2.5. *Let $X = \{x_1, ..., x_5\}$ be a zero-sequence in $\mathbb{Z}_p$ such that $1 \leq x_1 \leq ... \leq x_5 \leq \frac{p}{2}$. Then X is a basic zero-sequence.*

Proof. Assume that $x_i = \frac{p-y_i}{2}$ for some $1 \leq y_i \leq p-2$ where y_i is an odd integer. Then it is easy to see that $y_5 \leq ... \leq y_1$. Note since $x_i \leq \frac{p}{2}$ that $\sigma(X) < 3p$. If $\sigma(X) = 1p$ the sequence is basic and we are done, hence we can assume that $\sigma(X) = 2p$. Then we have

$$\frac{p-y_1}{2} + \dots + \frac{p-y_5}{2} = 2p$$

which implies that

$$y_1 + \dots + y_5 = p$$

Now notice that $(p-2)x_i = (p-2)\frac{p-y_i}{2} = \frac{p^2-p(2+y_i)+2y_i}{2} = p\frac{p-(2+y_i)}{2} + y_i$. Then since $y_i \leq p-2$ and is an odd integer, $2+y_i \leq p$ and odd. Hence $\frac{p-(2+y_i)}{2}$ is an integer and $mod((p-2)x_i, p) = y_i$. Thus if we take $m = p-2$ and let $\phi(x_i) = (mx_i)$, we get that $\sigma(\phi(X)) = mod(mx_1, p) + \dots + mod(mx_5, p) = y_1 + \dots + y_5 = p$. Hence X is a basic zero-sequence, and the theorem is proved. □

Therefore if a sequence of length p has height 5 and all terms are less than $\frac{p}{2}$ we can assume it has a basic subsequence. We remark that there exist minimal zero-sequences of length 5 in $\mathbb{Z}_p$ with only one term greater than $\frac{p}{2}$ that are non-basic ($\{1,3,7,8,15\}$ in $\mathbb{Z}_{17}$ for example). We conclude this section with an example where we are not concerned with the actual height of the sequence, rather that all terms are sufficiently small.

Example 3.2.6. *Suppose that $v(1) = H(X)$. Let $X = \{x_1, .., x_p\}$ be a sequence in $\mathbb{Z}_p$ such that $1 \leq x \leq H(X)+1$ for every $x \in X$. We show X has a basic subsequence. If we can find a subsequence $1 \notin Y$ of X such that $p - H(X) \leq \sigma(Y) \leq p$ note that the sequence $\{Y, 1^{p-\sigma(Y)}\}$ has sum p and is therefore basic. Then we note that since $\sigma(X) \geq p$ there exists a j such that*

$$\sum_{i=1}^{j} x_i < p - H(X)$$

but

$$p - H(X) \leq \sum_{i=1}^{j+1} x_i \leq p$$

and we are done.

3.2.2 Large Heights

In this section we concern ourselves with sequences with properties similar to that of having a large repetition value. Note that the general benefit of having a large height is that we may assume that $v(1)$ is large, and hence any sequence with no basic subsequence could only contain a relatively small amount of possible elements. In this section we generalize this concept to consider not only the height of the sequence, but also the largest continuous interval of integer sums in the sumset $\Sigma(X)$ of the form $[1, M]$. Before we do this, we present one last example of a result involving the height of a sequence.

Lemma 3.2.7. *If X is a sequence in $\mathbb{Z}_p$ with $|X| = p$ and $H(X) > \frac{p}{2}$, then X has a basic subsequence.*

Proof. Let $H(X) = \frac{p+k}{2} > \frac{p}{2}$ for some $1 \leq k \leq p$. After transformation by a suitable group automorphism, we assume that $v(1) = \frac{p+k}{2}$. Let $\bar{X} = \{x_1, ..., x_{\frac{p-k}{2}}\}$ be the subsequence of X obtained by removing the 1's. Note that if $\bar{X}$ has a subsequence Y such that $\frac{p-k}{2} \leq \sigma(Y) < p$, then this subsequence with an appropriate number of 1's ($p - \sigma(Y)$ to be exact) forms a basic subsequence of X. We show that this subsequence always exists. For any $x_i \in \bar{X}$, if $x_i \geq \frac{p-k}{2}$ we are done, so assume that $x_i \leq \frac{p-k-2}{2}$ for all i. Then note that since $\sigma(\bar{X}) \geq 2\frac{p-k}{2}$, there exists an m such that $\sigma(x_1, ..., x_m) < \frac{p-k}{2}$ but $\frac{p-k}{2} \leq \sigma(x_1, ..., x_{m+1}) < \frac{p-k}{2} + \frac{p-k-2}{2} = p - k - 1 < p$, and we are done. $\square$

As an immediate consequence of Lemma 3.2.7 and Corollary 3.2.4 we can conclude that for any $p < 11$ that Conjecture 1.3.2 holds. For larger primes however, we can significantly improve on the bound offered by this Lemma. We do so by considering the largest continuous interval of small sums in a sequence X, which is a new concept to consider. To clarify this concept, note any sequence X that does not have a basic subsequence must satisfy the property that $v(x) = 0$ for all $p - H \leq x \leq p$, otherwise we could easily make a sequence with sum p. For example, if we have a sequence X in $\mathbb{Z}_{17}$ with $H = 5$, we can assume $v(x) = 0$ for each $x \geq 12$. Another way to think of this is to note that $y \in \Sigma(X)$ for each integer $y \leq H$. So we know that $[1, H]$ is always a continuous interval in $\Sigma(X)$. Note if $[1, M]$ is a continuous interval in $\Sigma(X)$ for any integer M with $M \geq H$ then we can assume that $v(x) = 0$ for each $x \geq p - M$. For example, if X is a sequence with height H and we assume $v(1) = H$, then if $2 \in X$ then $[1, H+2]$ is a continuous interval in $\Sigma(X)$, so $M \geq H+2$. In general, we denote the size of this interval in the sumset by M. We note the important observations that $M \neq H + 1$ and if $M > 0$ then $1 \in X$. We will now establish a bound on $M(X)$ for some X contradicting Conjecture 1.3.2. We set up the theorem as follows.

Let $X = \{1, ..., 1, a_1, ..., a_r, b_1, ..., b_s, c_1, ..., c_t\}$ be a sequence of length p ordered by size with $v(1) = H$ and $r + s + t = p - H$. Set a_i such that the sumset $\Sigma(1, ..., 1, a_1, ..., a_r) = [1, H + \sum_{i=1}^{r} a_i]$. We assume that $b_1 \geq H + \sum_{i=1}^{r} a_i + 2$ else we can form a larger continuous interval of sums. Also assume that $b_s \leq \frac{p-1}{2}$ and $c_1 \geq \frac{p+1}{2}$. Set $M = H + \sum_{i=1}^{r} a_i$. Clearly then if $M \geq p$ then X must have a basic zero-sequence. Also if $c_t \geq p - M$ then $\{1, .., 1, c_t\}$ is basic with $v(1) = p - c_t$. Thus assume that $c_t \leq p - M - 1$.

Theorem 3.2.8. *Let $p \geq 11$. If X is a sequence in $\mathbb{Z}_p$ with $|X| = p$ and $M \geq \frac{p-2}{3}$, then X has a basic zero subsequence.*

Proof. Note if $M \geq \frac{p-1}{2}$ and if $s + t > 0$ then let x be any term in $\{b_1, .., b_s, c_1, ..., c_t\}$. Then $x \geq M + 2 \leq p - M$. Hence $p - x \in [1, M]$. Then since there exists a subsequence Y of $\{1, .., 1, a_1, .., a_r\}$ with sum $p - x$, and so $\{Y, x\}$ has sum p.

Now assume $\frac{p-2}{3} \leq M \leq \frac{p-3}{2}$ Note if $s \geq 2$ then $p - M \leq 2(M + 2) \leq b_1 + b_2 \leq 2\frac{p-1}{2} < p$. Similarly to above we can find a subsequence Y of $\{1, .., 1, a_1, .., a_r\}$ so $\{Y, b_1, b_2\}$ has sum p. So assume that $s \leq 1$. Then $t \geq p - \frac{p-3}{2} - 1 > 1$. Similarly to above assume that

$$\frac{p+1}{2} \leq c_1 \leq \ldots \leq c_t \leq p - M - 1$$

Now let $c_i = \frac{p+x_i}{2}$ for each $i \in [1, t]$. Then $x_i = 2c_i - p$ and so $1 \leq p - 2M - 2$ and $2c_i = x_i$ is odd mod p. Let $t_0 = 2\lfloor\frac{t-1}{2}\rfloor + 1$. Then $0 \leq t - t_0 \leq 1$. Set

$$R = \sum_{i=1}^{t_0} x_i = \sum_{i=1}^{t_0} 2c_i$$

If $R \leq p - 2M - 1$ then $t_0 \leq R \leq p - 2M - 1$. Hence $p = h + r + s + t \leq M + s + t \leq M + s + t \leq M + s + t_0 + 1 \leq M + 1 + p - 2M < p$, a contradiction. Therefore $R \geq p - 2M$. If $R < p$ then one can find a subsequence Y of $\{1, ., 1, a_1, ..., a_r\}$ such that $\{Y, c_1, .., c_{t_0}\}$ has sum p. So assume that $R \geq p$, in fact we can assume that $R \geq p + 1$. Since R is odd we can actually assume that $R \geq p + 2$. We can drop the last two terms of R and put

$$R_1 = \sum_{i=1}^{t_0-2} x_1 = \sum_{i=1}^{t_0-2} 2c_i$$

Then R_1 is odd and $R_1 \geq R - 2(p - 2M - 2) \geq p + 2 - 2(p - 2M - 2) \geq p - 2M$. If $R_1 \leq p + 1$ then similarly to above we can prove the lemma. Otherwise $R_1 \geq p + 2$. We can drop the last two terms of R_1 and put $R_2 = \sum_{i=1}^{t_0-4} x_1 = \sum_{i=1}^{t_0-4} 2c_i$. Continue the same progress as above and we can prove the Lemma. □

Note that since $H \leq M$ for any sequence X, we assume that any sequence with $H \geq \frac{p-2}{3}$ also has a basic subsequence. Thus we have proven that any sequence of length p acting as a counterexample to Conjecture 1.3.2 must have $5 \leq H \leq M \leq \frac{p-2}{3}$. When we demonstrate that the conjecture holds for a given p, we typically do so but verifying each of these possible values of M. We can then assume X has certain properties based on this assumption, if necessary. For example, we can collect all the elements $(1, .., M + 1)$ together and assume that we have at most M of these elements. Note we can always assume that $H \geq 5$ for prime integers, so if $M = 5$ or $M = 6$ we can assume $M = H$. Our goal now is to examine the nature of the zero-subsequences that we know must exist in $X = \{x_1, ..., x_p\}$. Since our ultimate objective is to prove Conjecture 1.3.2 holds for these p, we want to show that at least one of these zero-subsequences must be basic.

3.3 Verification of the KL Conjecture

The basic strategy of verifying Conjecture 1.3.2 is to first assume that X is a sequence in $\mathbb{Z}_p$ which can act as a counterexample. That is, we assume that X is a sequence of length p with no basic subsequence. Based on previous results and other easy observations, we can then assume that X satisfies certain properties. We mentioned before that we assume that $5 \leq H \leq M \leq \frac{p-2}{3}$ and that $v(x) = 0$ for each $x \in X$ satisfying $p - M \leq x \leq p$. In general, we assume X has a given value of M then proceed to eliminate the possibility of elements appearing in the sequence based on this assumption. By eliminating enough elements, we can derive a contradiction to the fact that X must have p elements. After eliminating the obvious elements, we typically separate the remaining ones into "boxes".

Definition 2. *Define a box $(x_1, .., x_l)$ in a sequence X to be a collection of elements from X. A set of boxes which contains every possible element of a sequence X is a partition of X. The cardinality of a box (on X) denoted $v(x_1, .., x_l)$ is the number of elements from the box appearing in the sequence X.*

Note that $v(x_1, .., x_l) \neq l$ necessarily. We often partition a sequence into boxes of the form $(\frac{p-k}{2}, \frac{p+k}{2})$. For example, if X is a sequence of length 23 in $\mathbb{Z}_{23}$, the box $(10, 13)$ can not have both elements 10 and 13. If we assume $H(X) = 5$ then the box $(10, 13)$ has cardinality $v(10, 13) \leq 5$, and if we assume $v(13) = 0$ then $v(10, 13) \leq 1$ since if $v(10) \geq 2$ we could form the basic sequence $\{10, 10, 1, 1, 1\}$.

We often look to specific basic sequences to justify why we can assume a counterexample to Conjecture 1.3.2 cannot contain certain elements. It is not always immediately obvious that these minimal zero-sequences are basic. Although we can always check a sequence for the basic property by multiplying by relatively prime integers modulo n until we find a suitable one, this is not always necessary. The following example is a consequence of the Four Sum Theorem proved earlier. It is a very useful trick for limiting how many times an element can appear.

Example 3.3.1. *Let $Y = \{a_1, a_2, a_3, 1, ..., 1\}$ be a minimal zero-sequence in $\mathbb{Z}_p$ for some integers $a_1, a_2, a_3 > \frac{p}{2}$, and let $v_Y(1) = d$ equal the number of 1's in Y, with $d \le \frac{p-2}{3}$. Then Y is a basic sequence.*

Proof. We can assume that each $a_i < p - d$ otherwise Y is not minimal. So we have $a_1 + a_2 + a_3 \le 3(p - d)$ so $\sigma(Y) = a_1 + a_2 + a_3 + d \le 3p - 2d$, so since Y is a zero-sequence we can assume $a_1 + a_2 + a_3 + d = 2p$. Then since $p < 2a_i < 2p$ we know $mod(2a_i, p) = 2a_i - p$. Therefore $\sigma(2Y) = 2a_1 - p + 2a_2 - p + 2a_3 - p + 2d = 4p - 3p = p$ and so Y is basic. $\square$

For any integer n, one can always easily check if any given sequence X in $\mathbb{Z}_n$ is basic by simply finding an appropriate m such that mX has sum n. However, when proving no counterexample to the conjecture exists for a given a value of M, knowing this value of M does not necessarily tell us the make-up of the elements which create these sums. For example, consider $\mathbb{Z}_{19}$. If we assume $M = 7$ then the sequence $\{1^2, 4, 8^4\}$ is basic after multiplication by 5. However, the sequence $\{1^3, 3, 8^4\}$ is actually non-basic. Hence if we want to verify a sequence for a value of M rather than H we need to take caution. Luckily, by using the following Lemma which is easily seen to be true, we can demonstrate that any sequence that is basic under the assumption $H = s$ is also basic under the assumption $M = s$. Thus if X

is a sequence which contains no basic subsequence, and some sequence of the form $\{1^s, Y\}$ is basic, we can assume that Y is not a subsequence of X whenever $M \geq s$.

Lemma 3.3.2. *If the sequence $X = \{1^s, Y\}$ in $\mathbb{Z}_n$ is a basic sequence for some subsequence Y with $s < n$, then $\bar{X} = \{Z, Y\}$ is basic for any Z with $\sigma(Z) = s$.*

Proof. Note that if X is basic then there exists an m such that $\sigma(mX) = n$. Since $mX = \{m^s, mY\}$ for some sequence mY we must have that $ms \leq n$. Then if Z is any sequence with $\sigma(Z) = s$ clearly $\sigma(mZ) = ms$ so $\{mZ, mY\}$ has sum n. $\square$

Remark 3.3.3. *Note that if there is a basic subsequence $\{1^s, Y\}$ of X, where $Y \subseteq X - (1, 2, .., M)$ and $M = M(X) \geq s$, then X has a basic subsequence (Z, Y) where $Z \subseteq X \Cap (1, 2, .., M)$ and $\sigma(Z) = s$. So for example, suppose that X is a sequence in $\mathbb{Z}_{29}$ which has no basic subsequence. Then since $\{20^4, 1^7\}$ is basic, by Lemma 3.3.2 we can assume $\{20^4\}$ is not a subsequence of X whenever $M \geq 7$. Note that this implies that $v_X(20) \leq 3$. This remark is used repeatedly in the remainder of this chapter and in section 4.6.*

We now set to verify Conjecture 1.3.2 for all prime integers less than 40. Note that if $\frac{p-2}{3} \leq 5$ then Corollary 3.2.4 and Theorem 3.2.8 imply that Conjecture 1.3.2 is always satisfied. Thus the conjecture holds for all $p \leq 17$. The first case we need to handle is thus $p = 19$.

3.3.1 Case p=19

We assume that X is a sequence of length 19 in $\mathbb{Z}_{19}$ with no basic subsequence, contradicting Conjecture 1.3.2. By Theorem 3.2.8 and Corollary 3.2.4 we can assume that $5 \leq H \leq M < \frac{19-2}{3}$, which means we can assume that $H = M = 5$. Therefore we can assume that $v(1) = 5$ and our sequence is of the form $X = \{1^5, x_6, ..., x_{19}\}$. We can also assume that $v(x) = 0$ for $2 \leq x \leq 6$, otherwise we would have $6 \in \Sigma(X)$, contradicting that $M = 5$.

We can assume there is no subsequence Y of X with $1 \notin Y$ such that $14 \leq \sigma(Y) \leq 19$, otherwise $\{Y, 1^{19-\sigma(Y)}\}$ is a basic subsequence. Therefore we can assume that $v(x) = 0$ for every $x \geq 14$. We partition the remaining elements of X into boxes (1) (13) (7,12) (8,11) (9,10) and note the sum of the cardinalities of these boxes must be at least 19. The cardinality of each box of the form $(\frac{p-k}{2}, \frac{p+k}{2})$ is at most 5 (since $H = 5$). We also note that $v(x) \leq 1$ for any $7 \leq x \leq 9$ else we can take $Y = \{x, x\}$.

Since $\{13^4, 1^5\}$ is equivalent to $\{1^4, 3^5\}$ after applying the automorphism $\phi(x) = 3x$, it is a basic sequence and hence can not appear in X. Since we assumed that $v(1) = 5$, we can assume that $v(13) \leq 3$. Since $\{12^3, 1^2\}$ is a basic sequence (with $\phi(x) = 2x$) we can assume that $v(12) \leq 2$. Then since $v(7) \leq 1$, we assume that $v(7, 12) \leq 2$. Finally, since $\{11^3, 1^5\}$ is a basic sequence, and $v(8) \leq 1$, we can assume that $v(8, 11) \leq 2$.

Therefore the sum of the cardinalities of the boxes (1) (13) (7,12) (8,11) (9,10) can be at most $5 + 3 + 2 + 2 + 5 < 19$, a contradiction. Thus any sequence of length 19 in $\mathbb{Z}_{19}$ must have a basic subsequence, and so Conjecture 1.3.2 holds when $p = 19$.

3.3.2 Case p=23

We assume that X is a sequence of length 23 in $\mathbb{Z}_{23}$ with no basic subsequence. By Theorem 3.2.8 and Corollary 3.2.4 we can assume that $5 \leq H \leq M \leq 6$. If $H = 5$ then note that we cannot have $M = 6$ since $M > H$ implies $M \geq H + 2$. So we can assume that either $H = M = 5$ or $H = M = 6$. Note the following sequences are basic and cannot appear in X. We also note the implications of these sequences:

$\{17^4, 1\}$ implies $v(17) \leq 3$

$\{16^4, 1^5\}$ implies $v(16) \leq 3$

$\{15^3, 1\}$ implies $v(15) \leq 2$

$\{8^5, 1^6\}$ implies $v(8, 15) \leq 4$ if $M \geq 6$

$\{14^3, 1^4\}$ implies $v(9, 14) \leq 2$

$\{13^5, 1^4\}$ implies $v(10, 13) \leq 4$

If H=M=6 we can assume that $v(1) = 6$. We can also assume that $v(x) = 0$ for $2 \leq x \leq 7$, otherwise $7 \in \Sigma(X)$ which contradicts $M = 6$. We assume there is no subsequence Y of X with $1 \notin Y$ such that $17 \leq \sigma(Y) \leq 23$, so we can assume that $v(x) = 0$ for every $x \geq 17$. We partition the remaining elements into (1) (16) (8,15) (9,14) (10,13) (11,12) where the sum of the cardinalities of these boxes must be at least 23. We also have immediately that $v(x) \leq 1$ if $9 \leq x \leq 11$.

Now assume that $v(16) > 0$. Note the sequence $\{16, 13^2, 1^4\}$ is basic so in this case we can assume $v(10, 13) \leq 1$. So we partition to boxes (1) (16) (8,15) (9,14) (10,13) (11,12) with sum of the cardinalities at most $6+3+4+2+1+6 < 23$, a contradiction. Therefore we can assume $v(16) = 0$. Now we can partition the remaining elements into (1) (8,15) (9,14) (10,13) (11,12) and the sum of the cardinalities of these boxes can be at most $6 + 4 + 2 + 4 + 6 < 23$, a contradiction. Therefore if $M = 6$ we get a contradiction, so we can assume that $H = M = 5$.

If H=M=5 we assume that $v(1) = 5$. We can also assume that $v(x) = 0$ for $2 \leq x \leq 6$. We assume there is no subsequence Y of X with $1 \notin Y$ such that $18 \leq \sigma(Y) \leq 23$, so assume $v(x) = 0$ for every $x \geq 18$. We partition the remaining elements into boxes (1) (17) (7,16) (8,15) (9,14) (10,13) (11,12). We have immediately that $v(x) \leq 1$ if $9 \leq x \leq 11$ and $v(7) \leq 2$.

Assume $v(17) > 0$. Then since $\{17, 8^3, 1^5\}$ and $\{17, 12^2, 1^5\}$ are basic sequences, we must have that $v(8,15) \leq 2$ and $v(11,12) \leq 1$. Then the cardinalities of the partition (1)(17)(7,16)(8,15)(9,14)(10,13)(11,12) are at most $5+3+3+2+2+4+1 < 23$, a contradiction, so we can assume that $v(17) = 0$.

Now assume that $v(16) > 0$. Then since $\{16, 12^4, 1^5\}$ is basic we assume that $v(11,12) \leq 3$. Then the cardinalities of the boxes (1)(16)(8,15)(9,14)(10,13)(11,12) is at most $5+3+5+2+4+3 < 23$, and so we can assume $v(16) = 0$.

Now we partition the remaining elements into boxes (1)(7)(8,15)(9,14)(10,13)(11,12) with cardinalities at most $5+2+5+2+4+5 = 23$. Then if $v(7) < 2$, the sum of the cardinalities of these boxes is less than 23, so we can assume $v(7) = 2$. Then $\{11, 7, 1^5\}$ and $\{12, 7, 1^4\}$ are basic so we can assume $v(11,12) = 0$, a contradiction. So we can assume $M \neq 5$, and thus any sequence with 23 elements must have at least one basic subsequence, and so Conjecture 1.3.2 is true for $p = 23$.

3.3.3 Case p=29

We assume that X is a sequence in $\mathbb{Z}_{29}$ of length 29 with no basic subsequence. We can assume that $5 \leq H \leq M \leq 8$. Note the following basic sequences and their implications:

$\{22^5, 1^6\}$ implies $v(22) \leq 4$ when $M \geq 6$

$\{21^4, 1^3\}$ implies $v(21) \leq 3$

$\{20^4, 1^7\}$ implies $v(20) \leq 3$ when $M \geq 7$

$\{19^3, 1\}$ implies $v(19) \leq 2$

$\{18^3, 1^4\}$ implies $v(18) \leq 2$

$\{17^3, 1^7\}$ implies $v(17) \leq 2$ when $M \geq 7$

$\{17^5, 1^2\}$ implies $v(17) \leq 4$

$\{16^5, 1^7\}$ implies $v(16) \leq 4$ when $M \geq 7$

$\{10^5, 1^8\}$ implies $v(10) \leq 4$ when $M \geq 8$

$\{11^5, 1^3\}$ implies $v(11, 18) \leq 4$

We note that we can make the above assumptions when M is large enough by Lemma 3.3.2. This trick is used freely throughout the remainder of this chapter and in the end of Chapter 4.

If M=8 let $\alpha = (1, .., 9)$ be a box in X. Then $\sigma(\alpha) = M = 8$. We can assume there is no subsequence Y of $X - \alpha$ such that $21 \leq \sigma(Y) \leq 29$. We partition the remaining elements into boxes (α) (20) (10,19) (11,18) (12,17) (13,16) (14,15) and note the cardinality of each box is at most 8. Also note that $v(x) \leq 1$ for any $11 \leq x \leq 14$.

Assume $v(20) > 0$. We note that since $\{20, 15^2, 1^8\}$ is basic we can assume $v(14, 15) \leq 1$. The cardinalities of the boxes (α) (20) (10,19) (11,18) (12,17) (13,16) (14,15) is then at most $8 + 3 + 4 + 2 + 2 + 4 + 1 < 29$, so assume $v(20) = 0$. We can now partition the remaining boxes into (α) (10,19) (11,18) (12,17) (13,16) (14,15) with cardinalities at most $8 + 4 + 2 + 2 + 4 + 8 < 29$, a contradiction. Therefore we can assume that $5 \leq H \leq M \leq 7$.

If M=7 then let $\alpha = (1, .., 8)$ be a box in X. We assume that there is no subsequence Y of $X - \alpha$ such that $22 \leq \sigma(Y) \leq 29$. We partition the remaining elements into boxes (α) (21) (9,20) (10,19) (11,18) (12,17) (13,16) (14,15) where the cardinality of each box is at most 7. We also note that $v(x) \leq 1$ for $11 \leq x \leq 14$ and $v(9) \leq 2$.

Assume $v(21) > 0$. We note that $\{21, 10^3, 1^7\}$ is basic hence $v(10, 19) \leq 3$. Also $\{21, 15^2, 1^7\}$ is basic so assume that $v(14, 15) \leq 1$. Therefore the sum of the cardinalities of our boxes is at most $7+3+3+2+2+2+4+1 < 29$, and so we can assume that $v(21) = 0$.

Now assume that $v(20) > 0$. Then since $\{20, 15^4, 1^7\}$ is basic we can assume that $v(14, 15) \leq 3$. The sum of the cardinalities of the boxes (α) (20) (10,19) (11,18) (12,17) (13,16) (14,15) is then at most $7+3+7+2+2+4+3 < 29$, and so we must have $v(20) = 0$.

Now assume $v(19) > 0$. Then we can assume $v(x) = 0$ for $9 \leq x \leq 10$ since otherwise we can form a sequence with sum 29. We partition into boxes (α) (19) (11,18) (12,17) (13,16) (14,15). The sum of the cardinalities of these boxes is at most $7+2+2+2+4+7 < 29$, a contradiction, so we can assume $v(19) = 0$. Now assume that $v(18) > 0$. Then we can assume $v(x) = 0$ for $9 \leq x \leq 11$, and we partition into boxes (α) (18) (12,17) (13,16) (14,15) with sum of cardinalities at most $7+2+2+4+7 < 29$, a contradiction. Hence $v(18) = 0$. Now assume $v(17) > 0$. Then we can assume $v(x) = 0$ for any $9 \leq x \leq 12$, and we can partition into boxes (α) (17) (13,16) (14,15), each with cardinality at most 7, a contradiction, so $v(17) = 0$. Now we partition the remaining elements into boxes (α) (9) (10) (11) (12) (13,16) (14,15) with cardinalities at most $7+2+4+2+1+4+7 < 29$, a contradiction. Thus if $M(X) = 7$ we can find a contradiction, so assume $5 \leq H \leq M \leq 6$.

If M=6, we can assume $H = 6$ so assume $v(1) = 6$ and $v(x) = 0$ for $2 \le x \le 7$. We assume there is no subsequence Y of X with $1 \notin Y$ such that $23 \le \sigma(Y) \le 29$. We partition the remaining elements into boxes (1) (22) (8,21) (9,20) (10,19) (11,18) (12,17) (13,16) (14,15). We also note that $v(x) \le 1$ for $12 \le x \le 14$ and $v(8), v(9) \le 2$.

Assume $v(22) > 0$. Then $\{22, 20^3, 1^5\}$ implies $v(9,20) \le 2$; $\{22, 10^3, 1^6\}$ implies $v(10,19) \le 2$; $\{22, 16^2, 1^4\}$ implies $v(13,16) \le 1$; $\{22, 15^2, 1^6\}$ implies $v(14,15) \le 1$. The cardinalities of boxes (1) (22) (8,21) (9,20) (10,19) (11,18) (12,17) (13,16) (14,15) is then at most $6+4+3+2+2+4+4+1+1 < 29$, so assume $v(20) = 0$.

Assume $v(21) > 0$. Then $\{21, 20^3, 1^6\}$ implies $v(9,20) \le 2$; $\{21, 17^2, 1^3\}$ implies $v(12,17) \le 1$; $\{21, 16^2, 1^5\}$ implies $v(13,16) \le 1$; $\{21, 15^4, 1^6\}$ implies $v(14,15) \le 3$. The cardinalities of boxes (1) (21) (9,20) (10,19) (11,18) (12,17) (13,16) (14,15) is then at most $6+3+2+6+4+1+1+3 < 29$, a contradiction. Thus $v(21) = 0$.

Assume $v(20) > 0$, then $v(x) = 0$ for any $8 \le x \le 9$. Then $\{20, 11^3, 1^5\}$ implies $v(11,18) \le 2$; $\{20, 17^2, 1^4\}$ implies $v(12,17) \le 1$; $\{20, 16^2, 1^6\}$ implies $v(13,16) \le 1$; $\{20, 15^6, 1^6\}$ implies $v(14,15) \le 5$. The cardinalities of boxes (1) (20) (10,19) (11,18) (12,17) (13,16) (14,15) is then at most $6+6+6+2+1+1+5 < 29$ elements, a contradiction, so $v(20) = 0$.

Assume $v(19) > 0$, then $v(x) = 0$ for any $8 \le x \le 10$. Now note that $\{19, 17^2, 1^5\}$ is basic which implies $v(12,17) \le 1$. The cardinalities of boxes (1) (19) (11,18) (12,17) (13,16) (14,15) is then at most $6+2+4+1+6+6 < 29$, so $v(19) = 0$.

Assume $v(18) > 0$. Then $v(x) = 0$ for any $8 \le x \le 11$. Hence we can have at most $6+2+4+6+6 < 23$ elements from boxes (1) (18) (12,17) (13,16) (14,15), a contradiction, and so $v(18) = 0$. Assume $v(17) > 0$. Then $v(x) = 0$ for any $8 \leq x \leq 12$, and we can partition into boxes (1) (17) (13,16) (14,15) with cardinalities at most 6, a contradiction. Hence $v(17) = 0$. Assume $v(16) > 0$. Then $v(x) = 0$ for any $8 \leq x \leq 13$ and we can partition into boxes (1) (16) (14,15), a contradiction. Hence $v(16) = 0$.

Now we note the sum of the cardinalities of the remaining boxes (1) (8) (9) (10) (11) (12) (13) (14,15) can be at most $6+2+2+6+4+1+1+6 < 29$, a contradiction. Thus we can assume that $H = M = 5$.

If M=5 we can assume $H = 5$ so that $v(1) = 5$ and $v(x) = 0$ for $2 \leq x \leq 6$. We assume there is no subsequence Y of X with $1 \notin Y$ such that $24 \leq \sigma(Y) \leq 29$. We partition the remaining elements into boxes (1) (23) (7,22) (8,21) (9,20) (10,19) (11,18) (12,17) (13,16) (14,15) each with cardinality at most 5. We note that $v(x) \leq 1$ for $12 \leq x \leq 14$ and $v(7) \leq 3$, $v(8), v(9) \leq 2$.

Assume $v(23) > 0$. Then $\{23^5, 1\}$ implies $v(23) \leq 4$; $\{23, 21^3, 1\}$ implies $v(8,21) \leq 2$; $\{23, 20^3, 1^4\}$ implies $v(9,20) \leq 2$; $\{23, 10^3, 1^5\}$ implies $v(10,19) \leq 2$; $\{23, 17^2, 1\}$ implies $v(12,17) \leq 1$; $\{23, 16^2, 1^3\}$ implies $v(13,16) \leq 1$; $\{23, 15^2, 1^5\}$ implies $v(14,15) \leq 1$. We partition into boxes (1) (23) (7,22) (8,21) (9,20) (10,19) (11,18) (12,17) (13,16) (14,15) with sum of the cardinalities at most $5+4+5+2+2+2+4+1+1+1 < 29$ elements, contradiction. Thus $v(23) = 0$.

Assume $v(22) > 0$. Then $\{22, 21^3, 1^2\}$ implies $v(8,21) \leq 2$; $\{22, 20^3, 1^5\}$ implies $v(9,20) \leq 2$; $\{22, 11^3, 1^3\}$ implies $v(11,18) \leq 2$; $\{22, 17^2, 1^2\}$ implies $v(12,17) \leq 1$; $\{22, 16^2, 1^4\}$ implies $v(13,16) \leq 1$. The sum of the cardinalities of the boxes (1) (22) (8,21) (9,20) (10,19) (11,18) (12,17) (13,16) (14,15) is then at most $5+5+2+2+$

$5+2+1+1+5 < 29$, a contradiction. Thus $v(22) = 0$.

Assume $v(21) > 0$. Then $v(x) = 0$ for any $7 \leq x \leq 8$. Then $\{21^4, 1^3\}$ implies $v(21) \leq 3$; $\{21, 11^3, 1^4\}$ implies $v(11,18) \leq 2$; $\{21, 17^2, 1^3\}$ implies $v(12,17) \leq 1$; $\{21, 16^2, 1^5\}$ implies $v(13,16) \leq 1$. The sum of the cardinalities of the boxes (1) (21) (9,20) (10,19) (11,18) (12,17) (13,16) (14,15) is then at most $5+3+5+5+2+1+1+5 < 29$, a contradiction. Thus $v(21) = 0$.

Assume $v(20) > 0$. Then $v(x) = 0$ for any $7 \leq x \leq 9$. Then $\{20, 11^3, 1^5\}$ implies $v(11,18) \leq 2$; $\{20, 17^2, 1^4\}$ implies $v(12,17) \leq 1$; $\{20, 16^4, 1^3\}$ implies $v(13,16) \leq 3$. The sum of the cardinalities of the boxes (1) (20) (10,19) (11,18) (12,17) (13,16) (14,15) boxes is then at most $5+5+5+2+1+3+5 < 29$, so $v(20) = 0$.

Assume $v(19) > 0$. Then $v(x) = 0$ for any $7 \leq x \leq 10$. We partition into boxes (1) (19) (11,18) (12,17) (13,16) (14,15) with sum of the cardinalities at most $5+2+4+4+5+5 < 29$, so $v(19) = 0$. Assume $v(18) > 0$. Then $v(x) = 0$ for any $7 \leq x \leq 11$, and we partition into boxes (1) (18) (12,17) (13,16) (14,15), a contradiction, so $v(18) = 0$. Assume $v(17) > 0$. Then $v(x) = 0$ for any $7 \leq x \leq 12$, and we partition into boxes (1) (17) (13,16) (14,15), a contradiction. Thus $v(17) = 0$. Assume $v(16) > 0$. Then $v(x) = 0$ for any $8 \leq x \leq 13$ and we can partition into boxes (1) (7) (16) (14,15), a contradiction, so $v(16) = 0$. Assume $v(15) > 0$. Then $v(x) = 0$ for any $9 \leq x \leq 14$ and we can assume 29 elements from boxes (1) (7) (8) (15), a contradiction, so $v(15) = 0$.

Now since $x_i < \frac{p}{2}$ for each $x_i \in X$, by Theorem 3.2.5 we have a basic subsequence. Thus we can assume that $M \neq 5$ and therefore Conjecture 1.3.2 holds for $p = 29$.

3.3.4 Case p=31

We need only verify when $5 \leq H \leq M \leq 9$. Assume that X is a sequence of length 31 in $\mathbb{Z}_{31}$ that has no basic subsequence and that $v(1) \geq 5$. Note the following basic sequences and their implications on X for future reference:

$\{24^5, 1^4\}$ implies $v(24) \leq 4$

$\{23^4, 1\}$ implies $v(23) \leq 3$

$\{22^4, 1^5\}$ implies $v(22) \leq 3$

$\{20^3, 1^2\}$ implies $v(20) \leq 2$

$\{19^3, 1^5\}$ implies $v(19) \leq 2$

$\{18^5, 1^3\}$ implies $v(18) \leq 4$

$\{17^7, 1^5\}$ implies $v(17) \leq 6$

$\{11^5, 1^7\}$ implies $v(11, 20) \leq 4$ when $M \geq 7$.

If M=9 let $\alpha = (1, .., 10)$ be a box in X. We can assume there is no subsequence Y of $X - \alpha$ such that $22 \leq \sigma(Y) \leq 31$. We partition the remaining elements into boxes (α) (21) (11,20) (12,19) (13,18) (14,17) (15,16). We note that the cardinality of any such box is at most 9 and $v(x) \leq 1$ for $11 \leq x \leq 15$.

Assume $v(21) > 0$. Then since $\{21^4, 1^9\}$ is basic assume $v(21) \leq 3$. Also since $\{21, 16^2, 1^9\}$ is basic we can assume that $v(15, 16) \leq 1$. We partition into boxes (α) (21) (11,20) (12,19) (13,18) (14,17) (15,16) with sum of cardinalities at most $9 + 3 + 2 + 2 + 4 + 6 + 1 < 31$, so assume $v(21) = 0$.

Assume $v(20) > 0$. Then $\{20, 16^4, 1^9\}$ is basic we can assume that $v(15, 16) \leq 3$. We partition into boxes (α) (20) (12,19) (13,18) (14,17) (15,16) and the cardinalities of these boxes is at most $9+2+2+4+6+3<31$, so assume $v(20)=0$. Assume $v(19)>0$. Then $v(x)=0$ for $11 \leq x \leq 12$ and we can partition into boxes (α) (19) (13,18) (14,17) (15,16) where the cardinalities of these boxes is at most $9+2+4+6+9<31$, so let $v(19)=0$.

Now we partition the remaining elements so that the sum of the cardinalities of their boxes (α) (11) (12) (13,18) (14,17) (15,16) can be at most $9+1+1+4+6+9<31$. Thus if no counterexample exists when $M=9$ so assume $5 \leq H \leq M \leq 8$.

If M=8 let $\alpha=(1,..,9)$ be a box in X. We can assume there is no subsequence Y of $X-\alpha$ such that $23 \leq \sigma(Y) \leq 31$. We partition the remaining elements into boxes (α) (22) (10,21) (11,20) (12,19) (13,18) (14,17) (15,16). We note that $v(x) \leq 1$ for $12 \leq x \leq 15$ and $v(10) \leq 2$.

Assume $v(22)>0$. Then $\{22,21^3,1^8\}$ is basic which implies that $v(10,21) \leq 2$; $\{11^5,1^7\}$ is basic which implies $v(11,20) \leq 4$; $\{22,16^2,1^8\}$ is basic which implies that $v(15,16) \leq 1$. The cardinalities of the boxes (α) (22) (10,21) (11,20) (12,19) (13,18) (14,17) (15,16) is then at most $8+3+2+4+2+4+6+1<31$, so $v(22)=0$.

Assume $v(21)>0$. Then $\{21,11^3,1^8\}$ is basic which implies that $v(11,20) \leq 2$; $\{21,18^2,1^5\}$ is basic which implies that $v(13,18) \leq 1$; $\{21,17^2,1^7\}$ is basic which implies that $v(14,17) \leq 1$. The cardinalities of the boxes (α) (21) (11,20) (12,19) (13,18) (14,17) (15,16) is then at most $8+8+2+2+1+1+8<31$, so $v(21)=0$.

Assume $v(20)>0$. Then $v(x)=0$ for any $10 \leq x \leq 11$ and we partition into boxes (α) (20) (12,19) (13,18) (14,17) (15,16). The cardinalities of these boxes is then at most $8+2+2+4+6+8<31$, a contradiction. Hence $v(20)=0$. Assume $v(19) > 0$. Then $v(x) = 0$ for any $10 \leq x \leq 12$ and we partition into boxes (α) (19) (13,18) (14,17) (15,16). The cardinalities of these boxes is then at most $8+2+4+6+8 < 31$, a contradiction. Hence $v(19) = 0$. Assume $v(18) > 0$. Then $v(x) = 0$ for any $10 \leq x \leq 13$ and we partition into boxes (α) (18) (14,17) (15,16) and the cardinalities of these boxes is at most $8+4+6+8 < 31$, a contradiction, hence $v(18) = 0$.

Now we note the sum of the cardinalities of the remaining boxes (α) (10) (11) (12) (13) (14,17) (15,16) can be at most $8+2+4+1+1+6+8 < 31$. Thus if $M = 8$ we derive a contradiction, so assume that $5 \leq H \leq M \leq 7$.

If M=7 let $\alpha = (1,..,8)$ be a box in X. We can assume there is no subsequence Y of $X - \alpha$ such that $24 \leq \sigma(Y) \leq 31$. We partition the remaining elements into boxes (α) (23) (9,22) (10,21) (11,20) (12,19) (13,18) (14,17) (15,16). We note that the cardinality of any such box is at most 7 and $v(x) \leq 1$ for $12 \leq x \leq 15$ and $v(y) \leq 2$ for $9 \leq y \leq 10$.

Assume $v(23) > 0$. Then $\{23, 22^3, 1^4\}$ implies $v(22, 9) \leq 2$; $\{23, 21^3, 1^7\}$ is basic which implies $v(10, 21) \leq 2$; $\{23, 17^2, 1^5\}$ implies $v(14, 17) \leq 1$; $\{23, 16^2, 1^7\}$ is basic which implies $v(15, 16) \leq 1$. The cardinalities of the boxes (α) (23) (9,22) (10,21) (11,20) (12,19) (13,18) (14,17) (15,16) is then at most $7+3+3+2+4+2+4+1+1 < 31$, so $v(23) = 0$.

Assume $v(22) > 0$. Then we partition into boxes (α) (22) (10,21) (11,20) (12,19) (13,18) (14,17) (15,16). Then note that $\{22, 21^3, 1^7\}$ is basic which implies $v(10, 21) \leq 2$. Also $\{22, 17^2, 1^6\}$ is basic which implies that $v(14, 17) \leq 1$. The cardinalities of these boxes is then at most $7+3+2+4+2+4+1+7 < 31$, so $v(22) = 0$.

Assume $v(21) > 0$. Then $v(x) = 0$ for any $9 \leq x \leq 10$ and we partition into boxes (α) (21) (11,20) (12,19) (13,18) (14,17) (15,16). Then note that $\{21, 18^2, 1^5\}$ and $\{21, 17^2, 1^7\}$ are basic which implies that $v(13, 18) \leq 1$ and $v(14, 17) \leq 1$. The cardinalities of these boxes is then at most $7+7+4+2+1+1+7 < 31$, so $v(21) = 0$.

Assume $v(20) > 0$. Then $v(x) = 0$ for any $9 \leq x \leq 11$ and we partition into boxes (α) (20) (12,19) (13,18) (14,17) (15,16) with cardinalities at most $7+2+2+4+6+7 < 24$, so $v(20) = 0$. Assume $v(19) > 0$. Then $v(x) = 0$ for any $9 \leq x \leq 12$ and we partition into boxes (α) (19) (13,18) (14,17) (15,16) with cardinalities at most $7 + 2 + 4 + 6 + 7 < 31$, so $v(19) = 0$. Assume $v(18) > 0$. Then $v(x) = 0$ for any $9 \leq x \leq 13$ and we partition into boxes (α) (18) (14,17) (15,16), a contradiction, so $v(18) = 0$.

Now we note the sum of the cardinalities of the remaining boxes (α) (9) (10) (11) (12) (13) (14,17) (15,16) can be at most $7+2+2+4+1+1+6+7 < 31$. Thus if $M = 7$ we can assume that X must have a basic subsequence, so assume $5 \leq H \leq M \leq 6$.

If M=6 we can assume that $H = 6$ so that $v(1) = 6$ and $v(x) = 0$ for $2 \leq x \leq 7$. We assume there is no subsequence Y of X with $1 \notin Y$ such that $25 \leq \sigma(Y) \leq 31$. We partition the remaining elements into boxes (1) (24) (8,23) (9,22) (10,21) (11,20) (12,19) (13,18) (14,17) (15,16). We note that the cardinality of any such box is at most 6 and $v(x) \leq 1$ for $13 \leq x \leq 15$ and $v(y) \leq 2$ for $9 \leq y \leq 10$.

Assume $v(24) > 0$. Then $\{24, 8^4, 1^6\}]$ implies $v(8, 23) \leq 3$; $\{24, 22^3, 1^3\}$ implies $v(22, 9) \leq 2$; $\{24, 21^3, 1^6\}$ implies $v(10, 21) \leq 2$; $\{24, 11^3, 1^5\}$ implies $v(11, 20) \leq$; $\{24, 12^3, 1^2\}$ implies $v(12, 19) \leq 2$; $\{24, 17^2, 1^4\}$ implies $v(14, 17) \leq 1$; $\{24, 16^2, 1^6\}$ implies $v(15,16) \leq 1$. The cardinalities of the boxes (1) (24) (8,23) (9,22) (10,21) (11,20) (12,19) (13,18) (14,17) (15,16) is then at most $6+4+3+3+2+2+2+4+1+1 < 31$, so $v(24) = 0$.

Assume $v(23) > 0$. Then $\{23, 11^3, 1^6\}$ implies $v(11,20) \leq 2$; $\{23, 12^3, 1^3\}$ implies $v(12,19) \leq 2$; $\{23, 18^2, 1^3\}$ implies $v(13,18) \leq 1$; $\{23, 17^2, 1^5\}$ implies $v(14,17) \leq 1$; The cardinalities of the boxes (1) (23) (9,22) (10,21) (11,20) (12,19) (13,18) (14,17) (15,16) is then at most $6+3+3+6+2+2+1+1+6 < 31$, so $v(23) = 0$.

Assume $v(22) > 0$. Then $\{22, 11^6, 1^5\}$ implies $v(11,20) \leq 5$; $\{22, 12^3, 1^4\}$ implies $v(12,19) \leq 2$; $\{22, 16^6, 1^6\}$ implies $v(15,16) \leq 5$; $\{22, 18^2, 1^4\}$ implies $v(13,18) \leq 1$; $\{22, 17^2, 1^6\}$ implies $v(14,17) \leq 1$; The cardinalities of the boxes (1) (22) (10,21) (11,20) (12,19) (13,18) (14,17) (15,16) is then at most $6+3+6+5+2+1+1+5 < 31$, so $v(22) = 0$.

Assume $v(21) > 0$ so $v(x) = 0$ for any $8 \leq x \leq 10$. Then $\{21, 12^3, 1^5\}]$ implies $v(12,19) \leq 2$; $\{21, 18^2, 1^5\}$ implies $v(13,18) \leq 1$; $\{21, 17^4, 1^4\}$ implies $v(14,17) \leq 3$. The cardinalities of the boxes (1) (21) (11,20) (12,19) (13,18) (14,17) (15,16). is then at most $6+6+6+2+1+3+6 < 31$, so $v(21) = 0$.

Assume $v(20) > 0$ so $v(x) = 0$ for any $8 \leq x \leq 11$. Then $\{20, 18^2, 1^6\}$ is basic so $v(13,18) \leq 1$. The cardinalities of the boxes (1) (20) (12,19) (13,18) (14,17) (15,16) is then at most $6+2+6+1+6+6 < 31$, so $v(20) = 0$.

Assume $v(19) > 0$. Then $v(x) = 0$ for any $8 \leq x \leq 12$ and we partition into boxes (1) (19) (13,18) (14,17) (15,16) with cardinalities at most $6+2+4+6+6 < 31$ so $v(19) = 0$. Assume $v(18) > 0$. Then $v(x) = 0$ for any $8 \leq x \leq 13$ and we partition into boxes (1) (18) (14,17)(15,16), a contradiction, hence $v(18) = 0$. Assume $v(17) > 0$. Then $v(x) = 0$ for any $8 \leq x \leq 14$ and we partition into boxes (1) (17) (15,16), a contradiction. Hence $v(17) = 0$. Assume $v(16) > 0$. Then $v(x) = 0$ for any $9 \leq x \leq 15$. Then we partition into boxes (1) (8) (16), a contradiction, so $v(16) = 0$.

Now we note the sum of the cardinalities of the remaining boxes (1) (8) (9) (10) (11) (12) (13,14,15) can be at most $6+6+2+2+6+6+1 < 31$. Thus if $M = 6$ we derive a contradiction, hence we can assume that $5 \leq H \leq M \leq 5$.

If M=H=5 assume $v(1) = 5$ and $v(x) = 0$ for $2 \leq x \leq 6$. We assume there is no subsequence Y of X with $1 \notin Y$ such that $26 \leq \sigma(Y) \leq 31$. We partition the remaining elements into boxes (1) (25) (7,24) (8,23) (9,22) (10,21) (11,20) (12,19) (13,18) (14,17) (15,16). We note that the cardinality of any such box is at most 5 and $v(x) \leq 1$ for $13 \leq x \leq 15$ and $v(y) \leq 2$ for $9 \leq y \leq 10$.

Assume $v(25) > 0$. Then $\{25, 24^4, 1^3\}$ implies $v(7,24) \leq 3$; $\{25, 22^3, 1^2\}$ implies $v(22,9) \leq 2$; $\{25, 21^3, 1^5\}$ implies $v(10,21) \leq 2$; $\{25, 11^3, 1^4\}$ implies $v(11,20) \leq 2$; $\{25, 12^3, 1\}$ implies $v(12,19) \leq 2$; $\{25, 18^2, 1\}$ implies $v(13,18) \leq 1$; $\{25, 17^2, 1^3\}$ implies $v(14,17) \leq 1$; $\{25, 16^2, 1^5\}$ implies $v(15,16) \leq 1$; The cardinalities of the boxes (1) (25) (7,24) (8,23) (9,22) (10,21) (11,20) (12,19) (13,18) (14,17) (15,16) is then at most $5+5+3+5+2+2+2+2+1+1+1 < 31$. Hence $v(25) = 0$.

Assume $v(24) > 0$. Then $\{24, 22^3, 1^3\}$ implies $v(22,9) \leq 2$; $\{24, 11^3, 1^5\}$ implies $v(11,20) \leq 2$; $\{24, 12^3, 1^2\}$ implies $v(12,19) \leq 2$; $\{24, 18^2, 1^2\}$ implies $v(13,18) \leq 1$; $\{24, 17^2, 1^4\}$ implies $v(14,17) \leq 1$; $\{24, 16^4, 1^5\}$ implies $v(15,16) \leq 3$. The cardinalities of the boxes (1) (24) (8,23) (9,22) (10,21) (11,20) (12,19) (13,18) (14,17) (15,16) is then at most $5+4+5+2+5+2+2+1+1+3 < 31$, thus $v(24) = 0$.

Assume $v(23) > 0$ so $v(x) = 0$ for any $7 \leq x \leq 8$. Then $\{23, 22^3, 1^4\}$ implies $v(22, 9) \leq 2$; $\{23, 12^3, 1^3\}$ implies $v(12, 19) \leq 2$; $\{23, 18^2, 1^3\}$ implies $v(13, 18) \leq 1$; $\{23, 17^2, 1^5\}$ implies $v(14, 17) \leq 1$; The cardinalities of the boxes (1) (23) (9,22) (10,21) (11,20) (12,19) (13,18) (14,17) (15,16) is then at most $5+3+2+5+5+2+1+1+5 < 31$, so $v(23) = 0$.

Assume $v(22) > 0$ so $v(x) = 0$ for any $7 \leq x \leq 9$. Then $\{22, 12^3, 1^4\}$ implies $v(12, 19) \leq 2$; $\{22, 18^2, 1^4\}$ implies $v(13, 18) \leq 1$; $\{22, 17^4, 1^3\}$ implies $v(14, 17) \leq 3$; The cardinalities of the boxes (1) (22) (10,21) (11,20) (12,19) (13,18) (14,17) (15,16) is then at most $5+3+5+5+2+1+3+5 < 31$, and so $v(22) = 0$.

Assume $v(21) > 0$ so $v(x) = 0$ for any $7 \leq x \leq 10$. Then $\{21, 12^3, 1^5\}$ implies $v(12, 19) \leq 2$; $\{21, 18^2, 1^5\}$ implies $v(13, 18) \leq 1$; $\{21, 17^4, 1^4\}$ implies $v(14, 17) \leq 3$. The cardinalities of the boxes (1) (21) (11,20) (12,19) (13,18) (14,17) (15,16) is then at most $5+5+5+2+1+3+5 < 31$, hence $v(21) = 0$.

Assume $v(20) > 0$. Then $v(x) = 0$ for any $7 \leq x \leq 11$ and we partition into boxes (1) (20) (12,19) (13,18) (14,17) (15,16), but each has cardinality at most 6, a contradiction. So $v(20) = 0$. Assume $v(19) > 0$. Then $v(x) = 0$ for any $7 \leq x \leq 12$ and we partition into boxes (1) (19) (13,18) (14,17) (15,16), a contradiction so $v(19) = 0$. Assume $v(18) > 0$. Then $v(x) = 0$ for any $8 \leq x \leq 13$. and we can partition into boxes (1) (7) (18) (14,17) (15,16), a contradiction, so $v(18) = 0$. Assume $v(17) > 0$. Then $v(x) = 0$ for any $9 \leq x \leq 14$. Then we partition into boxes (1) (7) (8) (17) (15,16), a contradiction, so $v(17) = 0$. Assume $v(16) > 0$. Then $v(x) = 0$ for any $10 \leq x \leq 15$. Then we partition into boxes (1) (7) (8) (9) (16), a contradiction, so $v(16) = 0$.

Now all elements are less than $\frac{p}{2}$ hence a basic subsequence exists by Theorem 3.2.5. Thus in all cases we have a basic sequence, and Conjecture 1.3.2 is proved for $p = 31$.

3.3.5 Case p=37

Assume that X is a sequence of length 37 in $\mathbb{Z}_{37}$ with no basic subsequence. We need to verify when $5 \leq H \leq M \leq 11$. Note the following basic zero-sequences and their implications:

$\{30^6, 1^5\}$ implies $v(30) \leq 5$

$\{29^5, 1^3\}$ implies $v(29) \leq 4$

$\{27^4, 1^3\}$ implies $v(27) \leq 3$

$\{26^4, 1^7\}$ implies $v(26) \leq 3$ for $M \geq 7$

$\{24^3, 1^2\}$ implies $v(24) \leq 2$

$\{23^3, 1^5\}$ implies $v(23) \leq 2$

$\{22^3, 1^8\}$ implies $v(22) \leq 2$ for $M \geq 8$

$\{21^5, 1^6\}$ implies $v(21) \leq 4$ for $M \geq 6$

$\{20^7, 1^8\}$ implies $v(20) \leq 6$ for $M \geq 8$

$\{14^5, 1^4\}$ implies $v(14) \leq 4$

If M=11 let $\alpha = (1, .., 12)$ be a box in X. Then we can assume that there is no subsequence Y of $X - \alpha$ such that $26 \leq \sigma(Y) \leq 37$. We partition the remaining elements into boxes (α) (25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) where the cardinality of any such box is at most 11. We have immediately that $v(x) \leq 1$ if $13 \leq x \leq 18$.

Then note that $\{25^4, 1^{11}\}$ is basic which implies $v(25) \leq 3$; $\{21^3, 1^{11}\}$ is basic which implies $v(16, 21) \leq 2$; $\{20^5, 1^{11}\}$ is basic which implies $v(17, 20) \leq 4$. Hence the cardinalities of the boxes (α) (25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) are at most $11+3+2+2+2+2+4+11 = 37$. Thus we can assume that $v(25) = 3$ and $v(19) = 11$ (since $v(18) \leq 1$) so then $\{25^2, 19, 1^4\}$ is a basic subsequence, a contradiction. Hence we can assume $5 \leq H \leq M \leq 10$.

If M=10, let $\alpha = (1, .., 11)$ be a box in X with $v(\alpha) \leq 10$. We can assume there is no subsequence Y of $X - \alpha$ such that $27 \leq \sigma(Y) \leq 37$. We partition the remaining elements into boxes (α) (26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19). We have immediately that $v(x) \leq 1$ if $14 \leq x \leq 18$ and $v(12) \leq 2$. Note $\{25^7, 1^{10}\}$ is basic which implies $v(12, 25) \leq 6$, and $\{13^5, 1^9\}$ is basic which implies $v(13) \leq 4$.

Assume $v(26) > 0$. Then $\{26, 13^3, 1^9\}$ is basic which implies $v(13, 24) \leq 2$, and $\{26, 19^2, 1^{10}\}$ is basic which implies $v(18, 19) \leq 1$. The cardinalities of the boxes (α) (26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most $10 + 3 + 6 + 2 + 2 + 2 + 4 + 6 + 1 < 37$ so assume $v(26) = 0$.

Assume $v(25) > 0$. Then note $\{25, 13^3, 1^{10}\}$ is basic which implies $v(13, 24) \leq 2$ and $\{25, 19^4, 1^{10}\}$ is basic which implies $v(18, 19) \leq 3$. The cardinalities of the boxes (α) (25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most $10 + 6 + 2 + 2 + 2 + 4 + 6 + 3 < 37$ so assume $v(25) = 0$.

Assume $v(24) > 0$ so that $v(x) = 0$ for any $12 \le x \le 13$. Then we partition into boxes (α) (24) (14,23) (15,22) (16,21) (17,20) (18,19) and the cardinalities of these boxes is at most $10 + 2 + 2 + 2 + 4 + 6 + 10 < 37$ so assume $v(24) = 0$. Assume $v(23) > 0$ so that $v(x) = 0$ for any $12 \le x \le 14$ and we partition into boxes (α) (23) (15,22) (16,21) (17,20) (18,19). The cardinalities of these boxes is then at most $10+2+2+4+6+10 < 37$ so $v(23) = 0$. Now assume $v(22) > 0$ so that $v(x) = 0$ for any $12 \le x \le 15$. We partition into boxes (α) (22) (16,21) (17,20) (18,19). The cardinalities of these boxes is then at most $10+2+4+6+10 < 37$ so $v(22) = 0$. Assume $v(21) > 0$ so that $v(x) = 0$ for any $12 \le x \le 16$. We partition into boxes (α) (21) (17,20) (18,19). The cardinalities of these boxes is then at most $10 + 4 + 6 + 10 < 37$ so $v(21) = 0$.

Now we note the sum of the cardinalities of the remaining boxes (α) (12) (13) (14) (15) (16) (17,20) (18,19) can be at most $10 + 2 + 4 + 1 + 1 + 1 + 6 + 10 < 37$, a contradiction. Hence we can assume $5 \le H \le M \le 9$.

If M=9 let $\alpha = (1, .., 10)$ be a box in X. We can assume is no subsequence Y of $X - \alpha$ such that $28 \le \sigma(Y) \le 37$. We partition the remaining elements into boxes (α) (27) (11,26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19). We have immediately that $v(x) \le 1$ if $14 \le x \le 18$, and $v(y) \le 2$ for $11 \le y \le 12$ and recall $\{13^5, 1^9\}$ is basic so we still assume $v(13) \le 4$.

Assume $v(27) > 0$. Then note $\{27, 25^3, 1^9\}$ is basic which implies $v(12, 25) \le 2$ and $\{27, 19^2, 1^9\}$ is basic which implies $v(18, 19) \le 1$. The cardinalities of the boxes (α) (27) (11,26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most $9 + 3 + 3 + 2 + 4 + 2 + 2 + 4 + 6 + 1 < 37$ so $v(27) = 0$.

Assume $v(26) > 0$. Then note $\{26, 25^6, 1^9\}$ is basic which implies $v(12, 25) \leq 5$; $\{26, 19^4, 1^9\}$ is basic which implies $v(18, 19) \leq 3$; and $\{26, 21^2, 1^6\}$ is basic which implies $v(16, 21) \leq 1$. The cardinalities of the boxes (α) (26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most $9+3+5+4+2+2+1+6+3 < 37$ so $v(26) = 0$.

Assume $v(25) > 0$ so that $v(11) = 0$. Then $\{25, 19^6, 1^9\}$ is basic which implies $v(18, 19) \leq 5$ and $\{25, 20^2, 1^9\}$ is basic which implies $v(17, 20) \leq 1$. The cardinalities of the boxes (α) (25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most $9+9+4+2+2+4+1+5 < 37$, so $v(25) = 0$.

Assume $v(24) > 0$. Then $v(x) = 0$ for any $11 \leq x \leq 13$ and we partition into boxes (α) (24) (14,23) (15,22) (16,21) (17,20) (18,19) with cardinalities at most $9+2+2+2+4+6+9 < 37$. Thus $v(24) = 0$. Assume $v(23) > 0$. Then $v(x) = 0$ for any $11 \leq x \leq 14$ and we partition into boxes (α) (23) (15,22) (16,21) (17,20) (18,19) with cardinalities at most $9+2+2+4+6+9 < 37$. Thus $v(23) = 0$. Assume $v(22) > 0$. Then $v(x) = 0$ for any $11 \leq x \leq 15$ and we partition into boxes (α) (22) (16,21) (17,20) (18,19) with cardinalities at most $9+2+4+6+9 < 37$. Thus $v(22) = 0$. Assume $v(21) > 0$. Then $v(x) = 0$ for any $11 \leq x \leq 16$ and we partition into boxes (α) (21) (17,20) (18,19) with cardinalities at most $9+4+6+9 < 37$, a contradiction. So assume $v(21) = 0$.

Now we note the sum of the cardinalities of the remaining boxes (α) (11) (12) (13) (14,15) (16) (17,20) (18,19) can be at most $9+2+2+4+1+1+6+9 < 37$. Thus we can assume $5 \leq H \leq M \leq 8$.

If M=8 let $\alpha = (1,..,9)$ be a box in X. We can assume there is no subsequence Y of $X - \alpha$ such that $29 \leq \sigma(Y) \leq 37$. We partition the remaining elements into boxes (α) (28) (10,27) (11,26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) and note the cardinality of each of these boxes is at most 8. We have immediately that $v(x) \leq 1$ if $15 \leq x \leq 18$, and $v(y) \leq 2$ for $10 \leq y \leq 12$. Note that $\{28^5, 1^8\}$ is basic which implies $v(28) \leq 4$, and $\{13^8, 1^7\}$ is basic which implies $v(13) \leq 7$.

Assume $v(28) > 0$. Then $\{28, 25^3, 1^8\}$ is basic which implies $v(12, 25) \leq 2$; $\{28, 13^3, 1^7\}$ is basic which implies $v(13, 24) \leq 2$; $\{28, 19^2, 1^8\}$ is basic which implies $v(18, 19) \leq 1$; $\{28, 20^2, 1^6\}$ is basic which implies $v(17, 20) \leq 1$; $\{28, 21^2, 1^4\}$ is basic which implies $v(16, 21) \leq 1$. The cardinalities of the boxes (α) (28) (10,27) (11,26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most $8+8+3+3+2+2+4+2+1+1+1 < 37$. So $v(28) = 0$.

Assume $v(27) > 0$. Then $\{27, 13^3, 1^8\}$ is basic which implies $v(13, 24) \leq 2$; $\{27, 25^6, 1^8\}$ is basic which implies $v(12, 25) \leq 5$; $\{27, 19^4, 1^8\}$ is basic which implies $v(18, 19) \leq 3$; $\{27, 20^2, 1^7\}$ is basic which implies $v(17, 20) \leq 1$. The cardinalities of the boxes (α) (27) (11,26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most $8+3+3+5+2+4+2+4+1+3 < 37$. So $v(27) = 0$.

Assume $v(26) > 0$ so that $v(x) = 0$ for any $10 \leq x \leq 11$. Then $\{26, 21^2, 1^6\}$ is basic which implies $v(16, 21) \leq 1$; $\{26, 20^2, 1^8\}$ is basic which implies $v(17, 20) \leq 1$; $\{26, 19^6, 1^8\}$ is basic which implies $v(18, 19) \leq 5$; $\{26, 13^6, 1^7\}$ is basic which implies $v(13, 24) \leq 5$. The cardinalities of the boxes (α) (26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most $8+3+8+5+4+2+1+1+5 = 37$. So we can assume $v(24) = 5$ and $\{26, 24, 24\}$ is a basic subsequence, a contradiction. Hence $v(26) = 0$.

Assume $v(25) > 0$ so that $v(x) = 0$ for any $10 \leq x \leq 12$. Then $\{25, 21^2, 1^7\}$ is basic which implies $v(16, 21) \leq 1$; $\{25, 20^4, 1^6\}$ is basic which implies $v(17, 20) \leq 3$; $\{25, 19^8, 1^8\}$ is basic which implies $v(18, 19) \leq 7$; $\{25, 14^3, 1^7\}$ is basic which implies $v(14, 23) \leq 2$; $\{25, 13^6, 1^8\}$ is basic which implies $v(13, 24) \leq 5$; The cardinalities of the boxes (α) (25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most $8+8+5+2+2+1+3+7 < 37$. So $v(25) = 0$.

Assume $v(24) > 0$. Then $v(x) = 0$ for any $10 \leq x \leq 13$ and we can partition into boxes (α) (24) (14,23) (15,22) (16,21) (17,20) (18,19) with cardinalities at most $8+2+4+2+4+6+8 < 37$. So $v(24) = 0$. Assume $v(23) > 0$. Then $v(x) = 0$ for any $10 \leq x \leq 14$ and we can partition into boxes (α) (23) (15,22) (16,21) (17,20) (18,19). The cardinalities of these boxes is then at most $8+2+2+4+6+8 < 37$. So $v(23) = 0$. Assume $v(22) > 0$. Then $v(x) = 0$ for any $10 \leq x \leq 15$ and we partition as (α) (22) (16,21) (17,20) (18,19) with cardinalities at most $8+2+4+6+8 < 37$. So $v(22) = 0$. Assume $v(21) > 0$. Then $v(x) = 0$ for any $10 \leq x \leq 16$ and partition as (α) (21) (17,20) (18,19), a contradiction, so $v(21) = 0$. Assume $v(20) > 0$. Then $v(x) = 0$ for any $10 \leq x \leq 17$ and we have boxes (α)(20) (18,19), a contradiction so $v(20) = 0$.

Now we partition the remaining elements into boxes (α) (10) (11) (12) (13) (14) (15,16,17) (18,19). Note that any two elements from $(15, 16, 17)$ clearly gives a sum from 30 to 44. So the sum of the cardinalities of these boxes can be at most $8+2+2+2+7+4+1+8 < 37$. Thus we can assume that $5 \leq H \leq M \leq 7$.

If M=7 let $\alpha = (1, .., 8)$ be a box in X. We can assume there is no subsequence Y of $X - \alpha$ such that $30 \leq \sigma(Y) \leq 37$. We partition the remaining elements into boxes

(α) (29) (9,28) (10,27) (11,26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19), and note the cardinality of each of these boxes is at most 7. We have immediately that $v(x) \leq 1$ if $15 \leq x \leq 18$, and $v(y) \leq 2$ for $10 \leq y \leq 12$, and $v(9) \leq 3$.

Assume $v(29) > 0$. Then $\{29, 28^4, 1^7\}$ is basic which implies $v(9,28) \leq 3$; $\{29, 25^3, 1^7\}$ is basic which implies $v(12,25) \leq 2$; $\{29, 22, 22, 1\}$ implies $v(15,22) \leq 1$; $\{29, 13^3, 1^6\}$ is basic which implies $v(13,24) \leq 2$; $\{29, 20^2, 1^5\}$ is basic which implies $v(17,20) \leq 1$; $\{29, 19^2, 1^7\}$ is basic which implies $v(18,19) \leq 1$. The cardinalities of the boxes (α) (29) (9,28) (10,27) (11,26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most $7+4+3+3+3+2+2+4+1+4+1+1 < 37$. So $v(29) = 0$.

Assume $v(28) > 0$. Then $\{28, 25^6, 1^7\}$ is basic which implies $v(12,25) \leq 5$; $\{28, 22^2, 1^2\}$ implies $v(15,22) \leq 1$; $\{28, 13^3, 1^7\}$ is basic which implies $v(13,24) \leq 2$; $\{28, 21^2, 1^4\}$ is basic which implies $v(16,21) \leq 1$; $\{28, 20^2, 1^6\}$ is basic which implies $v(17,20) \leq 1$; $\{28, 19^4, 1^7\}$ is basic which implies $v(18,19) \leq 3$; $\{28, 14^3, 1^4\}$ is basic which implies $v(14,23) \leq 2$. The cardinalities of the boxes (α) (28) (10,27) (11,26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) is then at most $7+7+3+3+5+2+2+1+1+1+3 < 37$. So $v(28) = 0$.

Assume $v(27) > 0$ so that $v(x) = 0$ for any $9 \leq x \leq 10$. Then $\{27, 22^2, 1^3\}$ implies $v(15,22) \leq 1$; $\{27, 13^6, 1^6\}$ is basic which implies $v(13,24) \leq 5$; $\{27, 21^2, 1^5\}$ implies $v(16,21) \leq 1$; $\{27, 20^2, 1^7\}$ implies $v(17,20) \leq 1$; $\{27, 19^6, 1^7\}$ implies $v(18,19) \leq 5$; $\{27, 14^3, 1^5\}$ implies $v(14,23) \leq 2$. The cardinalities of the boxes (α) (27) (11,26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most $7+3+3+7+5+2+1+1+1+5 < 37$. So $v(27) = 0$.

Assume $v(26) > 0$ so that $v(x) = 0$ for any $9 \leq x \leq 11$. Then $\{26, 22^2, 1^4\}$ implies $v(15, 22) \leq 1$; $\{26, 13^6, 1^7\}$ implies $v(13, 24) \leq 5$; $\{26, 20^4, 1^5\}$ is basic which implies $v(17, 20) \leq 3$; $\{26, 21^2, 1^6\}$ is basic which implies $v(16, 21) \leq 1$; $\{26, 14^3, 1^6\}$ is basic which implies $v(14, 23) \leq 2$; The cardinalities of the boxes (α) (26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most $7+3+7+5+2+1+1+3+7 < 37$, so $v(26) = 0$.

Assume $v(25) > 0$ so that $v(x) = 0$ for any $9 \leq x \leq 12$. Then $\{25, 22^2, 1^5\}$ implies $v(15, 22) \leq 1$; $\{25, 20^4, 1^6\}$ is basic which implies $v(17, 20) \leq 3$; $\{25, 21^2, 1^7\}$ is basic which implies $v(16, 21) \leq 1$; $\{25, 14^3, 1^7\}$ is basic which implies $v(14, 23) \leq 2$. The cardinalities of the boxes (α) (25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most $7 + 7 + 7 + 2 + 1 + 1 + 3 + 7 < 37$. So $v(25) = 0$.

Assume $v(24) > 0$. Then $v(x) = 0$ for any $9 \leq x \leq 13$ and we partition into boxes (α) (24) (14,23) (15,22) (16,21) (17,20) (18,19) with cardinalities at most $7 + 2 + 4 + 2 + 4 + 7 + 7 < 37$, so $v(24) = 0$. Assume $v(23) > 0$. Then $v(x) = 0$ for any $9 \leq x \leq 14$ and we partition into boxes (α) (23) (15,22) (16,21) (17,20) (18,19) with cardinalities at most $7 + 1 + 2 + 4 + 4 + 7 + 7 < 37$, so $v(23) = 0$. Assume $v(22) > 0$. Then $v(x) = 0$ for any $9 \leq x \leq 15$ and we partition into boxes (α) (22) (16,21) (17,20) (18,19), a contradiction, so $v(22) = 0$. If $v(21) > 0$ then $v(x) = 0$ for any $9 \leq x \leq 16$ and we partition into boxes (α) (21) (17,20) (18,19), a contradiction, so $v(21) = 0$. If $v(20) > 0$ then $v(x) = 0$ for any $10 \leq x \leq 17$ and we partition into boxes (α) (9) (17,20) (18,19), a contradiction so $v(20) = 0$. If $v(19) > 0$ then $v(x) = 0$ for any $11 \leq x \leq 18$ and we partition into boxes (α) (9) (10) (19), a contradiction so $v(19) = 0$.

Now we note the sum of the cardinalities of the remaining boxes (α) (9) (10) (11) (12) (13) (14) (15) (16) (17) (18) can be at most 7+3+2+2+2+7+4+1+1+1+1 < 37. Thus we can assume that $5 \leq H \leq M \leq 6$.

If M=6 we can also assume $H = 6$, so let $v(1) = 6$. We can assume there is no subsequence Y of X with $1 \notin Y$ such that $31 \leq \sigma(Y) \leq 37$, and that $v(x) = 0$ for $2 \leq x \leq 7$. We partition the remaining elements into boxes (1) (30) (8,29) (9,28) (10,27) (11,26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19), and note the cardinality of each of these boxes is at most 6. We have immediately that $v(x) \leq 1$ if $16 \leq x \leq 18$, and $v(y) \leq 2$ for $11 \leq y \leq 12$, $v(9) \leq 3$.

Assume $v(30) > 0$. Then $\{30, 29^4, 1^2\}$ implies $v(8, 28) \leq 3$; $\{30, 28^4, 1^6\}$ implies $v(9, 28) \leq 3$; $\{30, 26^3, 1^3\}$ implies $v(11, 26) \leq 2$; $\{30, 25^3, 1^6\}$ implies $v(12, 25) \leq 2$; $\{30, 22, 22\}$ implies $v(22) \leq 1$; $\{30, 21^2, 1^2\}$ implies $v(16, 21) \leq 1$; $\{30, 13^3, 1^5\}$ implies $v(13, 24) \leq 2$; $\{30, 20^2, 1^4\}$ implies $v(17, 20) \leq 1$; $\{30, 19^2, 1^6\}$ implies $v(18, 19) \leq 1$. The cardinalities of the boxes (1) (30) (8,29) (9,28) (10,27) (11,26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most $6 + 5 + 3 + 3 + 3 + 2 + 2 + 2 + 4 + 6 + 1 + 1 + 1 = 39$. So $v(30) \geq 3$, but then $\{30, 30, 14\}$ implies $v(14) = 0$ and $\{30^4, 23, 1^5\}$ is basic after multiplication by 5, so $v(30) = 3$. The cardinalities of the boxes is then at most 6+3+3+3+3+2+2+2+4+6+1+1+1 < 37, so $v(30) = 0$.

Assume $v(29) > 0$. Then $\{29, 28^4, 1^7\}$ implies $v(9, 28) \leq 3$; $\{29, 26^3, 1^4\}$ implies $v(11, 26) \leq 2$; $\{29, 22, 22, 1\}$ implies $v(15, 22) \leq 1$; $\{29, 21^2, 1^3\}$ implies $v(16, 21) \leq 1$; $\{29, 20^2, 1^5\}$ implies $v(17, 20) \leq 1$; $\{29, 13^3, 1^6\}$ implies $v(13, 24) \leq 2$; $\{29, 19^2, 1^7\}$ implies $v(18, 19) \leq 1$. The cardinalities of the boxes (1) (29) (9,28) (10,27) (11,26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most $6 + 4 + 3 + 3 + 2 + 6 + 4 + 6 + 1 + 1 + 1 < 37$. So $v(29) = 0$.

Assume $v(28) > 0$ so that $v(x) = 0$ for any $8 \leq x \leq 9$. Then $\{28, 26^3, 1^5\}]$ implies $v(11, 26) \leq 2$; $\{28, 22^2, 1^2\}$ implies $v(22) \leq 1$; $\{28, 15^3, 1\}$ implies $v(15, 22) \leq 2$; $\{28, 13^6, 1^5\}$ implies $v(13, 24) \leq 5$; $\{28, 21^2, 1^4\}$ implies $v(16, 21) \leq 1$; $\{28, 20^2, 1^6\}$ implies $v(17, 20) \leq 1$; $\{28, 19^6, 1^6\}$ implies $v(18, 19) \leq 5$; $\{28, 14^3, 1^4\}$ implies $v(14, 23) \leq 2$. The cardinalities of the boxes (1) (27) (11,26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most $6+3+2+6+5+2+2+1+1+5 < 37$. So $v(28) = 0$.

Assume $v(27) > 0$ so that $v(x) = 0$ for any $8 \leq x \leq 10$. Then $\{27, 26^3, 1^5\}$ implies $v(11, 26) \leq 2$; $\{27, 22^2, 1^3\}$ implies $v(22) \leq 1$; $\{27, 15^3, 1^2\}$ implies $v(15, 22) \leq 2$; $\{27, 21^2, 1^5\}$ implies $v(16, 21) \leq 1$; $\{27, 20^4, 1^4\}$ implies $v(17, 20) \leq 3$; $\{27, 14^3, 1^5\}$ implies $v(14, 23) \leq 2$; $\{27, 13^6, 1^6\}$ implies $v(13, 24) \leq 5$. The cardinalities of the boxes (1)(27) (11,26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most $6 + 3 + 2 + 6 + 5 + 2 + 2 + 1 + 3 + 6 < 37$, a contradiction so $v(27) = 0$.

Assume $v(26) > 0$ so that $v(x) = 0$ for any $8 \leq x \leq 11$. Then $\{26, 22^2, 1^4\}$ implies $v(22) \leq 1$; $\{26, 15^3, 1^3\}$ implies $v(15, 22) \leq 2$; $\{26, 20^4, 1^5\}$ implies $v(17, 20) \leq 3$; $\{26, 21^2, 1^6\}$ implies $v(16, 21) \leq 1$; $\{26, 14^3, 1^6\}$ implies $v(14, 23) \leq 2$. The cardinalities of the boxes (1) (26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most $6+6+6+6+2+2+1+3+6 = 38$. So $v(19), v(26) > 2$, a contradiction since $\{26, 26, 19, 1, 1, 1\}$ is basic. Hence $v(26) = 0$.

Assume $v(25) > 0$ so that $v(x) = 0$ for any $8 \leq x \leq 12$. Then $\{25, 15^3, 1^4\}$ implies $v(15) \leq 2$; $\{25, 21^4, 1^2\}$ implies $v(16, 21) \leq 3$; $\{25, 22^2, 1^5\}$ implies $v(15, 22) \leq 2$; $\{25, 20^4, 1^6\}$ implies $v(17, 20) \leq 3$; $\{25, 14^6, 1^2\}$ implies $v(14, 23) \leq 5$. The cardinalities of the boxes (1) (25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most $6 + 6 + 6 + 5 + 2 + 3 + 3 + 6 = 37$. So $v(19), v(25) > 2$ and $\{25^2, 19, 1^5\}$ is basic so $v(25) = 0$.

Assume $v(24) > 0$. Then $v(x) = 0$ for any $8 \le x \le 13$ and we partition into boxes (1) (24) (14,23) (15,22) (16,21) (17,20) (18,19) with cardinalities at most $6+2+6+6+4+6+6 < 37$, so $v(24) = 0$. If $v(23) > 0$ then $v(x) = 0$ for any $8 \le x \le 14$ and we partition into boxes (1) (23) (15,22) (16,21) (17,20) (18,19), a contradiction so $v(23) = 0$. If $v(22) > 0$ then $v(x) = 0$ for any $9 \le x \le 15$ and we partition into boxes (1) (8) (22) (16,21) (17,20) (18,19), a contradiction so $v(22) = 0$. If $v(21) > 0$ then $v(x) = 0$ for any $10 \le x \le 16$ and we partition into boxes (1) (8) (9) (21) (17,20) (18,19), a contradiction so $v(21) = 0$. If $v(20) > 0$ then $v(x) = 0$ for any $11 \le x \le 17$ and we partition into boxes (1) (8) (9) (10) (17,20) (18,19), a contradiction, so $v(20) = 0$. If $v(19) > 0$ then $v(x) = 0$ for any $12 \le x \le 18$ and we partition into boxes (1) (8) (9) (10) (11) (19), a contradiction so $v(19) = 0$.

Thus we have $1 \le x_i \le 18$ for every $x_i \in X$. We partition into boxes (1) (8,9) (10) (11,12) (13) (14) (15) (16,17,18). Note $v(16, 17, 18) \le 1$, $v(11, 12) \le 2$ and $v(8, 9) \le 3$. The maximum cardinalities of these boxes is thus $6+3+6+2+6+4+6+1 < 37$. Hence we can assume that $H = M = 5$.

If M=H=5, assume $v(1) = 5$. Then we assume there is no subsequence Y of X with $1 \notin Y$ such that $32 \le \sigma(Y) \le 37$ and $v(x) = 0$ for $2 \le x \le 6$. We partition the remaining elements into boxes (1) (31) (7,30) (8,29) (9,28) (10,27) (11,26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19), and note the cardinality of each of these boxes is at most 5. We have immediately that $v(x) \le 1$ if $16 \le x \le 18$, and $v(y) \le 2$ for $11 \le y \le 12$.

Assume $v(31) > 0$. Then $\{31, 29^4, 1\}$ implies $v(8,29) \leq 3$; $\{31, 28^4, 1^5\}$ implies $v(9,28) \leq 3$; $\{31, 10^4, 1^3\}$ implies $v(10,27) \leq 3$; $\{31, 26^3, 1^2\}$ implies $v(11,26) \leq 2$; $\{31, 25^3, 1^5\}$ implies $v(12,25) \leq 2$; $\{31, 21, 21, 1\}$ implies $v(16,21) \leq 1$; $\{31, 13^3, 1^4\}$ implies $v(13,24) \leq 2$; $\{31, 14^3, 1\}$ implies $v(14,23) \leq 2$; $\{31, 15^5, 1^5\}$ implies $v(15,22) \leq 4$; $\{31, 20^2, 1^3\}$ implies $v(17,20) \leq 1$; $\{31, 19^2, 1^5\}$ implies $v(18,19) \leq 1$.

The cardinalities of the boxes (1) (31) (7,30) (8,29) (9,28) (10,27) (11,26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most $5+5+5+3+3+3+2+2+2+2+4+1+1+1 = 39$. So $v(31) \geq 3$ and $v(7,30) \geq 3$. Note if $v(7) \geq 1$ then $\{31^2, 7, 1^5\}$ is basic, so assume $v(30) \geq 3$. Then $\{31^3, 30^3, 1^2\}$ is basic, a contradiction. Hence $v(31) = 0$.

Assume $v(30) > 0$. Then $\{30, 29^4, 1^2\}$ implies $v(8,29) \leq 3$; $\{30, 26^3, 1^3\}$ implies $v(11,26) \leq 2$; $\{30, 22, 22\}$ implies $v(15,22) \leq 1$; $\{30, 21^2, 1^2\}$ implies $v(16,21) \leq 1$; $\{30, 13^3, 1^5\}$ implies $v(13,24) \leq 2$; $\{30, 20^2, 1^4\}$ implies $v(17,20) \leq 1$; $\{30, 10^4, 1^4\}$ implies $v(10,27) \leq 3$; $\{30, 14^3, 1^2\}$ implies $v(14,23) \leq 2$; $\{30, 19^4, 1^5\}$ implies $v(18,19) \leq 3$.

The cardinalities of the boxes (1) (30) (8,29) (9,28) (10,27) (11,26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most $5+5+3+5+3+2+5+2+2+1+1+1+3 = 38$. So $v(30) \geq 4$ and $v(19) \geq 2$. But then $\{30^3, 19\}$ is a basic subsequence, hence $v(30) = 0$.

Assume $v(29) > 0$ so that $v(x) = 0$ for any $7 \leq x \leq 8$. Then $\{29, 26^3, 1^4\}$ implies $v(11,26) \leq 2$; $\{29, 15^3\}$ implies $v(15) \leq 2$; $\{29, 22, 22, 1\}$ implies $v(15,22) \leq 2$; $\{29, 21^2, 1^3\}$ implies $v(16,21) \leq 1$; $\{29, 20^2, 1^5\}$ implies $v(17,20) \leq 1$; $\{29, 10^4, 1^5\}$ implies $v(10,27) \leq 3$; $\{29, 14^3, 1^3\}$ implies $v(14,23) \leq 2$.

The cardinalities of the boxes (1) (29) (9,28) (10,27) (11,26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most 5+4+5+3+2+5+5+2+2+1+1+5 = 40. Then note if $v(19) \geq 1$ then $v(x) = 0$ for $21 \leq x \leq 26$ since $\{29, 19, x, 1, ..1, \}$ is basic after multiplication by 2. Also $v(y) = 0$ for $13 \leq y \leq 18$ since then we take $Y = \{19, y\}$. Hence $v(13, 24), v(14, 23)v(15, 22), v(16, 21) = 0$ a contradiction. So assume $v(19) = 0$. Thus the cardinalities of these boxes is then at most $5+4+5+3+2+5+5+2+1+2+1+1 < 37$, so $v(29) = 0$.

Assume $v(28) > 0$ so that $v(x) = 0$ for any $7 \leq x \leq 9$. Then $\{28, 26^3, 1^5\}$ implies $v(11, 26) \leq 2$; $\{28, 15^3, 1\}$ implies $v(15) \leq 2$; $\{28, 22^2, 1^2\}$ implies $v(15, 22) \leq 2$; $\{28, 21^2, 1^4\}$ implies $v(16, 21) \leq 1$; $\{28, 20^4, 1^3\}$ implies $v(17, 20) \leq 3$; $\{28, 14^3, 1^4\}$ implies $v(14, 23) \leq 2$. The cardinalities of the boxes (1) (27) (11,26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most 5+5+2+5+5+2+1+2+3+5 < 37. So $v(28) = 0$.

Assume $v(27) > 0$ so that $v(x) = 0$ for any $7 \leq x \leq 10$. Then $\{27, 26^3, 1^5\}$ implies $v(11, 26) \leq 2$; $\{27, 15^3, 1^2\}$ implies $v(15) \leq 2$; $\{27, 22^2, 1^3\}$ implies $v(15, 22) \leq 2$; $\{27, 21^2, 1^5\}$ implies $v(16, 21) \leq 1$; $\{27, 20^4, 1^4\}$ implies $v(17, 20) \leq 3$; $\{27, 14^3, 1^5\}$ implies $v(14, 23) \leq 2$. The cardinalities of the boxes (1) (27) (11,26) (12,25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most 5+3+2+5+5+2+2+1+3+5 < 37. So $v(27) = 0$.

Assume $v(26) > 0$ so that $v(x) = 0$ for any $7 \leq x \leq 11$. Then $\{26, 15^3, 1^3\}$ implies $v(15) \leq 2$; $\{26, 22^2, 1^4\}$ implies $v(15, 22) \leq 2$; $\{26, 20^4, 1^5\}$ implies $v(17, 20) \leq 3$; $\{26, 21^4, 1\}$ implies $v(16, 21) \leq 3$. The cardinalities of the boxes (1) (26) (25,12) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most $5+5+5+5+4+2+$

$3+3+5 = 37$, a contradiction since then $\{26^2, 19, 1^3\}$ is a basic subsequence. So $v(26) = 0$.

Assume $v(25) > 0$ so that $v(x) = 0$ for any $7 \leq x \leq 12$. Then $\{25, 21^4, 1^2\}$ implies $v(21) \leq 3$; $\{25, 22^2, 1^5\}$ implies $v(22) \leq 1$; $\{25, 15^3, 1^4\}$ implies $v(22, 15) \leq 2$. The cardinalities of the boxes (1) (25) (13,24) (14,23) (15,22) (16,21) (17,20) (18,19) is then at most $5+5+5+4+2+3+5+5 < 37$, so $v(25) = 0$.

Assume $v(24) > 0$. Then $v(x) = 0$ for any $8 \leq x \leq 13$ and we partition into boxes (1) (7) (24) (14,23) (15,22) (16,21) (17,20) (18,19) with cardinalities at most $5+4+2+4+5+5+5+5 < 37$, so $v(24) = 0$. Assume $v(23) > 0$. Then $v(x) = 0$ for any $9 \leq x \leq 14$, $v(7) \leq 1$, and we partition into boxes (1) (7) (8) (23) (15,22) (16,21) (17,20) (18,19) with cardinalities at most $5+1+3+2+5+5+5+5 < 37$, so $v(23) = 0$. If $v(22) > 0$ then $v(x) = 0$ for any $10 \leq x \leq 15$, $v(7) \leq 1$, and we partition into boxes (1) (7) (8) (9) (22) (16,21) (17,20) (18,19) with cardinalities at most $5+1+3+3+4+5+5 < 37$, so $v(22) = 0$. If $v(21) > 0$ then $v(x) = 0$ for any $11 \leq x \leq 16$, $v(7), v(8) \leq 1$ and we partition into boxes (1) (7) (8) (9) (10) (21) (17,20) (18,19) with cardinalities at most $5+1+1+3+5+4+5+5 < 37$, so $v(21) = 0$. If $v(20) > 0$ then $v(x) = 0$ for any $12 \leq x \leq 17$, $v(7), v(8) \leq 1$, and we partition into boxes (1) (7) (8) (9) (10) (11) (17,20) (18,19) with cardinalities at most $5+1+1+3+5+2+5+5 < 37$, so $v(20) = 0$. If $v(19) > 0$ then $v(x) = 0$ for any $13 \leq x \leq 18$, $v(7), v(8), v(9) \leq 1$, and we partition into boxes (7) (8) (9) (10) (11) (12) (19) with cardinalities at most $5+1+1+1+5+2+2+5 < 37$, so $v(19) = 0$.

Now $x_i < \frac{p}{2}$ for every $x_i \in X$, hence by Theorem 3.2.5 we have a basic zero sequence. Thus we can assume that $H, M \neq 5$, and in all cases we have satisfied the conjecture. Thus if $p = 37$ Conjecture 1.3.2 holds.

Chapter 4

Sequences of Length n in $\mathbb{Z}_n$

4.1 Introduction

In this chapter we more closely examine sequences of length n in $\mathbb{Z}_n$ for composite integers n. We initially set to verify the KL conjecture for all composite integers n less than or equal to 25. We restate Conjecture 1.3.4 in terms of some of the new terminology obtained.

Conjecture 4.1.1. *Every sequence of length n in $\mathbb{Z}_n$ has a basic zero-subsequence. Moreover, for every $d|n$, every sequence in $\mathbb{Z}_n$ of length n has a subsequence of length at most d equivalent to one with sum divisible by d and dividing n.*

Although the conjecture holds for the first 21 values of n, computational evidence eventually proved the conjecture to be false in the case of $n = 22$. That is, we are able to demonstrate a sequence in Z_{22} of length 22 which has no basic subsequence. We have now generalized this counterexample to prove the existence of a series of values of n that always fail the conjecture. In this chapter, we always assume that n is not prime since otherwise Conjecture 1.3.4 is an immediate consequence of the preceding chapter and Conjecture 1.3.2. Note that if $X \in MZS(\mathbb{Z}_n)$ is basic and $|X| \leq d$ then Conjecture 1.3.4 is satisfied, so if we assume $d = n$ we need to show that every sequence of length n in $\mathbb{Z}_n$ has a basic subsequence. If we assume $d \neq n$, the conjecture and its verification becomes sightly different than in the proceeding chapter. In fact, notice that certain subsequences which are not even zero-sequences may still satisfy the conjecture. However, the longest and most difficult case remains when $d = n$. Dealing with small values of d is generally easier, and section 4.3 provides some proven results which can handle some special cases of d values for general values of n. We begin this chapter by generalizing some notions we used for sequences in $\mathbb{Z}_p$ to the general case of $\mathbb{Z}_n$.

4.2 Heights in $\mathbb{Z}_n$

In Chapter 3 we defined the height of a sequence in $\mathbb{Z}_p$ to be the largest repetition value of any non-zero element in the sequence. In general, we define the height of a sequence X in $\mathbb{Z}_n$ denoted $H(X)$ to be the largest occurring multiplicity of any element in X relatively prime to n. The *divisor height* for a given divisor d of n denoted $H_d(X)$ is the largest occurring multiplicity of any element $x \in X$ with $gcd(x, n) = d$. For example, consider $X = \{1, 4, 4, 4, 5, 5, 8\}$ in $\mathbb{Z}_{10}$. Then $H(X) = 1$, $H_2(X) = 3$ and $H_5(X) = 2$. We can still assume that $v(1) = H(X)$ for any sequence just as we did in the previous chapter, however we generally need to consider more values for $H(X)$ when dealing with composite n. Notice that an arbitrary sequence in $\mathbb{Z}_n$ can have many integers coprime to n, or no integers coprime to n. In some situations, it can be useful if H_d is large to assume that $v(d) = H_d$, but if we do this we can no longer assume that $v(1) = H$, only that $v(1) \leq H$. We still consider the largest continuous interval $[1, M]$ in the sumset $\Sigma(X)$, which has the same definition as Chapter 3. Using this M value, we again try to establish a bound on the possible heights of sequences which may be counterexamples to Conjecture 1.3.4. Some easy results are still true when we expand the notion of M from $\mathbb{Z}_p$ to $\mathbb{Z}_n$.

Lemma 4.2.1. *Let X be a sequence in $\mathbb{Z}_n$ of length n. If $M \geq \frac{n}{2} - 1$ then X has a basic subsequence.*

Proof. Let $X = \{a_1, ..., a_s, b_1, ..., b_t\}$ such that $s+t = n$ and the sub-sumset $\Sigma(a_1, .., a_s) = [1, M]$. We have immediately that $b_i \geq \frac{n}{2} + 1$, otherwise $b_i \in [1, M]$. Then if $t \neq 0$, then $M \geq \frac{n}{2} - 1$ forces $b_i \neq x$ for any $\frac{n}{2} + 1 \leq x \leq n$ otherwise X has a basic subsequence. Hence $t = 0$, and so $s = n$ and $M \geq n$. □

If we isolate the case where n is an odd integer, then the proof of the following theorem is nearly identical to Theorem 3.2.8 as long as we replace p with n (and is therefore omitted). This is a useful result used often in the remainder of the computations.

Theorem 4.2.2. *Let $n \geq 11$ be odd. If X is a sequence in $\mathbb{Z}_n$ with $|X| = n$ and $M \geq \frac{n-2}{3}$, then X has a basic zero subsequence.*

The stipulation that n is odd is necessary since the proof relies on $\phi(x) = 2x$ being a valid automorphism. This is clearly not the case if n is even. In fact, when n is even it is not easy to determine a stronger bound on the height. This is mostly due to the fact that we cannot easily determine a small m that we know will be coprime to n. The following result improves on 4.2.1 slightly for even integers n.

Lemma 4.2.3. *Let X be a sequence in $\mathbb{Z}_n$ of length n. If $M \geq \frac{n}{2} - 2$ then X has a basic subsequence.*

Proof. In view of Theorem 4.2.2 and Lemma 4.2.1 we may assume n is even and $M = \frac{n}{2} - 2$. Then we can assume $v(x) = 0$ for each $\frac{n}{2} + 2 \leq x \leq n$. We can assume that $v(\frac{n}{2} - 1) = 0$ else $[1, M + 1]$ is a continuous interval. We assume $v(\frac{n}{2}) \leq 1$ and $v(\frac{n}{2} + 1) < \frac{n}{2}$ (otherwise there exists an automorphism ϕ such that $\phi(\frac{n}{2} + 1) = d$ for $d = gcd(\frac{n}{2}, n)$, and $d < \frac{n}{2}$), hence a non-basic sequence can have at most $M + \frac{n}{2} + 1 = n - 1 < n$ elements. □

Remark 4.2.4. *We note that the above result may be best possible. We later demonstrate a sequence of length n with $M = \frac{n}{2} - 3$ which has no basic subsequence for certain even values of n.*

The above results imply that we can restrict M (and therefore H) but only in the case when $d = n$. The results imply that we have a basic sequence, but if the length of this sequence is greater than d, we still have not satisfied Conjecture 1.3.4. However, we should note that for any value of d we can always assume that $M < d$ since otherwise we clearly have a sequence which sums to d and satisfies Conjecture 1.3.4. Since we will will soon demonstrate that we only need to handle cases where $d \leq \frac{n}{3}$ or $d = n$, this fact has little consequence here.

4.3 Automorphism Classes

Part of the benefit in dealing with sequences in $\mathbb{Z}_p$ is that we can effectively turn any element in a sequence into any other element after applying a suitable automorphism. This is not the case in $\mathbb{Z}_n$. In fact, as we will demonstrate, automorphisms in $\mathbb{Z}_n$ turn elements into other elements sharing the same *gcd* value with n. This is demonstrated by the following theorem and allows us to structure the elements on Z_n into *automorphism classes*, useful in verifying Conjecture 1.3.4. Our objective is to justify the following theorem, which we do by showing a series of results.

Theorem 4.3.1. *If X is a sequence in $\mathbb{Z}_n$, then for each pair of elements $x, y \in X$ with $gcd(x, n) = gcd(y, n) = d$, there exists an automorphism ϕ such that $\phi(x) = y$. Moreover, $\phi(x) \neq z$ for any automorphism ϕ where z is any element with $gcd(z, n) \neq d$.*

We note that if n is prime the theorem is obviously true, so we can assume n is not prime. We start by showing that an element x in $\mathbb{Z}_n$ can be transformed under automorphism into its greatest common divisor (d) with n. It is an easy consequence of this that an element can be transformed into any other element sharing the same greatest common divisor with n.

Lemma 4.3.2. *If $gcd(x, n) = d$ then there exists an m with $gcd(m, n) = 1$ such that $mod(mx, n) = d$.*

Proof. Let $d = p_1^{l_1}...p_k^{l_k}$ for some distinct primes p_i. Since $gcd(a, n) = d$, there exists an integer t such that $t\frac{a}{d} + s\frac{n}{d} = 1$ which implies that $gcd(t, \frac{n}{d}) = 1$. So we have that $ta \equiv d \ mod \ n$. Now if $gcd(t, n) = 1$ we are done. So assume $gcd(t, n) = d_1$. Then $d_1|d$. Let $x = \prod p_j$ for $1 \leq j \leq k$ with $p_j \nmid d_1$. Assume that $gcd(t + x\frac{n}{d}, n) = l$. We claim that $l = 1$. If $l \neq 1$, assume that $q|l$ for some prime q. Then $q \nmid \frac{n}{d}$, otherwise $q|t$ which contradicts $gcd(\frac{n}{d}, t) = 1$. So let $q = p_i$ for some $1 \leq i \leq k$. We have two cases, first if $p_i|d_1|t$. Then $p_i|x\frac{n}{d}$. Then since $gcd(p_i, x) = 1$ we know $p_i|\frac{n}{d}$, a contradiction. Second, if $p_i \nmid d_1$, then we must have $p_i|x$. So $p_i|t$ which implies $p_i|d_1 = gcd(t, n)$ a contradiction. So we must have $l = 1$. Now since $ax\frac{n}{d} = \frac{a}{d}xn \equiv 0 \ mod \ n$, we know that $ta \equiv (t + x\frac{n}{d})a \equiv d \ mod \ n$. So we take $m = t + x\frac{n}{d}$ and we are done.

□

Corollary 4.3.3. *If $gcd(x, n) = gcd(y, n) = d$ then there exists an m with $gcd(m, n) = 1$ such that $mod(mx, n) = y$.*

Proof. By the above Lemma, we find an m_1 and m_2 such that $mod(m_1x, n) = mod(m_2y, n) = d$. Then $mod(m_1m_2^{-1}x, n) = y$. □

The following property is important to note, and will complete the justification of Theorem 4.3.1:

Property 4.3.4. *If $gcd(x, n) = d$ then for every m with $gcd(m, n) = 1$ if $mod(mx, n) = k$ then $gcd(k, n) = d$. This is clear by the properties of the gcd. Recall that if m is any integer, then $gcd(x + mn, n) = gcd(x, n)$. Hence if $mod(mx, n) = mx - ln$ for some integer l then $gcd(mx - ln, n) = gcd(mx, n) = d$ since $gcd(m, n) = 1$.*

This property effectively says that transformation under an automorphism will not change the *gcd* of an element. Together with the above Lemma, this implies that we can separate the elements of $\mathbb{Z}_n$ sets of elements which have the same *gcd* and are closed under taking automorphisms. That is, we group the elements of $\mathbb{Z}_n$ into classes $\Gamma(d_i) = \{x \in \mathbb{Z}_n | gcd(x,n) = d_i\}$ which we call automorphism classes. Then, after multiplication by an m coprime to n, we know our resulting element will still be in the same class, and we can turn any element into any other element in the same class. The following observation is used as well, ensuring that the idea of partitioning into boxes of the form $(x, n-x)$ and partitioning into automorphism classes can be used in conjunction.

Lemma 4.3.5. *If $a+b=n$ then $gcd(a,n) = gcd(b,n)$.*

Proof. Note that $(a,n) = d$ implies that $(a, a+b) = d$ which implies that $(a,b) = d$. Then since $(a,b) = (a+b,b) = d$ we have $(b,n) = d$. □

4.4 Special Values of d

In this section we handle some special values of d for arbitrary n, so that we do not need to consider them in the next section. In particular if d is very small, we can prove that Conjecture 1.3.4 holds. We start with the following example:

Example 4.4.1. *Conjecture 1.3.4 holds for any value of n with $d = 1$. Given a sequence of length n in $\mathbb{Z}_n$, we require a single element in Z_n which is automorphic to an element divisible by 1 which divides n. Since every element x is automorphic to $gcd(x,n)$ by Theorem 4.3.1, this is clearly true for each x. Thus the conjecture holds for any n with $d = 1$.*

We now prove the conjecture true for any value of n for the special cases where $d = 2$ and $d = 3$.

Theorem 4.4.2. *Let X be a sequence of length n in $\mathbb{Z}_n$ for n even. Then there exists a subsequence of X of length at most 2, equivalent to a subsequence with sum divisible by 2 and dividing n.*

Proof. Assume to the contrary that X has length n and no such subsequence exists. We can assume that if $x \in X$ with $2|gcd(x,n) = d$ that $v(x) = 0$, else simply choose m so that $mod(mx,n) = d$ and we have a sequence of length one equivalent to one with sum divisible by 2 and dividing n. Note also for any $x \in \Gamma(d)$ with $2 \nmid d$ that we can assume $v(x) = 1$ else take m such that $mod(mx,n) = d$, so $\{mod(mx,n), mod(mx,n)\}$ is the desired sequence, with sum $2d$, which is divisible by 2 and divides n. Therefore, for each divisor d of n with $2|d$, we can assume that any element not in $\Gamma(d)$ appears at most once and $\Gamma(d)$ is empty, so X has length necessarily less than n, a contradiction. □

Theorem 4.4.3. *Let X be a sequence of length n in $\mathbb{Z}_n$ for n divisible by 3. Then there exists a subsequence of X with length at most 3, equivalent to a sequence with sum divisible by 3 and dividing n.*

Proof. Assume to the contrary that X has length n and no such subsequence exists. We again assume $v(x) = 0$ for any $x \in \Gamma(d)$ whenever $3|d$. We can also assume $v(x) \leq 2$ for any $x \in \Gamma(d)$ with $3 \nmid d$ else take m such that $mod(mx,n) = d$, then $\{mod(mx,n)^3\}$ has sum $3d$, which is divisible by 3 and divides n. So for each d with $3 \nmid d$ we can partition all the elements not in $\Gamma(d)$ into boxes of the form $(x, n-x)$. Since there exist some d for which $\Gamma(d)$ are empty, we have strictly less than $\frac{n}{2}$ boxes, each box with cardinality at most 2, hence $|X| < 2\frac{n}{2} = n$, a contradiction. □

We now introduce an interesting result, which will actually save us from having to verify Conjecture 1.3.4 for the instance where $d = \frac{n}{2}$, so long as we verify it for $n = d$.

Theorem 4.4.4. *Let $X = \{x_1, ..., x_k\}$ be a minimal zero-sequence in $\mathbb{Z}_n$ of length k for n even, with $\frac{n}{2} < k \leq n$ and $\sigma(X) = n$. Then $s \in \Sigma(X)$ for every $1 \leq s \leq n$. In other words, $M = n$.*

Proof. Note $1 \in X$, since otherwise clearly $\sigma(X) > n$. Now assume that $M < n$. Then let $Y = \{x_1, ..., x_t\}$ with $\sigma(Y) = M$ and $Z = X - Y$ so $|Z| = k - t$. Then clearly $1 \notin Z$ and $1 \in Y$ so that for each $z \in Z$ we know $|z| \geq M + 2$. Note Z is nonempty, and if $|Z| = 1$ then $|Y| \geq \frac{n}{2}$, a contradiction since then for $z \in Z$ we would have $z > \frac{n}{2} + 1$ and thus $\sigma(X) > n$. So assume $|Z| \geq 2$. Then $n = \sigma(X) = \sigma(Y) + \sigma(Z) \geq M + 2(M+2)$ which implies $M \leq \frac{n-4}{3}$. This implies that $|Y| \leq \frac{n-4}{3}$ and therefore that $|Z| \geq \frac{n}{2} + 1 - \frac{n-4}{3} = \frac{n+14}{6}$. Thus, if $M \geq 4$ we know each $z \in Z$ has $z \geq 6$ so $\sigma(Z) \geq 6 * \frac{n+14}{6} > n$, a contradiction. Clearly $M = 2$ implies we have at least $\frac{n}{2} - 1$ terms of size at least 4, a contradiction. Similarly if $M = 3$ we have at least $\frac{n}{2} - 2$ terms of size at least 5, a contradiction. So $M \geq 4$, and the theorem proved. □

Corollary 4.4.5. *If Conjecture 1.3.4 is true for $d = n$, then it is true for $d = \frac{n}{2}$.*

Proof. Assuming the conjecture is true for $d = n$, we know any sequence X of length n has a basic subsequence. If this subsequence has length less than or equal to $\frac{n}{2}$, then it satisfies Conjecture 1.3.4 for $d = \frac{n}{2}$. If it has length greater than $\frac{n}{2}$, then by the above theorem $\frac{n}{2} \in \Sigma(X)$, hence there exists a sequence of length at most d with sum d, and Conjecture 1.3.4 is again satisfied. □

4.5 Counter Example to KL Conjecture

In this section we prove that Conjecture 1.3.4 and therefore the original Conjecture 1.3.1 fails to hold for all n. The result comes as a result of being unable to verify the case $n = 22$ by hand. This discovery led to the observation that the sequence $\{1^8, 11, 12^{10}, 13^3\}$ had no basic subsequence in $\mathbb{Z}_{22}$. From this sequence we generalized from the case $n = 2 * 11$ to the case $n = 2 * p$ for any $p \geq 11$ by demonstrating a sequence of length $2p$ which exhibits no basic subsequence.

Theorem 4.5.1. *If $p \geq 11$ is prime, $n = 2p$, then the sequence $X = \{1^{p-3}, p, p + 1^{p-1}, (p+2)^3\}$ has no basic subsequence.*

Proof. Note that X has no subsequence with sum $2p$, so assume $1 < m < 2p$. We prove that there exists no m coprime to n such that mX has a subsequence with sum n. We distinguish four cases.

Case 1: Assume $m < \frac{p}{2}$. Then $mX = \{m^{p-3}, p, (p+m)^{p-1}, (p+2m)^3\}$. Then if Y is a subsequence of X with $\sigma(Y) = n$, note if $v_Y(p) = 1$ (we assume now that for any x, $v(x)$ means $v_Y(x)$) then $v(p+m), v(p+2m) = 0$ since then $\sigma(Y) > 2p = n$. Then if $v(m) = n_0$, we can assume $Y = \{m^{n_0}, p\}$ and therefore that $n_0 m + p = 2p$ which implies $n_0 m = p$. This implies that $m|p$, a contradiction since $(m,n) = 1$. So assume $v(p) = 0$. Likewise, if $v(p+m) > 0$ we can assume that $v(p+m) = 1$ and we can assume that $v(p+2m) = 0$, and if $v(m) = n_0$ we have $n_0 m + p + m = 2p$ which implies that $m(n_0+1) = p$, a contradiction since $m \nmid p$. If $v(p+2m) > 0$, we assume that $v(p+2m) = 1$ and $v(m) = n_0$, so we have $n_0 m + p + 2m = 2p$, and this again implies that $m|p$. Hence we can assume our sequence Y contains only the term m, a contradiction since then $n_0 m = 2p$ which implies $m|p$. Therefore, we can assume $m > \frac{p}{2}$.

Case 2: If $\frac{p}{2} < m < p$. Then $mX = \{m^{p-3}, p, (p+m)^{p-1}, (2m-p)^3\}$. Then let Y be a subsequence of X with $\sigma(Y) = n$. Note $v(m) \leq 4$ since if $v(m) \geq 4$ we have $4m > 2p$. Also note $v(2m-p) \leq 3$ by assumption.

Subcase 1: If $v(p) = 1$, we can assume $v(p+m) = 0$ and $v(m) \leq 1$ (since $(p+m)+m > n$). Let $v(m) = n_0$ with $0 \leq n_0 \leq 1$ and $v(2m-p) = n_1$ with $0 \leq n_1 \leq 3$.

Then assume $Y = \{m^{n_0}, p, (2m-p)^{n_1}\}$ so $\sigma(Y) = 2p$ implies $mn_0+p+(2m-p)n_1 = 2p$ which implies $mn_0 + (2m-p)n_1 = p \star$.

If $n_0 = 0$ then $\star$ implies $2n_1 m = p(1+n_1)$. Since $p \nmid 2n_1$ since $2n_1 \leq 6$, we must have $p|m$, a contradiction since $p > m$. So we can assume that $n_0 = 1$. Then $\star$ implies $m + (2m-p)n_1 = p \star\star$. So if $n_1 = 0$ we have $m = p$, a contradiction. If $n_1 = 1$ we have $m + (2m-p) = p$ which implies $3m = 2p$ a contradiction since $gcd(m,n) = 1$. If $n_1 = 2$ we have $m + 4m - 2p = p$ which implies $5m = 3p$ so $p|m$ or $p|5$, a contradiction. Lastly if $n_1 = 3$ we have $m + (2m-p)3 = p$ so $7m = 4p$, again a contradiction. So we can assume that $v(p) = 0$.

Subcase 2: If $v(p+m) > 0$ we know $v(m) = 0$ since otherwise $\sigma(Y) \geq (p+m)+m > n$. So we assume $Y = \{p+m, (2m-p)^{n_1}\}$ so that $p+m+n_1(2m-p) = 2p$ which implies $m + n_1(2m-p) = p$. Note that this equation is equivalent to $\star\star$ which has no solution from above, so we can assume that $v(p+m) = 0$.

Subcase 3: We can assume $Y = \{m^{n_0}, (2m-p)^{n_1}\}$ so that $n_0 m + n_1(2m-p) = 2p$. If we assume $n_1 = 0$ then $n_0 m = 2p$, a contradiction since $gcd(m, 2p) = 1$. If $n_1 = 1$ we have $n_0 m + 2m - p = 2p$ so $m(n_0+2) = 3p$. Then since $p \nmid n_0 + 2$ (since $n_0 \leq 4$) we must have $p|m$, a contradiction. If $n_1 = 2$ we have $n_0 m + 2(2m-p) = 2p$ so $m(n_0+4) = 4p$. Again, $p \nmid n_0 + 4$ implies $p|m$, a contradiction. Lastly, if $n_1 = 3$, we have $n_0 m + 3(2m-p) = 2p$ which implies $m(n_0+6) = 5p$. Since $p \nmid n_0 + 6$ we must have $p|m$, a contradiction. Therefore we find a contradiction in all cases, so we can assume $m > p$.

Case 3: If $p < m < \frac{3p}{2}$. Then $mX = \{m^{p-3}, p, (m-p)^{p-1}, (2m-p)^3\}$. Let Y be a subsequence of X with $\sigma(Y) = n$ and note $v(m) \leq 1$ and $v(2m-p) \leq 1$.

Now if $v(p) > 0$ we can assume that $v(m) = 0$, $v(2m-p) = 0$. So if $v(m-p) = n_2$, we have that $p + n_2(m-p) = 2p$. This implies that $n_2(m-p) = p$. So $n_2|p$, but $n_2 < p$ so we must have $n_2 = 1$. This implies $m - p = p$, a contradiction, so assume $v(p) = 0$.

If $v(m) > 0$ then assume $v(m) = 1$, hence $v(2m-p) = 0$. So if $v(m-p) = n_2$, we have that $m + n_2(m-p) = 2p$. This implies that $m(1+n_2) = p(2+n_2)$. Since we assume m is odd, this is a contradiction since p and m are both odd and $n_2 + 2$ and $n_2 + 1$ have opposite parity. Therefore we can assume $v(m) = 0$.

If $v(2m-p) > 0$ then assume $Y = \{(m-p)^{n_2}, (2m-p)\}$ and so $n_2(m-p)+2m-p = 2p$. This implies that $m(n_2+2) = p(n_2+1)$, a contradiction since p and m are both odd and n_2+2 and n_2+1 have opposite parity. Therefore we can assume our sequence is of the form $Y = \{(m-p)^{n_2}\}$ and then $n_2(m-p) = 2p$ so $n_2 m = p(n_2+2)$. This implies $p|m$ or $p|n_2$, a contradiction, so we can assume that $m > \frac{3p}{2}$.

Case 4: If $\frac{3p}{2} < m < 2p$. Then $mX = \{m^{p-3}, p, (m-p)^{p-1}, (2m-3p)^3\}$. Note $v(m) \leq 1$.

Subcase 1: If we assume $v(p) > 0$ we know $v(m) = 0$. If $v(2m-3p) = n_1$, $v(m-p) = n_2$ so that $Y = \{p, (m-p)^{n_2}, (2m-3p)^{n_1}\}$. Then assume $p+n_1(2m-3p)+n_2(m-p) = 2p$ which implies that $m(n_2+2n_1) = p(1+3n_1+n_2)$. Since $p \nmid m$ we must have $p|n_2+2n_1$, but $n_2+2n_1 \leq p-1+6 = p+5$. So we must have $n_2+2n_1 = p$. Then $m(n_2+2n_1) = p(1+3n_1+n_2)$ implies $m = (1+3n_1+n_2) = 1+3n_1+p-2n_1 = 1+n_1+p$. But $1+n_1+p \leq p+4$, a contradiction since $m > \frac{3p}{2}$ and $p \geq 11$. So assume $v(p) = 0$.

Subcase 2: If $v(m) > 0$, assume $v(m) = 1$, $v(2m-3p) = n_1$, $v(m-p) = n_2$, then

$m+n_1(2m-3p)+n_2(m-p)=2p$ which implies that $m(1+n_2+2n_1)=p(2+3n_1+n_2)$. So we can assume that $p|1+n_2+2n_1$. Again, since $1+n_2+2n_1<2p$ we assume $1+n_2+2n_1=p$. Therefore $m=2+3n_1+n_2=2+3n_1+p-1-2n_1=p+n_1+1$. Again this is a contradiction, so assume $v(m)=0$.

Subcase 3: Our sequence only has the terms $2m-3p$ and $m-p$. So assume that $n_1(2m-3p)+n_2(m-p)=2p$ which implies that $m(2n_1+n_2)=p(2+3n_1+n_2)$. So we can assume that $p|2n_1+n_2$. Again we assume that $2n_1+n_2=p$. This implies that $m=2+3n_1+n_2$. Note if $n_1\leq 2$ that $m\leq p+5$ a contradiction. So we can assume $n_1=3$, then $n_2=p-6$ and $m=p+5$, a contradiction. So in in all cases, the theorem is true. □

We note that the theorem implies that the sequence $\{1^8, 11, 12^{p-1}, 13^3\}$ has no basic subsequence in $\mathbb{Z}_{22}$ and is the smallest counterexample to the Kleitman and Lemke Conjecture. Interestingly, the above sequence was shown to in fact be the only sequence in $\mathbb{Z}_{22}$ of length 22 with no basic subsequence. The theorem also gives other sequences, such as $\{1^{10}, 13, 14^{12}, 15^3\}$ in $\mathbb{Z}_{26}$ and $\{1^{14}, 17, 18^{16}, 19^3\}$ in $\mathbb{Z}_{34}$ which can be verified to have no basic subsequence. We also remark that the result is likely true if we replace p with any odd integer in the statement of the theorem. This would imply that any integer n of the form $n=4k+1$ for sufficiently large k the conjecture fails to hold. Computational evidence suggests this is in fact the case, and Gao now reports to have a proof for this situation.

4.6 Verification of the KL Conjecture

In this section we try to get a better understanding of when the basic subsequence condition holds for composite values of n by proving that Conjecture 1.3.4 holds for all $n\leq 21$ and $n=24,25$. Many of the tactics used to verify Conjecture 1.3.2

in Chapter 3 are similarly applied here. We still follow the general procedure of presuming that a counterexample to the conjecture exists, and then eliminating the possibilities of elements occurring in such a sequence for a given M value until our sequence has length necessarily less than n, which gives us a contradiction. We still use the observation that if a sequence $Y = \{Z, 1^s\}$ is basic for some $s \leq M$ then we can assume $Z \not\subseteq X$. We still usually partition elements of $\mathbb{Z}_n$ in pairs which sum to n, but in addition we consider the partitioning of elements into their automorphism classes. We can take advantage of easy observations such as if $dk = n$, and $x \in \Gamma(d)$ then $v(x) < k$. Note this is the case since if $v(x) \geq k$ the sequence $\{x^k\}$ is equivalent to $\{d^k\}$ which has sum n. However with larger values of n, this bound can still leave many possible elements in $\Gamma(d)$. For example, in $\mathbb{Z}_{22}$, $\Gamma(2) = \{(2,20)(4,18)(6,16)(8,14)(10,12)\}$. So if we assume that each box has cardinality at most 11, that means we can only assume that $|\Gamma(2)| \leq 55$. The following theorem actually limits the size of these automorphism classes by a much better margin. Note that it is dependent on Conjecture 1.3.4 being verified for some smaller value of n.

Theorem 4.6.1. *Let $kd = n$. Then if every sequence of length of length k in $\mathbb{Z}_k$ has a basic subsequence, X is a sequence of length n in $\mathbb{Z}_n$, and $|\Gamma(d)| \geq k$ for any d, then X has a basic subsequence.*

Proof. Let $Y = \{x_1, x_2, ..., x_k\}$ be a set of k elements in $\Gamma(d)$. Then $x_i = sd$ for some integer $1 \leq s \leq k-1$. So define a map $\pi : \mathbb{Z}_n \to \mathbb{Z}_k$ such that $\pi(sd) = s$. Note π is invertible. Then we have a sequence of length k in $\mathbb{Z}_k$, which has a basic subsequence mZ with $\sigma(mZ) = k$ for some m coprime to k. Then if m is coprime to n, $\phi(x) = mx$ is an automorphism in $\mathbb{Z}_n$, and $\pi^{-1}(\phi(Z)) \in \mathbb{Z}_n$ and has sum $kd = n$, hence is basic. The problem is that this m may not coprime, and we handle such situations as follows:

Assume $gcd(m, n) = y$ for some y and let $d = \prod p_i^{l_i}$ for some primes p_i. Then let $x = \prod p_j$ such that $p_j | d$ but $p_j \nmid y$. We claim that $gcd(m + x\frac{n}{d}, n) = 1$. Otherwise, assume that $p|gcd(m + x\frac{n}{d}, n)$ for some prime p. If $p|\frac{n}{d}$ then $p|m$ since $p|n$. Hence, $p|gcd(m, \frac{n}{d})$ a contradiction, so assume that $p \nmid \frac{n}{d}$. Then, since $p|n$ we can assume that $p|d$. Then if $p|y$ we know that $p|m$ and hence $p|x\frac{n}{d}$. But $p|y$ implies that $p \nmid x$ so we must have $p|\frac{n}{d}$, a contradiction. If $p \nmid y$ then we can assume that $p|x$. Then $p|m$, which implies that $p|y$, a contradiction. Therefore $gcd(m + x\frac{n}{d}, n) = 1$. Now observe that for $a \in \Gamma(d)$ then $ax\frac{n}{d} = \frac{a}{d}xn \equiv 0 \bmod n$, so we know that $a(m + x\frac{n}{d}) \equiv am \bmod n$. Therefore we take $t = m + x\frac{n}{d}$ and then $\sigma(\pi^{-1}(tZ)) = kd = d$ and we are done. □

We now begin verifying Conjecture 1.3.4. Recall that we can always assume that $d \neq 1, 2, 3$ or $\frac{n}{2}$ by the previous section (so long as we can verify it for the case $d = n$). If $n = 4$ we need only consider the case where $d = n = 4$. Then since $(\mathbb{Z}_4, k)$ is a basic pair for any k by Theorem 2.2.1, any sequence X in $\mathbb{Z}_4$ will have a basic zero-subsequence satisfying Conjecture 1.3.4. The following examples shows how our results on M in section 4.2 imply the conjecture is true for small enough n.

Example 4.6.2. *If $n = 6$ we need only consider when $d = n = 6$. Assume that X is a sequence in $\mathbb{Z}_6$ of length 6 that contains no basic subsequence. We can partition to elements of X into 3 automorphism classes: $\Gamma(1) = \{(1, 5)\}$, $\Gamma(2) = \{(2, 4)\}$, and $\Gamma(3) = \{(3)\}$. Then from Theorem 4.6.1 we can assume that $|\Gamma(2)| \leq 2$ and $|\Gamma(3)| \leq 1$, hence we must have that $|\Gamma(1) \geq 3$. Thus we can assume that $H(X) \geq 3$. Then Lemma 4.2.3 tells us that X must have a basic subsequence, a contradiction. Thus Conjecture 1.3.4 holds for $n = 6$.*

Example 4.6.3. *If $n = 8$ we only need to consider when $d = n = 8$. Assume that X is a sequence in $\mathbb{Z}_8$ of length 8 with no basic subsequence. We partition the elements of X into classes $\Gamma(1) = \{(1, 7)(3, 5)\}$, $\Gamma(2) = \{(2, 6)\}$, and $\Gamma(4) = \{(4)\}$. Then $|\Gamma(4)| \leq 1$ and $|\Gamma(2)| \leq 3$ by Theorem 4.6.1, so can assume $|\Gamma(1)| \geq 4$ and then by the pigeonhole principle we can assume $H \geq 2$. However Lemma 4.2.3 implies that* *for any* $H \geq \frac{8}{2} - 2 = 2$ *that* X *has a basic subsequence, a contradiction, and we are done. So Conjecture 1.3.4 holds for* $n = 8$.

Example 4.6.4. *If* $n = 9$ *we only need to consider when* $d = n = 9$. *Again assume* X *is a sequence in* $\mathbb{Z}_9$ *of length 9 with no basic subsequence. We partition into classes* $\Gamma(1) = \{(1,8)(2,7)(4,5)\}$, $\Gamma(3) = \{(3,6)\}$. *Then we can assume* $|\Gamma(3)| \leq 2$ *which implies* $\Gamma(1) \geq 7$. *Then by the pigeonhole principle we must have that* $H \geq 3$, *but we can assume from Lemma 4.2.3 that* $H \leq \frac{9-2}{3} < 3$, *a contradiction. So Conjecture 1.3.4 holds for* $n = 9$.

The case $n = 10$ is not quite so simple, and more illustrative of the general process we use. We only need to consider when $d = n = 10$. We assume that X is a sequence in $\mathbb{Z}_{10}$ of length 10 with no basic subsequence and we partition elements into classes:

- $\Gamma(1) = \{(1,9)(3,7)\}$
- $\Gamma(2) = \{(2,8)(4,6)\}$
- $\Gamma(5) = \{(5)\}$

Then $|\Gamma(5)| \leq 1$ and $|\Gamma(2)| \leq 4$ from Theorem 4.6.1, which means that we can assume there are at least 5 elements in X from $\Gamma(1)$. Then by the pigeonhole principle we can assume that $3 \leq H \leq M \leq 4$. We always assume $v(1) \geq 3$ and take possible values of M case by case.

If M=4 then we can assume that $H = 4$ since we cannot have $M = H + 1$. Hence we assume $v(x) = 0$ for $2 \leq x \leq 5$ since otherwise $M > 4$. We can also assume $v(x) = 0$ for $6 \leq x \leq 10$ otherwise we have a subsequence with sum 10. This is a contradiction so we can assume $M \neq 4$. **If M=3** then we can assume $H = 3$ so that $v(x) = 0$ for $2 \leq x \leq 4$ and $7 \leq x \leq 10$. Thus we can partition the remaining elements into boxes $(1)(5)(6)$, but these boxes have cardinalities at most $3+1+4 < 10$, a contradiction. Therefore Conjecture 1.3.4 holds when $n = 10$. We now set to verify the larger values of n that have previously been unverified.

4.6.1 Case n=12

We need to consider when $d = 4$ or when $d = 12$. We assume X to be a sequence of length 12 in $\mathbb{Z}_{12}$ with properties contradictory to Conjecture 1.3.4. That is, if $d = 4$ we assume X has no sequence of length at most 4 equivalent to one with sum 4 or 12. If $d = 12$ we assume that X has no basic subsequence. We then partition elements of X into automorphism classes:

- $\Gamma(1) = \{(1, 11)(5, 7)\}$
- $\Gamma(2) = \{(2, 10)\}$
- $\Gamma(3) = \{(3, 9)\}$
- $\Gamma(4) = \{(4, 8)\}$
- $\Gamma(6) = \{6\}$

If $d = 4$ then we can assume that $|\Gamma(1)| \leq 6$ otherwise $H \geq 4$ and there is a subsequence equivalent to $\{1^4\}$. Likewise we can assume that $|\Gamma(2)| \leq 1$, $|\Gamma(3)| \leq 3$, $|\Gamma(4)| = 0$ and $|\Gamma(6)| \leq 1$. The sum of the cardinalities of these classes in X less than 12, a contradiction, hence the conjecture holds when $n = 12$ and $d = 4$.

If $d = 12$, we can assume $|\Gamma(2)| \leq 5$, $|\Gamma(3)| \leq 3$, $|\Gamma(4)| \leq 2$, $|\Gamma(6)| \leq 1$. Therefore we can assume that $|\Gamma(1) > 0$ and so by the pigeonhole-principle and Lemma 4.2.3 we can assume $1 \leq H \leq M \leq 3$.

If M=3 let $\alpha = (1,..,4)$ be a box in X. We can assume that $v(\alpha) \leq 3$ and that $v(x) = 0$ for $9 \leq x \leq 12$. We can then partition into boxes (α) (8) (5,7) (6) with cardinalities at most 3+2+3+1 < 12, a contradiction. So we can assume $1 \leq H \leq M \leq 2$.

If M=2 then we can assume that $H = 2$. Hence $v(1) = 2$ and $v(x) = 0$ for $10 \leq x \leq 12$ and $2 \leq x \leq 3$. We then partition into boxes (1) (4,8) (5,7) (6) (9). The sum of the cardinalities of these boxes is then at most $2+2+2+1+3 < 12$, a contradiction, so assume $H = M = 1$.

If M=1, then we can assume that $H = 1$ so that $v(1) = 1$ and $v(x) = 0$ for $11 \leq x \leq 12$ and $x = 2$. We partition into boxes (1) (3,9) (4,8) (5,7) (6) (10) with cardinalities at most $1+3+2+1+1+5 = 13$. So we can assume that $v(10) \geq 4$. Then since $\{10^2, 3, 1\}$ and $\{10^3, 9^2\}$ are basic sequences, we must have $v(3) = 0$ and $v(9) \leq 1$, hence $v(3,9) \leq 1$. The sum of the cardinalities of our boxes is then at most $1+1+2+1+1+5 < 12$, a contradiction. So we can assume that $M, H \neq 1$ and thus in all cases Conjecture 1.3.4 holds for $n = 12$.

4.6.2 Case n=14

We need only consider when $d = 14$. Let X be a sequence in $\mathbb{Z}_{14}$ of length 14 with no basic subsequence. We partition the elements of X into classes:

- $\Gamma(1) = \{(1,13)(3,11)(5,9)\}$
- $\Gamma(2) = \{(2,12)(4,10)(6,8)\}$
- $\Gamma(7) = \{(7)\}$

Since $|\Gamma(2)| \leq 6$ and $|\Gamma(7)| \leq 1$ we can assume that $|\Gamma(1)| \geq 7$. Therefore we can assume $3 \leq H \leq M \leq 4$. **If M=4** we can assume that $H = 4$, so $v(1) = 4$, and

$v(x) = 0$ for $2 \leq x \leq 5$ and $10 \leq x \leq 14$. We partition the remaining elements into boxes (1) (6,8) (9) (7). Note $\{9^3, 1\}$ is basic so $v(9) \leq 2$. Then the cardinalities of these boxes is then at most $4 + 6 + 2 + 1 < 14$, a contradiction, so $M \neq 4$.

If M=3 we can assume that $H = 3$. So $v(1) = 3$ and $v(x) = 0$ for $2 \leq x \leq 4$ and $11 \leq x \leq 14$. So we partition into boxes (1) (6,8) (5,9) (10) (7) with cardinalities at most $3 + 6 + 3 + 6 + 1 = 19$. Then since $v(1), v(5,9) \leq 3$ and $v(7) \leq 1$, we must have that $v(6,8,10) \geq 7$, a contradiction since Theorem 4.6.1 implies $|\Gamma(2)| \leq 6$. So $M \neq 3$ and Conjecture 1.3.4 holds for $n = 14$.

4.6.3 Case n=15

We need to consider when $d = 5$ or when $d = 15$. Assume X is a sequence in $\mathbb{Z}_{15}$ of length 15 with properties contradictory to Conjecture 1.3.2. We partition the elements of X into classes:

- $\Gamma(1) = \{(1, 14)(2, 13)(4, 11)(7, 8)\}$
- $\Gamma(3) = \{(3, 12)(6, 9)\}$
- $\Gamma(5) = \{(5, 10)\}$

If $d = 5$ we can assume that $|\Gamma(5)| = 0$. Then since $|\Gamma(3)| \leq 4$ we can assume that $3 \leq H \leq M \leq 4$. If $M = 4$ then assume $H = 4$, so we have no subsequence Y in X such that $1 \notin Y$, $11 \leq \sigma(Y) \leq 15$ and $v(x) = 0$ for $2 \leq x \leq 5$. So we can partition into boxes (1) (7,8) (6,9) with cardinalities at most $4 + 4 + 4 < 15$, a contradiction. If $M = 3$ we can assume that $H = 3$, so we have no subsequence Y of X such that with $1 \notin Y$ such that $12 \leq \sigma(Y) \leq 15$ and $v(x) = 0$ for $2 \leq x \leq 4$. So we can partition into boxes (1) (7,8) (6,9) (11) with cardinalities at most $3 + 3 + 4 + 3 < 15$, a contradiction, so assume $d \neq 5$.

If $d = 15$, since $|\Gamma(3)| \leq 4$ and $|\Gamma(5)| \leq 2$ we can assume that $3 \leq H \leq M \leq 4$. **If M=4** then we can assume that $H = 4$ so we have no subsequence $1 \notin Y$ with $11 \leq \sigma(Y) \leq 15$ and $v(x) = 0$ for $2 \leq x \leq 5$. So we can partition into boxes $(1)(10)(7,8)(6,9)$ with cardinalities at most 4+2+4+4 < 15, a contradiction. **If M=3** we can assume that $H = 3$, so we have no subsequence $1 \notin Y$ with $12 \leq \sigma(Y) \leq 15$ and $v(x) = 0$ for $2 \leq x \leq 4$. So we can partition into boxes (1) (5,10) (7,8) (6,9) (11). Now $\{6^2, 1^3\}$ and $\{9^3, 1^3\}$ are basic sequences so we can assume that $v(6,9) \leq 2$. The cardinalities of these boxes is then at most $3+2+3+2+3 < 15$, a contradiction, so Conjecture 1.3.4 holds for $n = 15$.

4.6.4 Case n=16

We only need to consider when $d = 4, 16$. We assume X is a sequence in $\mathbb{Z}_{16}$ of length 16 contradicting Conjecture 1.3.4. We partition the elements of X into classes:

- $\Gamma(1) = \{(1,15)(3,13)(5,11)(7,9)\}$
- $\Gamma(2) = \{(2,14)(6,10)\}$
- $\Gamma(4) = \{(4,12)\}$
- $\Gamma(8) = \{8\}$

We note if $d = 4$ then we can assume that $|\Gamma(1)| \leq 12$, $|\Gamma(2)| \leq 1$, $|\Gamma(4)| = 0$, $|\Gamma(8)| = 0$. The sum of the cardinalities of these classes is then less than 16, a contradiction, hence the conjecture is true for $d = 4$.

If $d = 16$ then we have $|\Gamma(2)| \leq 7$, $|\Gamma(4)| \leq 3$, $|\Gamma(8)| \leq 1$, so we can assume that $2 \leq H \leq M \leq 5$. We note that since $\{10^3, 1^2\}$ is basic we can always assume that $v(10) \leq 2$. **If M=5** let $\alpha = (1,..,6)$ be a box in X. We can assume no sequence Y in

$X-\alpha$ with $11 \leq \sigma(Y) \leq 16$ and we partition into boxes (α) (7,9) (8) (10) with cardinalities at most 5+5+1+2 < 16, a contradiction. So we can assume $2 \leq H \leq M \leq 4$.

If M=4 let $\alpha = (1,..,5)$ be a box in X. We can assume no sequence Y in $X-\alpha$ with $12 \leq \sigma(Y) \leq 16$ and we partition into boxes (α) (7,9) (8) (6,10) (11) with cardinalities at most $4+4+1+2+4 < 16$ (note $v(6) \leq 1$). This is a contradiction, so we can assume $2 \leq H \leq M \leq 3$.

If M=3 we can assume that $H = 3$, so assume X has no subsequence Y with $1 \notin Y$ and $13 \leq \sigma(Y) \leq 16$. We also assume that $v(x) = 0$ for $2 \leq x \leq 4$. So we can partition into boxes (1) (7,9) (8) (6,10) (5,11) (12) with cardinalities at most $3+3+1+7+3+3 = 20$. If $v(10) > 0$ then note that $v(6,10) \leq 2$ which gives us a contradiction. So we can assume that $v(10) = 0$ and thus that $v(6) \geq 2$. Then $\{6,12,12,1,1\}$ implies $v(12) \leq 1$, and $\{6^5, 1^2\}$ implies $v(6) \leq 4$, so the cardinalities of the boxes (1) (7,9) (8) (6) (5,11) (12) is then at most $3+3+1+4+3+1 < 16$. This is a contradiction, so we can assume $H = M = 2$.

If H=M=2 we can assume X has no subsequence Y with $1 \notin Y$ such that $14 \leq \sigma(Y) \leq 16$, and $v(2), v(3) = 0$. So we can partition into boxes (1) (7,9) (8) (6,10) (5,11) (4,12) (13) with cardinalities at most $2+2+1+7+2+3+2 = 19$. Again, if $v(10) > 0$ then $v(6,10) \leq 2$, a contradiction. So we can assume that $v(10) = 0$ and that $v(6) \geq 3$. Then $\{6,6,4\}$ and $\{6^3, 12, 1^2\}$ imply $v(4,12) = 0$ and as above $\{6^5, 1^2\}$ means we can assume $v(6) \leq 4$. Therefore the cardinalities of the boxes (1) (7,9) (8) (6) (5,11) (13) is at most $2+2+1+4+2+2 < 16$, so we cannot have $M = H = 2$. Hence Conjecture 1.3.4 holds for $n = 16$.

4.6.5 Case n=18

We must verify the conjecture is true for $d = 6, 18$. Let X be a sequence of length 18 in $\mathbb{Z}_{18}$ contradictory to Conjecture 1.3.4. We partition the elements of X into classes:

- $\Gamma(1) = \{(1,17)(5,13)(7,11)\}$
- $\Gamma(2) = \{(2,16)(4,14)(8,10)\}$
- $\Gamma(3) = \{(3,15)\}$
- $\Gamma(6) = \{(6,12)\}$
- $\Gamma(9) = \{(9)\}$

If $d = 6$ then $|\Gamma(6)| = 0$, $|\Gamma(2)| \leq 6$, $|\Gamma(3)| \leq 1$, $|\Gamma(9)| \leq 1$, so we can assume that $4 \leq H \leq M \leq 5$. If $M = 5$ then $H = 5$ so $v(x) = 0$ for $2 \leq x \leq 6$ and $13 \leq x \leq 18$. We partition into boxes (1) (7,11) (8,10) (9), and the sum of the cardinalities of the boxes is then less than 18. If $M = 4$, again $H = 4$ so $v(x) = 0$ for $2 \leq x \leq 5$ and $14 \leq x \leq 18$. We partition into boxes (1) (7,11) (8,10) (9) (13), and the sum of the cardinalities of the boxes is then less than 18. So assume $d \neq 6$.

If $d = 18$ then we can assume $1 \leq H \leq M \leq 6$. **If M=6** let $\alpha = (1,..,7)$ be a box in X. We assume no sequence Y in $X - \alpha$ with $12 \leq \sigma(Y) \leq 18$ and $v(\alpha) \leq 6$. So we can partition into boxes (α) (8,10) (9) (11). Then since $\{11^3, 1^3\}$ is basic we can assume $v(11) \leq 2$, so our boxes have cardinalities at most $6 + 8 + 1 + 2 < 18$, a contradiction. Hence we can always assume that $1 \leq H \leq M \leq 5$.

If M=5 let $\alpha = (1,..,6)$ be a box in X. We assume no sequence Y in $X - \alpha$ with $13 \leq \sigma(Y) \leq 18$ and we partition into boxes (α) (12) (7,11) (8,10) (9). Since $v(7) \leq 1$

and $v(11) \leq 2$ from above we know $v(7,11) \leq 2$. So our boxes have cardinalities at most $5+2+2+8+1 = 18$. Then if $v(11) \neq 2$, $v(7,11) < 2$ a contradiction, so assume $v(11) = 2$. Then $\{12^2, 11, 1\}$ is basic, so assume $v(12) < 2$, a contradiction. So we can always assume that $1 \leq H \leq M \leq 4$.

If M=4 let $\alpha = (1,..,5)$ be a box in X. We can assume no sequence Y in $X - \alpha$ with $14 \leq \sigma(Y) \leq 18$ and we can partition into boxes (α) (6,12) (7,11) (8,10) (9) (13) with cardinalities at most $4+2+4+8+1+4 = 23$. Since $v(8) \leq 1$, if $v(10) < 3$ then the sum of the cardinalities of the boxes is less than 18, so assume $v(10) \geq 3$. Then we can assume $v(x) = 0$ for $6 \leq x \leq 8$, $v(11) \leq 2$ from above, and $\{10, 13, 13\}$ implies $v(13) \leq 1$. Also $\{10^3, 12^2\}$ implies $v(12) \leq 1$, and the cardinalities of boxes (α) (12) (11) (10) (9) (13) is at most $4+1+2+8+1+1 < 18$, a contradiction. So we can assume $1 \leq H \leq M \leq 3$.

If M=3 let $\alpha = (1,..,4)$ be a box in X. We can assume no sequence Y in $X - \alpha$ with $15 \leq \sigma(Y) \leq 18$, and we partition into boxes (α) (14) (5,13) (6,12) (7,11) (8,10) (9) with cardinalities at most $3+8+3+2+3+8+1 = 28$. Then if $v(14) > 0$, $\{14, 11, 11\}$ implies $v(11) \leq 1$, $\{14, 10^4\}$ implies $v(8,10) \leq 3$. $\{14, 13^3, 1\}$ is basic so $v(5,13) \leq 2$. So we have boxes (α) (14) (5,13) (6,12) (7,11) (8,10) (9) with cardinalities at most $3+8+2+2+3+3+1 = 22$. Then we can assume that $v(14) \geq 4$. Then $\{14^2, 7, 1\}$ implies $v(7,11) \leq 1$. Also $\{14^3, 10^3,\}$ implies $v(8,10) \leq 2$ and $\{14^3, 12\}$ implies $v(12) = 0$. So our boxes are (α) (14) (5,13) (6) (7,11) (8,10) (9) with cardinalities at most $3+8+2+2+1+2+1 = 19$, which implies that $v(6) > 0$, so we can assume $v(10) = 0$, and therefore $v(7,11) = 0$. Lastly, $\{5^2, 6, 1^2\}$ and $\{13^2, 14^2\}$ imply that $v(5,13) \leq 1$, a contradiction. So assume $v(14) = 0$. So we can partition into boxes (α) (5,13) (6,12) (7,11) (8,10) (9) with cardinalities at most $3+3+2+3+8+1 = 20$. Then we can assume that $v(10) \geq 6$ else a contradiction, so $\{10, 6, 1^2\}$ and $\{10^6, 12\}$ imply that $v(6, 12) = 0$. This implies that $v(5, 13) = 3$, a contradiction since $\{10, 5, 1, 1, 1\}$, $\{10, 5, 1, 2\}$ or $\{13^2, 10\}$ must be a subsequence. Hence we can assume that $1 \leq H \leq M \leq 2$.

If M=2 we can also assume $H = 2$, so we have no sequence Y with $16 \leq \sigma(Y) \leq 18$ and $v(2, 3) = 0$. So we can partition into boxes (1) (15) (4,14) (5,13) (6,12) (7,11) (8,10) (9) with cardinalities at most 2+5+8+2+2+2+8+1 = 30. Then if $v(15) > 1$, $\{15, 10^2, 1\}$ implies $v(8, 10) \leq 1$. Also $\{15^2, 14^3\}$, $\{15^2, 4, 1^2\}$ imply $v(4, 14) \leq 2$. So we have boxes (1) (15) (4,14) (5,13) (6,12) (7,11) (8,10) (9) with cardinalities at most $2 + 5 + 2 + 2 + 2 + 2 + 1 + 1 < 18$, so assume $v(15) \leq 1$. So we can partition into boxes (1) (15) (4,14) (5,13) (6,12) (7,11) (8,10) (9) with cardinalities at most $2 + 1 + 8 + 2 + 2 + 2 + 8 + 1 = 26$. Then if $v(10) > 0$, $v(7) = 0$ and $\{7, 14^2, 1\}$, $\{10, 4^2\}$ implies $v(4, 14) \leq 1$. So we can assume $v(10) \geq 7$ we also have $\{10^6, 12\}$ and $\{10^3, 6\}$ which imply $v(6, 12) = 0$, a contradiction. So assume $v(10) = 0$. So we can partition into boxes (1) (15) (4,14) (5,13) (6,12) (7,11) (8) (9) with cardinalities at most $2 + 1 + 8 + 2 + 2 + 2 + 1 + 1 = 19$. So we can assume $v(14) \geq 7$, and $\{14^3, 12\}$ implies $v(12) = 0$ and so $v(6) > 0$. But then $\{14^6, 6\}$ is a basic sequence, a contradiction. So we can assume that $H, M \neq 2$.

If M=1 assume $H = 1$ note since $|\Gamma(1)| \leq 3$, $|\Gamma(3)| \leq 5$, $|\Gamma(6)| \leq 2$ and $|\Gamma(9)| \leq 1$ that we can assume that $7 \leq |\Gamma(2)| \leq 8$. This implies that $3 \leq H_2 \leq 8$ so we can assume our sequence has $v(2) \geq 3$. Hence assume $v(12) = 0$ and $v(6) \leq 1$, so $|\Gamma(6)| \leq 1$. So our classes have cardinality sums at most $3 + 8 + 5 + 1 + 1 = 18$. Then we can assume that $|\Gamma(2)| = 8$ else a contradiction, so we can assume $v(2) = 8$ otherwise we have a basic sequence, but then $|\Gamma(6)| = 0$, a contradiction. So we cannot have $M = H = 1$, and in all cases the conjecture is true for $n = 18$.

4.6.6 Case n=20

We need to verify the conjecture for $d = 4, 5, 20$. Let X be a sequence of length 20 in $\mathbb{Z}_{20}$ contradictory to Conjecture 1.3.2. We partition into classes:

- $\Gamma(1) = \{(1, 19)(3, 17)(7, 13)(9, 11)\}$
- $\Gamma(2) = \{(2, 18)(6, 14)\}$
- $\Gamma(4) = \{(4, 16)(8, 12)\}$
- $\Gamma(5) = \{(5, 15)\}$
- $\Gamma(10) = \{(10)\}$

If $d = 4$, we can assume that $|\Gamma(2)| \leq 2$, $|\Gamma(4)| = 0$, $|\Gamma(5)| \leq 3$ and $|\Gamma(10)| \leq 1$. Hence by the pigeonhole principle we can assume that $H \geq 4$, a contradiction. So we can assume that $d \neq 4$.

If $d = 5$, we can assume that $|\Gamma(2)| \leq 4$ (note we can assume no sequence Y with $\sigma(Y) = 10$), $|\Gamma(4)| \leq 4$, $|\Gamma(5)| = 0$ and $|\Gamma(10)| = 0$. So we can assume that $3 \leq H \leq 4$ and $|\Gamma(1)| \geq 12$. If $H = 4$ then $v(x) = 0$ for $2 \leq x \leq 10$ and $15 \leq x \leq 20$. We are left with boxes (1) (11) (12) (13) (14) with cardinalities at most $4 + 4 + 4 + 4 + 4 = 20$. Hence $\{1, 14\}$ is a subsequence equivalent to $\{3, 2\}$ after multiplication by 3. If $H = 3$ then $v(x) = 0$ for $2 \leq x \leq 5$, $7 \leq x \leq 10$, and $17 \leq x \leq 20$. We are left with boxes (1) (11) (13) (12,16) (6,14) with cardinalities at most $3 + 3 + 3 + 4 + 4 < 20$, so we can assume $d \neq 5$.

If $d = 20$, we can assume that $1 \leq H \leq M \leq 7$. **If** $M = 7$ let $\alpha = (1, .., 8)$ be a box in X. Assume there is no sequence Y in $X - \alpha$ with $13 \leq \sigma(Y) \leq 20$ and partition into boxes (α) (12) (9,11) (10) with cardinalities at most $7+4+7+1 < 20$, a contradiction. Thus we can assume that $1 \leq H \leq M \leq 6$.

If M=6 let $\alpha = (1,..,7)$ be a box in X. Then we can assume there is no sequence Y in $X - \alpha$ with $14 \leq \sigma(Y) \leq 20$ and partition into boxes (α) (13) (8,12) (9,11) (10), and note that $\{13^3, 1\}$ is basic which implies $v(13) \leq 2$ whenever $M > 0$. So our boxes have cardinalities at most $6 + 2 + 4 + 6 + 1 < 20$, a contradiction, so we can assume that $1 \leq H \leq M \leq 5$.

If M=5 let $\alpha = (1,..,6)$ be a box in X. Assume there is no sequence Y in $X - \alpha$ with $15 \leq \sigma(Y) \leq 20$ and partition into boxes (α) (14) (7,13) (8,12) (9,11) (10). Note $\{14^4, 1^4\}$ is basic so $v(14) \leq 3$ if $M \geq 4$ so our boxes have cardinalities at most $5 + 3 + 5 + 4 + 5 + 1 = 23$. Then we can assume $v(11) \geq 2$, and thus that $v(7) = v(8) = v(9) = 0$. So since $\{14^2, 11, 1\}$ is basic, we can assume $v(14) \leq 1$. Then we have boxes (α) (14) (13) (12) (11) (10) with cardinalities $5+1+5+4+5+1 = 21$. Note if $v(14) > 0$ then $v(13) \leq 1$ since $\{14, 13, 13\}$ is basic. So assume $v(14) = 0$. Then $\{13^3, 11, 10\}$ is a basic subsequence, so we can assume that $1 \leq H \leq M \leq 4$.

If M=4 let $\alpha = (1,..,5)$ be a box in X. Assume no sequence Y in $X - \alpha$ with $16 \leq \sigma(Y) \leq 20$, and partition into boxes (α) (15) (6,14) (7,13) (8,12) (9,11) (10) which have cardinalities at most $4+3+3+4+4+4+1 = 23$. Note if $v(6) > 0$ then $v(9, 11) = 0$ and $v(10) = 0$, a contradiction. So if $v(14) > 1$ then $\{14^2, 11, 1\}$ and $\{14, 14, 12\}$ imply $v(11) = v(12) = 0$, and $v(8), v(9) \leq 1$, a contradiction. So assume $v(14) \leq 1$. If $v(13) \geq 3$ we get a contradiction since $\{13^3, 1\}$ is basic, so assume $v(7) \geq 3$. Then $v(9, 11) = 0$, a contradiction, so assume that $1 \leq H \leq M \leq 3$.

If M=3 let $\alpha = (1,..,4)$ be a box in X. Assume there is no sequence Y in $X - \alpha$ with $17 \leq \sigma(Y) \leq 20$. Note that $\{14^7, 1, 1\}$ is basic so assume $v(14) \leq 6$. We partition into boxes (α) (16) (5,15) (6,14) (7,13) (8,12) (9,11) (10) which have cardinalities at most $3+4+3+6+3+4+3+1=27$. Note if $v(6)>0$ then assume $v(x)=0$ for $11 \leq x \leq 14$ and $v(7), v(9) \leq 1$. We then have boxes (α) (16) (5,15) (6) (7) (8) (9) (10) with sum at most $3+4+3+2+1+4+1+1<20$, so assume $v(6)=0$. If $v(7)>0$ then $v(x)=0$ for $10 \leq x \leq 13$, and $\{16^2, 7, 1\}$ implies that $v(16) \leq 1$. Then since $\{15^2, 7, 1^3\}$ is basic we can assume $v(5,15) \leq 1$ and we therefore have boxes (α) (16) (5,15) (14) (7) (8) (9) (10) with sums at most $3+1+1+6+3+4+1<20$ so $v(7)=0$. Now if $v(14)>1$ then $v(13) \leq 1$, $v(11), v(12)=0$ from above. Also $\{14^2, 16^2\}$ is basic so $v(16) \leq 1$ so we have boxes (α) (16) (15) (14) (13) (8) (9) (10) with sum at most $3+1+3+6+1+3+1+1<20$, so $v(14)=0$. We now have boxes (α) (16) (5,15) (13) (8,12) (9,11) (10) with sum at most $3+4+3+3+4+3+1=21$. Then $\{16,16,8\}$ or $\{12,12,16\}$ must be a subsequence, so assume $1 \leq H \leq M \leq 2$.

If M=2 we can assume $H=2$ so there is no sequence $1 \notin Y$ with $18 \leq \sigma(Y) \leq 20$, and $v(2,3)=0$. We partition into boxes (1) (17) (5,15) (6,14) (7,13) (4,8,12,16) (9,11) (10), with cardinalities at most $2+2+3+6+2+4+2+1=22$. We group $\Gamma(4)$ in a single box. Then since $v(6) \leq 2$ we must have $v(14) \geq 4$. This is a contradiction since $\{14,5,1\}$ and $\{14^2, 10, 1^2\}$ imply $v(5)=v(10)=0$, so $v(15) \geq 2$ and $\{15^2, 14^2, 1^2\}$ is a basic subsequence. So assume $H=M=1$.

If H=M=1 we know $|\Gamma(1)| \leq 4$, $|\Gamma(4)| \leq 4$, $|\Gamma(5)| \leq 3$ and $|\Gamma(10)| \leq 1$. So we can assume $|\Gamma(2)| \geq 8$. By Theorem 4.6.1 we can then assume that $8 \leq |\Gamma(2)| \leq 9$ and therefore that $4 \leq H_2 \leq 9$. If $H_2 \geq 8$ assume $v(2)=8$ and $v(6,14,18)=0$ so $|\Gamma(2)|=8$ and $|\Gamma(10)|=0$, $|\Gamma(4)|=0$, a contradiction. If $H_2=7$, again $|\Gamma(2)|=7$ and $|\Gamma(10)|=0$, $|\Gamma(4)| \leq 1$, a contradiction. If $H_2=6$, assume $v(2)=6$ and $v(6) \leq 1$, so $|\Gamma(2)|=7$ and $|\Gamma(10)|=0$, $|\Gamma(4)| \leq 1$ a contradiction. If $H_2=5$ then $v(2)=5$, $v(6) \leq 1$ so $|\Gamma(2)| \leq 6$ and $|\Gamma(10)|=0$, $|\Gamma(4)| \leq 2$, a contradiction. If $H_2=4$ then

$v(2) = 4$, $v(6) \leq 1$ so $|\Gamma(2)| \leq 5$ and $|\Gamma(4)| \leq 2$, a contradiction. Thus we can assume $H, M \neq 1$, and Conjecture 1.3.4 is true for $n = 20$.

4.6.7 Case n=21

We only need to consider when $d = 7, 21$. Let X be a sequence in $\mathbb{Z}_{21}$ of length 21 contradictory to Conjecture 1.3.4. We partition into classes:

- $\Gamma(1) = \{(1,20)(2,19)(4,17)(5,16)(8,13)(10,11)\}$
- $\Gamma(3) = \{(3,18)(6,15)(9,12)\}$
- $\Gamma(7) = \{(7,14)\}$

If $d = 7$ we can assume that $|\Gamma(7)| = 0$ and $|\Gamma(3)| \leq 6$, so assume $3 \leq H \leq 6$. Note the following sequences: $\{1, 13\}$ is equivalent to $\{2, 5\}$, which implies $v(13) = 0$; $\{1^2, 12\}$ is equivalent to $\{2^2, 3\}$ which implies $v(12) = 0$; $\{1^3, 11\}$ is equivalent to $\{2^3, 1\}$ implies $v(11) = 0$. Now if $H \geq 5$, we can assume $v(x) = 0$ for $16 \leq x \leq 21$ and $2 \leq x \leq 7$. So partition into boxes (1) (8) (9) (10) (15), a contradiction since $v(8), v(9), v(10) \leq 1$. If $H = 4$ then $v(x) = 0$ for $17 \leq x \leq 20$ and $3 \leq x \leq 7$. So partition into boxes (1) (2) (8) (9) (10) (15) (16), a contradiction since $v(2), v(8), v(9), v(10) \leq 1$. If $H = 3$ then $v(x) = 0$ for $18 \leq x \leq 20$ and $4 \leq x \leq 7$. So partition into boxes (1) (2) (3) (8) (9) (10) (15) (16) (17), a contradiction since $v(2), v(3), v(9), v(10) \leq 1$. So assume $d \neq 7$.

If $d = 21$ we can assume $3 \leq H \leq M \leq 6$. Note then that $\{13^3, 1^3\}$ and $\{12^5, 1^3\}$ imply we can always assume that $v(13) \leq 2$ and $v(12) \leq 4$. Then **if M=6** let $\alpha = (1,..,7)$ be a box in X. We can assume that $v(x) = 0$ for $15 \leq x \leq 21$ and we partition into boxes (α) (8,13) (9,12) (10,11) (14) with cardinalities at most $6 + 2 + 4 + 6 + 2 < 21$, so assume $3 \leq H \leq M \leq 5$. **If M=5** let $\alpha = (1,..,6)$ be a box in X. Assume $v(x) = 0$ for $16 \leq x \leq 21$ and partition into boxes (α) (15) (8,13) (9,12) (10,11) (7,14) with cardinalities at most $5+6+2+4+5+2 = 24$. Then we can assume that $v(11) > 0$ but then $\{11, 15, 15, 1\}$ implies $v(15) \leq 1$, a contradiction. So assume $3 \leq H \leq M \leq 4$.

If M=4 the we can assume $H = 4$. We can assume $v(1) = 4$ and $v(x) = 0$ for $2 \leq x \leq 5$ and $17 \leq x \leq 21$, so we partition into boxes (1) (16) (6,15) (8,13) (9,12) (10,11) (7,14) with cardinalities at most $4+4+6+4+4+4+2 = 28$. Then if $v(15) > 0$, $\{15, 16^3\}$ is basic so assume $v(16) \leq 2$; $\{15, 12^2, 1^3\}$ is basic which implies $v(9, 12) \leq 1$; $\{15, 11^4, 1^4\}$ implies $v(10, 11) \leq 3$; $\{15, 13^2, 1\}$ implies $v(13) \leq 1$. So if $v(8) = 0$ we have boxes (1) (16) (15) (13) (9,12) (10,11) (7,14) with cardinalities at most $4+2+6+1+1+3+2 < 21$. If $v(8) > 0$. Then $v(9, 12), v(10, 11) = 0$, and we have boxes (1) (16) (15) (8) (7,14) with cardinalities less than 21, a contradiction. Therefore assume $v(15) = 0$. So we have boxes (1) (16) (6) (8,13) (9,12) (10,11) (7,14) with cardinalities at most $4+4+2+4+4+4+2 = 24$. Then we can assume $v(16) > 0$, but then $\{16, 12^2, 1^2\}$ implies $v(9, 12) \leq 1$ and $\{16, 11^2, 1^4\}$ implies $v(10, 11) \leq 1$, a contradiction. So assume $H = M = 3$.

If H=M=3 we can assume $v(x) = 0$ for $2 \leq x \leq 4$ and $18 \leq x \leq 21$. So we partition into boxes (1) (17) (5,16) (6,9,12,15) (8,13) (10,11) (7,14) with cardinalities at most $3+3+3+6+3+3+2 = 23$. So if $v(8) > 0$ then $v(10, 11) = 0$, a contradiction. If $v(13) > 0$ then $v(13) \leq 2$, $v(7) = 0$ and $\{13, 14, 14, 1\}$ imply $v(7, 14) \leq 1$; and $\{13, 5, 1^3\}$ implies $v(5) = 0$. Therefore we can assume boxes (1) (17) (16) (6,9,12,15) (13) (10,11) (7,14) with cardinalities at most $3+3+3+6+2+3+1 = 21$. Then we can assume $\{16, 13^2\}$ is a subsequence, a contradiction. So $v(13) = 0$, and we are left with boxes (1) (17) (5,16) (6,9,12,15) (10,11) (7,14) with cardinalities at most $3+3+3+6+3+2 < 21$, a contradiction, so Conjecture 1.3.4 holds for $n = 21$.

4.6.8 Case n=22

We note that in this case the Conjecture 1.3.4 fails for $d = n$, here we attempt to verify it for the case $d = \frac{n}{2}$. We partition into classes:

- $\Gamma(1) = \{(1,21)(3,19)(5,17)(7,15)(9,13)\}$
- $\Gamma(2) = \{(2,20)(4,18)(6,16)(8,14)(10,12)\}$

Note $|\Gamma(11)| = 0$ since we are assuming that $d = 11$. Also note $|\Gamma(2)| \leq 10$ so we can assume $3 \leq H \leq 10$. If $H = 10$ we can assume no $12 \leq x \leq 22$ and no $2 \leq x \leq 10$, an obvious contradiction. If $H = 9$ we can assume no $13 \leq x \leq 22$ and no $2 \leq x \leq 10$, so we partition into boxes (1) (12) with cardinalities at most $9 + 10 < 22$, a contradiction. If $H = 8$ we can assume no $14 \leq x \leq 22$ and no $3 \leq x \leq 10$, so we partition into boxes (1) (2,12) (13). Note $\{13^5, 1\}$ is basic so we can always assume $v(13) \leq 4$. Thus our cardinalities are at most $8 + 10 + 4 = 22$, which means we can assume $v(2,12) = 10$, a contradiction since $v(2^5, 1)$ has sum 11, which implies $v(12) \geq 6$. However $\{12^3, 13^4\}$ is a basic sequence. If $H = 7$ we can assume no $15 \leq x \leq 22$ and no $4 \leq x \leq 10$, so we partition into boxes (1) (3) $\Gamma(2)$ (13) with cardinalities at most $7+1+10+4 = 22$. Therefore we must have $\{13, 3, 1^6\}$ as a subsequence, a contradiction.

If $H = 6$ we can assume no $16 \leq x \leq 22$ and no $5 \leq x \leq 10$, so we partition into boxes (1) (3) $\Gamma(2)$ (13) (15). Note $\{15^2, 1^3\}$ is equivalent to $\{1^2, 3^3\}$ so we can always assume $v(15) \leq 1$, and our cardinalities are at most $6+1+10+4+1 = 22$. Then we can assume $\{15, 3, 1^4\}$ is a subsequence, a contradiction. If $H = 5$ we can assume no $17 \leq x \leq 22$ and no $6 \leq x \leq 10$, so we partition into boxes (1) (3) (5) $\Gamma(2)$ (13) (15) and our cardinalities are at most $5 + 1 + 1 + 10 + 4 + 1 = 22$, a contradiction since then $\{5, 3, 1^3\}$ must be a subsequence.

If $H = 4$ we can assume no $18 \leq x \leq 22$ and no $7 \leq x \leq 10$, so we partition into boxes (1) (3,5) $\Gamma(2)$ (13) (15) (17). Note $v(3,5) \leq 2$. and our cardinalities are at most $4+2+1+10+4+1+4 = 26$. Note if $v(17) > 0$ then assume $v(x) = 0$ for $2 \leq x \leq 5$, and $\{17, 13^2, 1\}$ is basic so we can assume $v(13) \leq 1$. The sum of the cardinalities of the boxes (1) $\Gamma(2)$ (13) (15) (17) is then at most $4 + 10 + 1 + 1 + 4 < 22$, so assume $v(17) = 0$. We then have boxes (1) (3,5) $\Gamma(2)$ (13) (15) and our cardinalities are at most $4 + 2 + 1 + 10 + 4 + 1 = 22$, so we can assume $\{15, 3, 3\}$ is a subsequence a contradiction. So assume $H \neq 4$.

If $H = 3$ we can assume no $19 \leq x \leq 22$ and no $8 \leq x \leq 10$, so we partition into boxes (1) (3,5) (7) $\Gamma(2)$ (13) (15) (17) and our cardinalities are at most $3 + 2 + 1 + 1 + 10 + 3 + 1 + 3 = 24$. Then note if $v(17) > 0$ that $v(3,5) = 0$ and $v(13) \leq 1$ from above, a contradiction, so assume $v(17) = 0$. Then the sum of the cardinalities of the boxes is less that 22, so we can assume $H \neq 3$, and we are done.

4.6.9 Case n=24

Then we need only consider when $d = 4, 6, 8, 24$. Let X be a sequence of length 24 in $\mathbb{Z}_{24}$ contradictory to Conjecture 1.3.4, and we partition elements of X into classes:

- $\Gamma(1) = \{(1, 23)(5, 19)(7, 17)(11, 13)\}$
- $\Gamma(2) = \{(2, 22)(10, 14)\}$
- $\Gamma(3) = \{(3, 21)(9, 15)\}$
- $\Gamma(4) = \{(4, 20)\}$
- $\Gamma(6) = \{(6, 18)\}$

- $\Gamma(8) = \{(8, 16)\}$

- $\Gamma(12) = \{(12)\}$

Note the following sequences are basic; $\{15^3, 1^3\}$, $\{9^5, 1^3\}$, $\{14^5, 1^2\}$, $\{10^7, 1^2\}$ so we can assume that $v(15) \leq 2$ and $v(9) \leq 4$ when $M \geq 3$, and $v(14) \leq 4$, $v(10) \leq 6$ when $M \geq 2$.

If $d = 4$ we can assume that $|\Gamma(1)| \leq 12$, $|\Gamma(2)| \leq 2$, $|\Gamma(3)| \leq 6$, $|\Gamma(4)| = 0$,$|\Gamma(6)| \leq 1$, $|\Gamma(8)| = 0$ and $|\Gamma(12)| = 0$. So the sum of the cardinalities of the classes is less than 24, a contradiction. So assume $d \neq 4$.

If $d = 6$ we can assume that $|\Gamma(2)| \leq 4$, $|\Gamma(3)| \leq 2$, $|\Gamma(4)| \leq 2$, $|\Gamma(6)| = 0$, $|\Gamma(8)| \leq 2$ and $|\Gamma(12)| = 0$. So assume $|\Gamma(1)| \geq 14$ and therefore $4 \leq H \leq 5$. If $H = 5$ then we can assume that $v(x) = 0$ for $2 \leq x \leq 6$, $7 \leq x \leq 12$, $19 \leq x \leq 24$. So we partition into boxes (1) (17) (13) (14) (15) (16) with cardinalities at most $5+5+5+2+1+2 < 24$. If $H = 4$ then we can assume that $v(x) = 0$ for $2 \leq x \leq 6$, $8 \leq x \leq 12$, $20 \leq x \leq 24$. So we partition into boxes (1) (7,17) (13) (14) (15) (16) (19) with cardinalities at most $4+4+4+2+1+2+4 < 24$, a contradiction. So we assume that $d \neq 6$.

If $d = 8$ we can assume $|\Gamma(8)| = 0$ and $1 \leq H \leq M \leq 7$. Note $v(x) \leq 1$ for any $x \in \Gamma(4)$ and $v(x) \leq 3$ for any $x \in \Gamma(2)$. So if $M = 7$ let $\alpha = (1, .., 8)$ be a box in X. Then assume $v(\alpha) \leq 7$ and $v(x) = 0$ for $17 \leq x \leq 24$. We partition into boxes (α) (9,15) (10,14) (11,13) (12) with cardinalities at most $7+2+3+7+1 < 24$, a contradiction. If $M = 6$ let $\alpha(1, .., 7)$ and $v(x) = 0$ for $18 \leq x \leq 24$. We partition into boxes (α) (9,15) (10,14) (11,13) (12) (17) with cardinalities at most $6+2+3+6+1+6 = 24$. So we can assume $\{17, 15, 15, 1\}$ is a subsequence, contradiction. If $M = 5$ let $\alpha = (1, .., 6)$ we can assume $v(x) = 0$ for $7 \leq x \leq 8$ and $19 \leq x \leq 24$. We partition into boxes (α)

(9,15) (10,14) (11,13) (12) (17) with cardinalities at most $5+4+3+5+1+5 < 24$, a contradiction. If $M=4$ let $\alpha = (1,..,5)$ and we can assume $v(x) = 0$ for $6 \le x \le 8$ and $20 \le x \le 24$. We partition into boxes (α) (9,15) (10,14) (11,13) (12) (17) (18) (19) with cardinalities at most $4+4+3+4+1+4+3+4 = 27$. Then if $v(11) > 0$ we could assume $v(x) = 0$ for $9 \le x \le 13$ so then $v(11) \le 1$, $v(9,15) \le 2$ and $v(11,13) \le 1$ a contradiction. If $v(13) > 0$, $v(9) = 0$ and $\{13, 17, 17, 1\}$ implies $v(17) \le 1$, a contradiction. So assume $v(11,13) = 0$. We can therefore partition into boxes (α) (9,15) (10,14) (12) (17) (18) (19) with cardinalities at most $4+4+3+1+4+3+4 < 24$.

If $M=3$ let $\alpha = (1,..,4)$ and assume $v(x) = 0$ for $5 \le x \le 8$ and $21 \le x \le 24$. We partition into boxes (α) (9,15) (10,14) (11,13) (12) (17) (18) (19) (20) with cardinalities at most $3+4+3+3+1+3+3+3+1 = 24$. So $\{13, 17^2, 1\}$ is a basic subsequence, a contradiction. If $M=2$ we assume $H=2$ and $v(x)=0$ for $2 \le x \le 3$, $6 \le x \le 8$ and $22 \le x \le 24$. We partition into boxes (1) (9,15,21) (10,14) (11,13) (12) (17) (18) (5,19) (4,20) with cardinalities at most $2+7+3+2+1+2+3+2+1 < 24$, a contradiction. Finally if $M=1$ we assume $v(1)=1$ and $v(2), v(7), v(23) = 0$. We partition into boxes (1) (3,9,15,21) (10,14,22) (11,13) (12) (17) (6,18) (5,19) (4,20) with cardinalities at most $1+7+3+1+1+1+1+3+1+1 < 24$, a contradiction. So we can assume that $d \ne 8$.

If $d = 24$ we consider $0 \le H \le M \le 9$. **If M=9** let $\alpha = (1,..,10)$ be a box in X. We assume $v(x) = 0$ for $15 \le x \le 24$ and partition into boxes (α) (11,13) (12) (14) with cardinalities at most $9+9+1+4 < 24$, so assume $0 \le H \le M \le 8$. **If M=8** let $\alpha = (1,..,9)$ be a box in X. We assume $v(x) = 0$ for $16 \le x \le 24$ and partition into boxes (α) (11,13) (12) (10,14) (15) with cardinalities at most $8+8+1+4+2 < 24$. **If M=7** then let $\alpha = (1,..,8)$ be a box in X and we assume $v(x) = 0$ for $17 \le x \le 24$. So we partition into boxes (α) (11,13) (12) (10,14) (9,15) (16) with cardinalities at most $7+7+1+4+2+2<24$, a contradiction, so assume $0 \le H \le M \le 6$.

If M=6 let $\alpha = (1,..,7)$ be a box in X. Assume $v(x) = 0$ for $18 \le x \le 24$ and partition into boxes (α) (11,13) (12) (10,14) (9,15) (8,16) (17) with cardinalities at most $6+6+1+4+2+2+6 = 27$. So we can assume that $v(17) > 1$. Then $\{17,17,13,1\}$ is a basic subsequence, which implies $v(11,13) \le 1$, a contradiction. So assume $0 \le H \le M \le 5$.

If M=5 let $\alpha = (1,..,6)$ be a box in X. Assume $v(x) = 0$ for $19 \le x \le 24$ and partition into boxes (α) (11,13) (12) (10,14) (9,15) (8,16) (7,17) (18) with cardinalities at most $5+5+1+4+4+2+5+3 = 29$. Note if $v(17) > 1$ then $\{17,17,13,1\}$ implies $v(11,13) \le 1$, and $\{17,17,14\}$ implies $v(10,14) \le 1$, a contradiction. So assume $v(17) \le 1$ and $v(7,17) \le 2$. Then if $v(13) > 0$ note $v(x) = 0$ for $7 \le x \le 11$, so we can assume boxes (α) (13) (12) (14) (15) (16) (17) (18) with cardinalities at most $5+5+1+4+2+2+1+3 < 24$, so assume $v(13) = 0$. We can therefore assume boxes (α) (11) (12) (10,14) (9,15) (8,16) (7,17) (18) with cardinalities at most $5+1+1+4+4+2+2+3 < 24$, a contradiction. So assume $0 \le H \le M \le 4$.

If M=4 let $\alpha = (1,..,5)$ be a box in X. Assume $v(x) = 0$ for $20 \le x \le 24$ and we partition into boxes (α) (11,13) (12) (10,14) (9,15) (8,16) (7,17) (6,18) (19) with cardinalities at most $4+4+1+4+4+2+4+3+4 = 30$. Assume that $v(19) > 0$, and note since $\{19,14,14,1\}$ is basic we assume $v(10,14) \le 1$. If $v(13) > 0$ aswell, then $v(x) = 0$ for $7 \le x \le 11$, and $\{19,16,13\}$ and $\{19,15,13,1\}$ imply $v(15), v(16) = 0$. So we partition into boxes (α) (13) (12) (14) (17) (6,18) (19) with cardinalities at most $4+4+1+1+4+3+4 < 24$ so we can assume $v(13) = 0$ when $v(19) > 0$. So we have boxes (α) (11) (12) (10,14) (9,15) (8,16) (7,17) (6,18) (19) with cardinalities at most $4+1+1+1+4+2+4+3+4 = 24$. Then $\{19,19,17,17\}$ must be a subsequence, a contradiction. So assume that $v(19) = 0$. Now assume that $v(13) > 0$. Then $v(x) = 0$ for $7 \leq x \leq 11$. Also $\{13, 17, 17, 1\}$ is basic so $v(17) \leq 1$. So we can assume boxes (α) (13) (12) (14) (15) (16) (17) (6,18) with cardinalities at most $4 + 4 + 1 + 4 + 2 + 2 + 1 + 3 < 24$, so assume $v(13) = 0$. Therefore we can assume boxes (α) (11) (12) (10,14) (9,15) (8,16) (7,17) (6,18) with cardinalities at most $4 + 1 + 1 + 4 + 4 + 2 + 4 + 3 < 24$, so we can assume $0 \leq H \leq M \leq 3$.

If M=3 let $\alpha = (1, .., 4)$ be a box in X. Assume $v(x) = 0$ for $21 \leq x \leq 24$ and partition into boxes (α) (11,13) (12) (10,14) (9,15) (8,16) (7,17) (6,18) (5,19) (20) with cardinalities at most 3+3+1+6+4+2+3+3+3+5 = 33. If $v(20) > 0$, note $\{20, 14, 14\}$ implies $v(14) \leq 1$ and $\{20, 9, 9, 9, 1\}$ implies $v(9, 15) \leq 2$. Then if $v(10) > 0$ we can assume $v(11, 13), v(12) = 0$. Also $v(6), v(7) \leq 1$ and $\{20, 10, 18\}$ and $\{20, 10, 17, 1\}$ show $v(18), v(17) = 0$. Then we would have boxes (α) (10) (9,15) (8,16) (7) (6) (5,19) (20) with cardinalities at most $3+6+2+2+1+1+3+5 < 24$. So let $v(10) = 0$ when $v(20) > 0$. So when $v(20) > 0$ we can assume boxes (α) (11,13) (12) (14) (9,15) (8,16) (7,17) (6,18) (5,19) (20) with cardinalities at most 3+3+1+1+2+2+3+3+3+5 = 26. Then $\{20, 20, 20, 12\}$, $\{20, 20, 8\}$ and $\{20, 20, 16, 16\}$ shows that we must have either $v(8, 16) \leq 1$ and $v(12) = 0$ or $v(20) \leq 2$, both contradictions. So we assume $v(20) = 0$.

Now we have boxes (α) (11,13) (12) (10,14) (9,15) (8,16) (7,17) (6,18) (5,19) with cardinalities at most $3 + 3 + 1 + 6 + 4 + 2 + 3 + 3 + 3 = 28$. Note if $v(9) > 0$ then $v(x) = 0$ for $12 \leq x \leq 15$. We partition into boxes (α) (11) (10) (9) (8,16) (7,17) (6,18) (5,19) with cardinalities at most $3 + 1 + 6 + 4 + 2 + 3 + 3 + 3 = 25$. If $v(11) > 0$ also then $v(10) = 0$, a contradiction, so assume $v(11) = 0$. Then we must have $\{10, 10, 10, 9, 9\}$ as a basic subsequence so assume $v(9) = 0$. Now we have boxes (α) (11,13) (12) (10,14) (15) (8,16) (7,17) (6,18) (5,19) with cardinalities at most $3 + 3 + 1 + 6 + 2 + 2 + 3 + 3 + 3 = 26$. Note if $v(10) > 0$ then $(11, 13) = 0$, a contradiction, so assume $v(10) = 0$. So we can assume $v(14) > 0$ and hence $v(7) = 0$, so $\{17, 17, 14\}$ is a basic subsequence, a contradiction. So assume $0 \leq H \leq M \leq 2$.

If M=2 we can assume $H = 2$ so that $v(x) = 0$ for $22 \leq x \leq 24$ and we partition into boxes (1) (11,13) (12) (10,14) (9,15,21) (8,16) (7,17) (6,18) (5,19) (4,20) with cardinalities at most $2+2+1+6+7+2+2+3+2+5 = 32$.

If $v(20) > 0$ then $\{20, 14, 14\}$ implies $v(14) \leq 1$. If $v(7) > 0$ as well, $v(15), v(16)$ and $v(17) = 0$. Also, $v(20) \leq 1$ since $\{20, 20, 7, 1\}$ is basic. $\{20, 7, 10, 10, 1\}$ and $\{20, 14, 14\}$ imply $v(10, 14) \leq 1$. So we can assume boxes (1) (11,13) (12) (10,14) (9,21) (8) (7) (6,18) (5,19) (20) with cardinalities at most $2+2+1+1+7+2+2+3+2+1 < 24$. So $v(7) = 0$ when $v(20) > 0$. Now if $v(17) > 0$ as well, $v(10) = 0$ since $\{20, 17, 10, 1\}$ is basic. Also $v(6, 18) = 0$ since $\{20, 18, 10\}$ is basic. So we partition into boxes (1) (11,13) (12) (14) (9,15,21) (8,16) (17) (5,19) (20) with cardinalities at most $2+2+1+1+7+2+2+2+5 = 24$, so $\{17, 17, 14\}$ is a basic subsequence. So we assume $v(17) = 0$ when $v(20) > 0$. If $v(10) > 0$ as well, we can assume $v(13), v(12) = 0$, $v(6) \leq 1$. $\{20, 10, 18\}$ shows $v(18) = 0$. Also $\{20, 10^5, 1, 1\}$ is basic so $v(10) \leq 4$. So we partition into boxes (1) (11) (10) (9,15,21) (8,16) (6) (5,19) (20) with cardinalities at most $2+1+4+7+2+1+2+5 = 24$. So then $\{20, 20, 20, 11, 1\}$ is a basic subsequence, contradiction, so $v(10) = 0$ when $v(20) > 0$. So we partition into boxes (1) (11,13) (12) (14) (9,15,21) (8,16) (6,18) (5,19) (20) with cardinalities at most $2+2+1+1+7+2+3+2+5 = 25$. So we can assume $v(20) \geq 4$, then $\{20, 20, 20, 20, 16\}$ $\{20, 20, 8\}$ are basic so $v(8, 16) = 0$, a contradiction. Hence we can assume $v(20) = 0$.

Now if $v(4) > 0$, $v(18), v(19) = 0$. $\{10, 10, 4\}$ implies $v(10) \leq 1$ and $\{14, 14, 14, 4, 1, 1\}$ implies $v(10, 14) \leq 2$. So we partition into boxes (1) (11,13) (12) (10,14) (9,15,21)

(8,16) (7,17) (6) (5) (4) with cardinalities at most 2+2+1+2+7+2+2+2+2+5 = 27. So we can assume $v(4) \geq 2$. So we can assume $v(8,16) \leq 1$. This implies $v(4) \geq 3$. So $v(12), v(14) = 0$, which implies $v(10,14) \leq 1$. $v(10) \leq 1$ and $\{14, 14, 14, 4, 1, 1\}$ implies $v(10,14) \leq 2$. So we partition into boxes (1) (11,13) (10) (9,15,21) (8) (7,17) (6) (5) (4) with cardinalities at most $2+2+1+7+1+2+2+2+5 = 24$, which has $\{10, 8, 6\}$ as a basic sequence. So assume $v(4) = 0$.

So we partition into boxes (1) (11,13) (12) (10,14) (9,15,21) (8,16) (7,17) (6,18) (5,19) with cardinalities at most $2 + 2 + 1 + 6 + 7 + 2 + 2 + 3 + 2 = 27$. Then if $v(10) \geq 5$, $v(12) = 0$, and $\{10^3, 18\}$ and $\{10^4, 6, 1^2\}$ are basic so $v(6, 18) = 0$, a contradiction. So we assume $v(10, 14) \leq 4$. So we can assume $v(14) \geq 3$, so $\{18, 14, 14, 1, 1\}$ and $\{14, 14, 14, 6\}$ are basic, so $v(18, 6) = 0$, a contradiction, hence we can assume $H = M = 1$.

If H=M=1 we can assume $|\Gamma(1)| \leq 4$ so we know $|\Gamma(2)| \geq 2$ so we can assume $1 \leq H_2 \leq 11$. If $H_2 \geq 7$ then note we assume $7 \leq v(2) \leq 11$ and $v(x) = 0$ for any $x \geq 10$ with x even. So we partition into boxes (1,23) (2) (4,6,8) (5,19) (7,17) (3,9,15,21) (11,13) with cardinalities at most $1 + 11 + 2 + 1 + 1 + 7 + 1 = 24$. Note it is easy to see any 3 elements from $(4, 6, 8)$ will give an even sum from 10 to 24. So we can assume $v(2) \geq 10$ and so $v(4, 6, 8) = 0$, a contradiction. If $H_2 = 6$ we can assume $v(2) = 6$ so $v(x) = 0$ for $x \geq 12$ with x even. So we partition into boxes (1,23) (2) (4,6,8,10) (5,19) (7,17) (3,9,15,21) (11,13) with cardinalities at most $1+6+2+1+1+7+1 < 24$. Again we can see any 3 elements from $(4, 6, 8, 10)$ will give an even (sub)sum from 12 to 24. If $H_2 = 5$ we can assume $v(2) = 5$ so $v(x) = 0$ for $x \geq 14$ with x even. So we partition into boxes (1,23) (2) (4,6,8,10,12) (5,19) (7,17) (3,9,15,21) (11,13) with cardinalities at most $1 + 5 + 3 + 1 + 1 + 7 + 1 < 24$. It is clear any 4 elements from $(4, 6, 8, 10, 12)$ will give an even (sub)sum from 14 to 24.

If $H_2 = 4$ we can assume $v(2) = 4$ so $v(x) = 0$ for $x \geq 16$ with x even. So we partition into boxes (1,23) (2) (4,6,8,10,12) (14) (5,19) (7,17) (3,9,15,21) (11,13) with cardinalities at most $1 + 4 + 3 + 4 + 1 + 1 + 7 + 1 < 24$. If $H_2 = 3$ we can assume $v(2) = 3$ so $v(x) = 0$ for $x \geq 18$ with x even. So we partition into boxes (1,23) (2) (4,6,8,10,12) (14) (16) (5,19) (7,17) (3,9,15,21) (11,13) with cardinalities at most 1+3+4+3+2+1+7+1 < 24.If $H_2 = 2$ we can assume $v(2) = 2$ so $v(x) = 0$ for $x \geq 20$ with x even. So we partition into boxes (1,23) (2) (4,6,8,10,12) (14) (16) (18) (5,19) (7,17) (3,9,15,21) (11,13) with cardinalities at most 1+2+4+2+2+3+1+1+7+1 $= 24$. So assume $v(14)$ and $v(16) > 0$, then $v(4, 6, 8, 10) = 0$ and $v(12) \leq 1$, a contradiction.

If $H_2 = 1$ we can assume $v(22) = 0$ so we partition into boxes (1,23) (2)(10,14) (4,20) (8,16) (6,18) (12) (5,19) (7,17) (3,9,15,21) (11,13) with cardinalities at most 1+1+1+5+2+3+1+1+1+7+1 $= 24$. So we can assume that $v(4, 20) = 5$. If $v(4) = 5$ then $v(8, 16) = 0$ a contradiction. If $v(20) = 5$ then $\{20, 20, 8\}$ and $\{20, 20, 20, 16\}$ are basic, a contradiction. So we can assume $H_2 \neq 1$, and the conjecture holds for $n = 24$.

4.6.10 Case n=25

Let X be a sequence in Z_{25} of length 25 that contradicts Conjecture 1.3.2. We only need to consider the cases where $d = 5, 25$, so we partition the elements of X into classes:

- $\Gamma(1) = \{(1, 24)(2, 23)(3, 22)(4, 21)(6, 19)(7, 18)(8, 17)(9, 16)(11, 14)(12, 13)\}$
- $\Gamma(5) = \{(5, 20)(10, 15)\}$

If $d = 5$ then we can assume that $|\Gamma(5)| = 0$, so we can assume $3 \leq H \leq M \leq 4$. We note that since $\{13, 1, 1\}$ is equivalent to $\{1, 2, 2\}$ and $\{14, 1\}$ is equivalent to

$\{3,2\}$ that we can assume $v(14), v(13) = 0$. If $M = 4$ then we can assume that $H = 4$ and $v(x) = 0$ for $2 \leq x \leq 5$ and $21 \leq x \leq 25$. So we partition into boxes (1) (19,6) (18,7) (17,8) (16,9) (11) (12). Note $v(11), v(12) \leq 1$, so the sum of the cardinalities of the boxes is then at most $4+4+4+4+4+1+1 < 25$, a contradiction. If $M = 3$ then we can assume $H = 3$ and partition into boxes (1) (21) (6,19) (7,18) (8,17) (9,16) (11) (12) where the sum of cardinalities is at most $3+3+3+3+3+3+1+1 < 25$, a contradiction. So we can assume $d \neq 5$.

If $d = 25$ we can assume that $|\Gamma(5)| \leq 4$, so we need to consider when $3 \leq H \leq M \leq 7$. Note $\{18^4, 1^3\}$ is basic so $v(18) \leq 3$; $\{16^3, 1^2\}$ is basic so $v(16) \leq 2$. Also $\{15^3, 1^5\}$ and $\{14^5, 1^5\}$ are basic so we can assume that $v(15) \leq 2$ and $v(14) \leq 4$ when $M \geq 5$.

If M=7 let $\alpha = (1, .., 8)$ be a box in X. We can assume that there is no sequence Y in $X - \alpha$ with $18 \leq \sigma(Y) \leq 25$. We partition into boxes (α) (17) (9,16) (10,15) (11,14) (12,13). Note $\{17^4, 1^7\}$ is basic so assume $v(17) \leq 3$. The sum of the cardinalities of our boxes is then at most $7+3+2+2+4+7 = 25$. Thus we can assume that $v(17) = 3$ and $v(13) = 7$, a contradiction since $\{17^2, 13, 1^3\}$ is basic. So assume $3 \leq H \leq M \leq 6$.

If M=6 let $\alpha = (1, .., 7)$ be a box in X. We can have no sequence Y in $X - \alpha$ with $19 \leq \sigma(Y) \leq 25$ and we partition into boxes (α) (18) (17,8) (9,16) (10,15) (11,14) (12,13). Note $\{9^5, 1^5\}$ is basic so we can assume $v(9,16) \leq 4$. The sum of the cardinalities of our boxes is then at most $6+3+6+4+2+4+6 = 31$. Then if $v(17) > 0$, $\{17, 13^4, 1^6\}$ is basic so we can assume $v(12,13) \leq 2$; $\{17, 14^2, 1^5\}$ is basic so we can assume that $v(11,14) \leq 1$; $\{17, 15^2, 1^3\}$ is basic so we can assume $v(10,15) \leq 1$. Therefore our boxes have sum of cardinalities at most $6+3+6+4+1+1+3 < 25$, so assume

$v(17) = 0$. Now if $v(13) > 0$ then we can assume $v(8) = 0$ and $\{13, 18^2, 1\}$ is basic so we can assume $v(18) \leq 1$. Therefore in this situation we have boxes (α) (18) (9,16) (10,15) (11,14) (12,13) with sum of cardinalities at most $6+1+4+2+4+6 < 25$ so we can assume $v(13) = 0$. Therefore we can assume our boxes are partitioned into (α) (18) (8) (9,16) (10,15) (11,14) (12) with cardinalities at most 6+3+2+4+2+4+1 < 25, a contradiction, so assume $3 \leq H \leq M \leq 5$.

If M=5 let $\alpha = (1, .., 6)$ be a box in X. We assume no sequence Y in $X - \alpha$ with $20 \leq \sigma(Y) \leq 25$ and we partition into boxes (α) (19) (18,7) (17,8) (9,16) (10,15) (11,14) (12,13). The sum of the cardinalities of our boxes is then at most $5+5+3+5+5+2+4+5 = 34$. If $v(19) > 0$ then $\{19, 13^2, 1^5\}$ is basic so we can assume $v(12, 13) \leq 1$; $\{19, 14^2, 1^3\}$ is basic so $v(11, 14) \leq 1$; $\{19, 17^3, 1^5\}$ is basic so $v(17, 8) \leq 2$; so we have boxes (α) (19) (18,7) (17,8) (9,16) (10,15) (11,14) (12,13) with cardinalities at most $5+5+3+2+5+2+1+1 < 25$, so we can assume $v(19) = 0$. If $v(18) > 0$ then $\{18, 15^2, 1^2\}$ and $\{18, 14^2, 1^4\}$ is basic which implies $v(10, 15), v(11, 14) \leq 1$. So we partition into boxes (α) (18) (17,8) (9,16) (10,15) (11,14) (12,13), with cardinalities at most $5+3+5+5+1+1+5 = 25$. So we can assume $\{18, 18, 13, 1\}$ is a subsequence, but this is basic. So assume $v(18) = 0$. Now note if $v(13) > 0$ then $v(x) = 0$ for $7 \leq x \leq 12$, so we can partition into boxes (α) (17) (16) (15) (14) (13) with sum of cardinalities at most $5+5+2+2+4+5 < 25$, so assume $v(13) = 0$. So we can now partition into boxes (α) (7) (17,8) (9,16) (10,15) (11,14) (12) with sum of the cardinalities at most $5+2+5+5+2+4+1 < 25$, so we can assume $3 \leq H \leq M \leq 4$.

If M=4 we can assume that $H = 4$, so let $v(1) = 4$, $v(x) = 0$ for $2 \leq x \leq 5$ and we assume no subsequence Y of X with $1 \notin Y$ such that $21 \leq \sigma(Y) \leq 25$. So we partition into boxes (1) (19,6) (18,7) (17,8) (9,16) (10,15,20) (11,14) (12,13). The sum of the cardinalities of our boxes is then at most 4+4+3+4+4+4+4+4 = 31. Then if $v(19) > 0$, $\{19, 18^3, 1^2\}$ implies $v(18, 7) \leq 2$; $\{19, 14^2, 1^3\}$ so $v(11, 14) \leq 1$; $\{19, 13^4, 1^4\}$ implies $v(12, 13) \leq 3$; so our boxes have cardinalities at most 4+4+2+4+4+4+1+3 = 26. So we can assume $v(19) \geq 3$, a contradiction since $\{19^3, 16, 1^2\}$ and $\{19^2, 9, 1^3\}$ are basic which would imply $v(9, 16) = 0$. So we can assume $v(19) = 0$. If $v(18) > 0$ then $v(x) = 0$ for $6 \leq x \leq 7$. Note $\{18, 14^2, 1^4\}$ implies $v(11, 14) \leq 1$ so we have boxes (1) (18) (17,8) (9,16) (10,15,20) (11,14) (12,13) with cardinalities at most $4 + 3 + 4 + 4 + 4 + 1 + 4 < 25$ a contradiction, so let $v(18) = 0$. Then if $v(17) > 0$ we know that $v(x) = 0$ for $6 \leq x \leq 8$, and we can partition into boxes (1) (17) (9,16) (10,15,20) (11,14) (12,13) with cardinalities at most $4 + 4 + 4 + 4 + 4 + 4 < 25$, so let $v(17) = 0$. Then if $v(14) > 0$ we can assume $v(x) = 0$ for $7 \leq x \leq 11$, so we can partition into boxes (1) (6) (16) (10,15,20) (14) (12,13). The sum of the cardinalities of our boxes is then at most $4 + 3 + 2 + 4 + 4 + 4 < 25$, a contradiction, so let $v(14) = 0$. So we have boxes (1) (6) (7) (8) (9,16) (10,15,20) (11) (12,13). The sum of the cardinalities of our boxes is at most $4 + 3 + 2 + 2 + 4 + 4 + 1 + 4 < 25$, a contradiction, so we can assume $H = M = 3$.

If M=H=3 we can assume $v(1) = 3$, $v(x) = 0$ for $2 \leq x \leq 4$, and we have no subsequence Y of X with $1 \notin Y$ such that $22 \leq \sigma(Y) \leq 25$. We partition into boxes (1) (21) (6,19) (18,7) (17,8) (9,16) (5,10,15,20) (11,14) (12,13). The sum of the cardinalities of our boxes is at most $3 + 3 + 3 + 3 + 3 + 3 + 4 + 3 + 3 = 28$. Then if $v(21) > 0$ then $\{21, 14^2, 1\}$ and $\{21, 13^2, 1^3\}$ imply $v(12, 13), v(11, 14) \leq 1$. Our boxes then have cardinalities at most $3 + 3 + 3 + 3 + 3 + 3 + 4 + 1 + 1 < 25$, a contradiction, so assume $v(21) = 0$. Then we can partition into boxes (1) (6,19) (18,7) (17,8) (9,16) (5,10,15,20) (11,14) (12,13) with sum of the cardinalities at most $3 + 3 + 3 + 3 + 3 + 4 + 3 + 3 = 25$. So since $v(8) \leq 2$ we can assume $v(17) = 3$, then $\{17^2, 14, 1^2\}$ implies $v(11, 14) \leq 1$, a contradiction. So we can assume $M \neq 3$, a contradiction, and Conjecture 1.3.4 holds for $n = 25$.

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