Calculation of the Optical Conductivity for the Anderson Impurity Model Using the Slave Boson Technique

by

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ABSTRACT

The optical conductivity of the Anderson impurity model has been calculated by employing the slave boson technique and an expansion in powers of $1/N$, where $N$ is the degeneracy of the f electron level. This method has been used to find the effective mass of the conduction electrons for temperatures above and below the Kondo temperature. For low temperatures, the mass enhancement is found to be large while at high temperatures, the mass enhancement is small. The conductivity is found to be Drude like with frequency dependent effective mass and scattering time for low temperatures and frequency independent effective mass and scattering time for high temperatures. The behavior of both the effective mass and the conductivity is in qualitative agreement with experimental results.
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There is a class of alloys which exhibit radically different behavior than normal metals. Such alloys can be produced by dissolving a small concentration of a transition element, which has a partially occupied d or f orbital (from here on, denoted by an f orbital meaning d or f orbital) and thus a net magnetic moment, into a nonmagnetic host metal, for example iron in copper. These magnetic impurities in the host metal have localized f bands and act as scattering centres for the conduction electrons. It should be noted that Fe-Cu is a magnetic alloy for temperatures less than room temperature, whereas Fe-Al or Ni-Cu do not contain any localized moments (Heeger, 1969). Thus, existence of a moment of the free atom does not necessarily imply that the moment is retained in the alloy.

These alloys possess a minimum in the electrical resistivity at low temperatures. The temperature at which the resistivity is minimum is known as the Kondo temperature. For temperatures much larger than the Kondo temperature, the system behaves as a system of independent localized moments embedded in a sea of conduction electrons. This can be seen in the high temperature susceptibility which is proportional to the inverse of temperature. This is the behavior of the
Curie law which describes a set of ions, each with the same angular momentum. For temperatures much less than the Kondo temperature, the impurities lose their magnetic moment as if the f electrons become part of the Fermi sea, and the system behaves as a Fermi liquid with the quasiparticles having enormous mass. This is reflected in specific heat measurements. The low temperature specific heat is linear in temperature, where the slope $\gamma$ can be of the order of 1000 mJ/mol·K$^2$·impurity (Lee et al., 1986). whereas for a normal metal $\gamma$ is of the order of 1–10 mJ/mol·K$^2$. By writing $\gamma$ for a free electron model with an effective mass $m^*$ replacing the actual mass,

$$\gamma = \frac{k_B k_F m^*}{3\pi^2 \hbar^2},$$

where $k_B$ is Boltzmann's constant, $\hbar$ is Plank's constant divided by $2\pi$, and $k_F$ is the Fermi wave vector, the large value of $\gamma$ implies an effective mass of the order 100–1000 times that of the actual mass of an electron. The susceptibility $\chi$ at low temperatures becomes constant suggesting Pauli type paramagnetism. It is also greatly enhanced over that of normal metals: $\chi$ is of the order $10^{-2}$ emu/mol·impurity (Lee et al., 1986) and for normal metals it is of the order $10^{-5}$–$10^{-4}$ emu/mol. However the Wilson ratio $\chi/\gamma$ has a free electron gas value indicating that the enhancements of $\chi$ and $\gamma$ cancel each other.

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P.W. Anderson (1961) introduced a model to describe the magnetic impurity in a metallic host. He replaced the impurity with a single localized f level interacting with a sea of conduction electrons. The single impurity Anderson Hamiltonian, where single impurity means that the impurity concentration is dilute enough such that the conduction electrons interact with one impurity at a time, is

\[ H = H_{\text{band}} + H_f + H_U + H_{\text{mix}} , \]  

(1.1)

where

\[ H_{\text{band}} = \sum_{k,m} \varepsilon_{km} c_{km}^+ c_{km}, \]
\[ H_f = E_f \sum_m f_m^+ f_m, \]
\[ H_U = U \sum_{mm'} f_m^+ f_m f_{m'}^+ f_{m'}, \]
\[ H_{\text{mix}} = \sum_{k,m} (V_k c_{km}^+ f_m + V_k^* f_m^+ c_{km}), \]

and all energies are measured with respect to the Fermi energy. \( H_{\text{band}} \) describes the sea of conduction electrons with creation and destruction operators of electrons in spherical wave state \((k,m) c_{km}^+ \) and \( c_{km} \) respectively and energy \( \varepsilon_{km} \). The interaction between conduction electrons and between conduction electrons and the lattice can be included in an average way into \( \varepsilon_{km} \). In the absence of a magnetic field, the energy is independent of spin and is written \( \varepsilon_k \). \( H_f \) describes the localized f electrons with creation and destruction
operators $f^+_m$ and $f^-_m$ respectively and energy $E_f < 0$. The magnetic quantum number, $m$ runs from $-j$ to $+j$, where $j$ is the total angular momentum. The degeneracy of the $f$ orbital is then $N = 2j + 1$. $H_U$ is the Coulomb interaction between $f$ electrons, where the prime on the summation indicates to exclude the case $m = m'$. $H_{\text{mix}}$ describes the interaction between the conduction electrons and the $f$ electrons. This Hamiltonian has neglected spin-orbit effects and crystal field splitting. Even this simplified model has yet to yield all of its predictions despite extensive work.

For these systems, the Coulomb interaction $U$ is of the order of $5 eV$ (Lee et al., 1986). Since $|E_f|$ is of the order 1 eV (Lee et al., 1986), the $f$ orbital will tend to favour being no more than singly occupied since the energy of a second electron in the $f$ orbital will be increased by an amount $U$ over a first electron. With this in mind, the limit $U \rightarrow +\infty$ is taken and thus the $f$ orbital is either singly occupied or empty. This gives the so called infinite-$U$ Anderson Hamiltonian:

$$H = H_{\text{band}} + H_f + H_{\text{mix}}$$

where $H_{\text{band}}, H_f$ and $H_{\text{mix}}$ are as above, with the constraint

$$\sum_m f^+_m f^-_m \leq 1.$$ 

This is a quantum mechanical problem with a constraint. Barnes (1976) and Coleman (1984) introduced a zero energy
boson to represent the empty $f$ orbital state. With this so
called slave boson, the Hamiltonian becomes

$$H = H_{\text{band}} + H_f + H_{\text{mix}},$$

(1.2)

where $H_{\text{band}}$ and $H_f$ are as above and

$$H_{\text{mix}} = \sum_{km} (V_k c_{km}^+ b^+ f_m + V_k^* f_m^+ b c_{km}),$$

where $b^+$ and $b$ are the creation and destruction operators for
the slave boson respectively. The constraint now is

$$Q = b^+ b + \sum_m f_m^+ f_m = 1.$$

Since the constraint is now in the form of an equality, it
can be handled much easier than the inequality constraint. One
can just as easily represent the empty $f$ state with a slave
fermion. However this representation leads to an odd number
of fermion operators in $H_{\text{mix}}$ and Wick's theorem cannot be
applied. It can be shown that

$$-i \frac{dQ}{dt} = [Q, H] = 0,$$

and thus $Q$ is a constant of motion. The corresponding
symmetry of $H$ is the local gauge invariance, i.e., the
invariance of $H$ under the transformation

$$f_m \rightarrow f_m e^{i \varphi}, \quad b^+ \rightarrow b^+ e^{-i \varphi}, \quad c_{km} \rightarrow c_{km}.$$

This Hamiltonian has been examined for low temperatures
by mean field theory (Newns and Read, 1987), by replacing the boson operators with their averages:

\[ b \rightarrow \langle b \rangle = z, \]
\[ b^+ \rightarrow \langle b^+ \rangle = z^*. \]

Since the Hamiltonian (1.2) has the local gauge symmetry, the mean field theory breaks this symmetry by fixing the phase \( \phi \). This broken symmetry implies a phase transition, when in actual fact there is no phase transition (at least in the impurity case). Thus the mean field theory is unphysical. However it does give good results for low temperature thermodynamic properties.

This model can also be examined without assuming the broken symmetry state. Jin and Kuroda (1988) have done this and do obtain good results for both low and high temperature thermodynamics. Thus it is hoped that this model also gives good results for transport properties, specifically for the ac conductivity. The procedure for calculating operator averages, following Jin and Kuroda, using the slave boson technique is described in the next section and appendices. Also included in the second section is the calculation of boson occupation number and f electron occupation number, which will be needed in calculating any physical property. The third section calculates the ac conductivity using the slave boson technique.
CHAPTER 2  FORMALISM

This section describes how operator averages are calculated for the bosonized Anderson Hamiltonian with constraint. Also the Feynman-Dyson perturbation theory is described for this model and the $1/N$ expansion technique.

2.1 The method of including the constraint $Q=1$

To handle the constraint, define a new Hamiltonian $H(\lambda)$ as

$$H(\lambda) = H + \lambda Q = H_0(\lambda) + H_{mix},$$

where

$$H_0(\lambda) = H_{band} + \sum_m (E_m + \lambda) f_m^+ f_m + \lambda b^+ b.$$

Define the grand partition function for $H(\lambda)$ as

$$z(\lambda) = \text{tr}\left\{ e^{-\beta H(\lambda)} \right\},$$

and the average of an operator $A$ as

$$\langle A \rangle_\lambda = \frac{1}{z(\lambda)} \text{tr}\left\{ A e^{-\beta H(\lambda)} \right\}.$$

Since $H_0(\lambda)$ is diagonal and $H_{mix}$ contains an even number of
fermion operators, Wick's theorem can be applied to the non-interacting averages defined by

\[
\langle A \rangle_{\lambda}^{(\omega)} = \frac{1}{z(\lambda)} \operatorname{tr}\left\{ A e^{-\beta H_0(\lambda)} \right\},
\]

where

\[
z(\lambda) = \operatorname{tr}\left\{ e^{-\beta H_0(\lambda)} \right\}.
\]

Feynman-Dyson perturbation theory can be applied to calculate these \( \lambda \) dependent averages.

The problem is how to extract the physical averages from knowledge of \( \langle A \rangle_{\lambda} \) and \( z(\lambda) \). To do this, the entire Fock space, \( \mathcal{F}_\lambda = \mathcal{F}_\lambda^f \otimes \mathcal{F}_\lambda^b \), where \( \mathcal{F}_\lambda^f \) is the Fock space of all fermions and \( \mathcal{F}_\lambda^b \) is the Fock space of bosons, is split into an orthogonal sum of subspaces \( \mathcal{H}(q) \) of the eigenvalues of the Hermitian operator \( Q \), \( q=0,1,2,\ldots \):

\[
\mathcal{F}_\lambda = \bigoplus_{q=0}^{\infty} \mathcal{H}(q),
\]

with

\[
Q |\psi_q \rangle = q |\psi_q \rangle, \text{ for all } |\psi_q \rangle \in \mathcal{H}(q).
\]

Let \( \{ |q\rangle \} \) be a complete orthonormal basis in \( \mathcal{H}(q) \), then the trace over \( \mathcal{F}_\lambda \) can be written as

\[
\operatorname{tr}\{ \cdots \} = \sum_{q=0}^{\infty} \operatorname{tr}_q \{ \cdots \},
\]

\[8\]
where
\[ \text{tr}_q \{ \cdots \} = \sum_{\ell} \langle q \ell | \cdots | q \ell \rangle . \]

The grand partition function then can be written
\[ z(\lambda) = \sum_{q=0}^{\infty} \text{tr}_q \{ e^{-\beta H(\lambda)} \} . \quad (2.9) \]

Since \( H \) commutes with \( Q \),
\[ e^{-\beta (H + \lambda Q)} = e^{-\beta H} e^{-\beta \lambda Q} , \]
and
\[ e^{-\beta \lambda Q} | q \ell \rangle = e^{-\beta \lambda q} | q \ell \rangle , \]
then
\[ z(\lambda) = \sum_{q=0}^{\infty} e^{-\beta \lambda q} \text{tr}_q \{ e^{-\beta H} \} \]
\[ = \sum_{q=0}^{\infty} e^{-\beta \lambda q} \mathcal{Z}(q) , \quad (2.10) \]

where
\[ \mathcal{Z}(q) = \text{tr}_q \{ e^{-\beta H} \} . \quad (2.11) \]

From this the physical subspace of \( q=1 \) must be extracted, i.e., \( \mathcal{Z}(1) \). Differentiating both sides of (2.10) with respect to \( e^{-\beta \lambda} \) gives:
\[ \frac{\partial z(\lambda)}{\partial (e^{-\beta \lambda})} = \sum_{q=0}^{\infty} q e^{-\beta \lambda (q-1)} \mathcal{Z}(q) \]
\[ = \mathcal{Z}(1) + 2 e^{-\beta \lambda} \mathcal{Z}(2) + 3 e^{-2\beta \lambda} \mathcal{Z}(3) + \cdots . \]
Taking the limit $\lambda \to +\infty$ of both sides gives

$$\mathcal{Z}(1) = \lim_{\lambda \to +\infty} \frac{\partial \mathcal{Z}(\lambda)}{\partial (e^{-\beta \lambda})}. \quad (2.12)$$

$\mathcal{Z}(1)$ is precisely the physical grand partition function.

Following a similar procedure for $\langle Q \rangle_\lambda$ gives the relation

$$\lim_{\lambda \to +\infty} e^{-\beta \lambda} \langle Q \rangle_\lambda = \frac{\mathcal{Z}(1)}{\mathcal{Z}(0)}. \quad (2.13)$$

One has

$$\mathcal{Z}(0) = \text{tr}_{q=0} \{ e^{-\beta H(\lambda)} \} = \sum_{\ell} \langle q=0, \ell | e^{-\beta H} | q=0, \ell \rangle = \sum_{\ell} \langle q=0, \ell | e^{-\beta H_{\text{band}}} | q=0, \ell \rangle = z_{\text{band}},$$

since the state $|q=0, \ell \rangle$ contains no $f$ electrons or bosons. Thus (2.13) becomes

$$\mathcal{Z}(1) = z_{\text{band}} \lim_{\lambda \to +\infty} e^{-\beta \lambda} \langle Q \rangle_\lambda. \quad (2.14)$$

Given an operator $A$, consider the average, $\langle QA \rangle_\lambda$. Following the same procedure as above to extract the $q=1$ portion, and using (2.13), gives the physical average $\langle A \rangle$:

$$\langle A \rangle = \lim_{\lambda \to +\infty} \frac{\langle QA \rangle_\lambda}{\langle Q \rangle_\lambda}.$$

If $A|\psi\rangle = 0$ for any $|\psi\rangle \in \mathcal{H}(0)$, then this can be written as
\[ \langle A \rangle = \lim_{\lambda \to +\infty} \frac{\langle A \rangle_{\lambda}}{\langle Q \rangle_{\lambda}} \quad . \quad (2.15) \]

By using (2.14), the free energy \( F \) defined by
\[ F = -\frac{1}{\beta} \ln Z(1) \]
can be written as \( F = F_{\text{band}} + \Delta F \), where
\[ \Delta F = -\frac{1}{\beta} \ln \left\{ \lim_{\lambda \to +\infty} e^{\beta \lambda} \langle Q \rangle_{\lambda} \right\} , \quad (2.16) \]
and
\[ F_{\text{band}} = -\frac{1}{\beta} \ln z_{\text{band}} \quad . \]

As can be seen, to calculate any physical averages, the average \( \langle Q \rangle_{\lambda} \) must be known. The \( \lambda \) dependent averages are in the form of a problem that has no constraint and the standard Feynman-Dyson perturbation theory can be applied. The explanation of the Feynman-Dyson perturbation theory follows and then the calculation of \( \langle Q \rangle_{\lambda} \).

2.2 Feynman-Dyson perturbation theory and the \( 1/N \) expansion

The partition function \( z(\lambda) \) and other \( \lambda \) dependent averages are calculated by Feynman-Dyson temperature perturbation theory and Feynman diagrams. To this end, define
\[ S(\tau) = e^{\tau H_0(\lambda)} e^{-\tau H(\lambda)} \quad . \quad (2.17) \]

By differentiating both sides with respect to \( \tau \) and defining
for any operator \( A \),

\[
A(\tau) = e^{\tau H_0(\lambda) A} e^{-\tau H_0(\lambda)},
\]

(2.18)

(2.17) becomes

\[
\frac{\partial S(\tau)}{\partial \tau} = -H_{\text{mix}}(\tau) S(\tau),
\]

(2.19)

with the boundary condition \( S(0) = 1 \). This differential equation and the boundary condition have the formal solution

\[
S(\tau) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\tau d\tau_1 \cdots \int_0^\tau d\tau_n
\]

\[
\times \ P_\tau(H_{\text{mix}}(\tau_1) \cdots H_{\text{mix}}(\tau_n)),
\]

(2.20)

where \( P_\tau \) is Dyson's imaginary time ordering defined by

\[
P_\tau(a_1(\tau_1)a_2(\tau_2) \cdots a_m(\tau_m))
\]

\[= a_{p_1(\tau_1)} a_{p_2(\tau_2)} \cdots a_{p_m(\tau_m)},
\]

where

\[
\beta \geq \tau \geq 0.
\]

and the operators \( a \) can be fermion and/or boson operators.

Since \( H_{\text{mix}}(\tau) \) contains an even number of fermion operators, Dyson's imaginary time ordering in (2.20) can be replaced by Wick's imaginary time ordering \( T_\tau \) defined by

\[
T_\tau(a_1(\tau_1)a_2(\tau_2) \cdots a_m(\tau_m))
\]

\[= (-1)^{P_\tau} a_{p_1(\tau_1)} a_{p_2(\tau_2)} \cdots a_{p_m(\tau_m)},
\]

(2.21a)

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where
\[ \beta > \tau_{p_1} > \tau_{p_2} > \cdots > \tau_{p_m} > 0, \]  
(2.21b)

and \( p \) is the parity of permutation of fermion operators such that (2.21b) is satisfied. It can be shown that (2.20) can be written as the time ordered exponential
\[ S(\beta) = T_\tau e^{-\int_0^\beta d\tau H_{\text{mix}}(\tau)}, \]
(2.22)

where the time ordered exponential is defined as
\[ T_\tau e^{-\int_0^\beta d\tau A(\tau)} = \lim_{n \to +\infty} e^{-\epsilon A(\tau_n)} e^{-\epsilon A(\tau_{n-1})} \cdots e^{-\epsilon A(\tau_1)}, \]
with \( \epsilon = \tau/n \), and \( \tau_p = p\epsilon, p \in \{1, 2, \ldots, n\} \).

The noninteracting average of \( S(\beta) \) is given by
\[ \langle S(\beta) \rangle^\omega_\lambda = \frac{1}{z_0(\lambda)} \text{tr} \left\{ S(\beta) e^{-\beta H_0(\lambda)} \right\}. \]
(2.23)

Substituting for \( S(\beta) \) from (2.17) gives the relation
\[ z(\lambda) = z_0(\lambda) \langle S(\beta) \rangle^\omega_\lambda. \]
(2.24)

Thus \( z(\lambda) \) can be calculated if \( \langle S(\beta) \rangle^\omega_\lambda \) is found.

The calculation of \( \langle S(\beta) \rangle^\omega_\lambda \) involves evaluating averages of the form
\[ \langle T_\tau (H_{\text{mix}}(\tau_1) \cdots H_{\text{mix}}(\tau_n)) \rangle^\omega_\lambda. \]

This can be evaluated by means of Wick’s theorem. The statement of Wick’s theorem for an even number of \( \tau \) independent operators is
\[ \langle T_\tau(a_1(\tau_1) \cdots a_{2n}(\tau_{2n})) \rangle^{(\omega)} \]

\[ = \sum_{\text{all complete pairings}} \left\{ (-1)^p \right\} \langle T_\tau(a_{p_1}(\tau_{p_1}) a_{p_2}(\tau_{p_2})) \rangle^{(\omega)} \cdots \]

\[ \times \cdots \langle T_\tau(a_{p_{2n-1}}(\tau_{p_{2n-1}}) a_{p_{2n}}(\tau_{p_{2n}})) \rangle^{(\omega)} \]  \hspace{1cm} (2.25)

where the \((-1)^p\) factor is for the case of fermion operators and the factor 1 is for the case of bosons. A single complete pairing for the case of six operators for example is

\[ \langle a_1 a_2 a_3 a_4 a_5 a_6 \rangle^{(\omega)} : \langle a_1 a_2 \rangle^{(\omega)} \langle a_3 a_4 \rangle^{(\omega)} \langle a_5 a_6 \rangle^{(\omega)} \]

The basic averages that are needed are of the form

\[ \langle a a' \rangle^{(\omega)} = \frac{1}{Z_\omega(\lambda)} \text{tr} \left\{ a a' e^{-\beta H_\omega(\lambda)} \right\} , \]  \hspace{1cm} (2.26)

where \(a, a'\) can be conduction electron, \(f\) electron, or boson operators. Since

\[ a'(\beta) = e^{\beta H_\omega(\lambda)} a' e^{-\beta H_\omega(\lambda)} \]

can be shown (by the Baker-Hausdorff theorem) that

\[ e^A e^B = B + \frac{1}{1!} [A,B] + \frac{1}{2!} [A,[A,B]] + \cdots , \]

for any two operators \(A\) and \(B\), to be equal to

\[ a'(\beta) = e^{\pm \beta \varepsilon'} , \]  \hspace{1cm} (2.27)

where the upper sign is for the case when \(a'\) is a creation
operator and the lower sign is for the case when $a'$ is a destruction operator, and

$$s' = \begin{cases} 
\varepsilon_k & \text{if } a' \text{ is a conduction electron operator}, \\
E_f + \lambda & \text{if } a' \text{ is a f electron operator}, \\
\lambda & \text{if } a' \text{ is a boson operator}.
\end{cases}$$

Thus

$$a'e^{-\beta H_0(\lambda)} = e^{\pm \beta \varepsilon'} e^{-\beta H_0(\lambda)} a'.$$

Using the invariance of the trace under a cyclic permutation of factors gives

$$<aa'>^{(\infty)}_\lambda = e^{\pm \beta \varepsilon'} \frac{1}{Z_0(\lambda)} \text{tr} \left\{ a'e^{-\beta H_0(\lambda)} a' \right\}$$

$$= e^{\pm \beta \varepsilon'} <a'a>^{(\infty)}_\lambda.$$ (2.28)

Since the $a$ operators may be of the Fermi type or the Bose type, consider first the case of $a$ and $a'$ are both fermion operators, then

$$a'a = \{a, a'\} - aa',$$

where $\{a, a'\} = aa' + a'a$ is a complex number. Substituting this into (2.28) gives

$$<aa'>^{(\infty)}_\lambda = \frac{\{a, a'\}}{1 + e^{\pm \varepsilon' \beta}}.$$ (2.29)

If $a$ and $a'$ are both boson operators, then

$$<aa'>^{(\infty)}_\lambda = \frac{[a, a']}{1 - e^{\pm \varepsilon' \beta}}.$$ (2.30)
For the case of commuting a and a', by (2.28)

\[ \langle aa' \rangle_\lambda^{(0)} (1 - e^{\pm \beta \varepsilon'}) = 0. \]

For \( T > 0 \) and \( \varepsilon' \neq 0 \), this implies that

\[ \langle aa' \rangle_\lambda^{(0)} = \langle a'a \rangle_\lambda^{(0)} = 0. \]  \hspace{1cm} (2.31)

Thus more specifically to the present problem,

\[ \langle c_{km} b \rangle_\lambda^{(0)} = \langle c_{km} b^+ \rangle_\lambda^{(0)} = \langle c_{km}^+ b \rangle_\lambda^{(0)} = \langle c_{km}^+ b^+ \rangle_\lambda^{(0)} = 0, \]

and similarly for the combinations of f electron operators and boson operators. The analogous Wick's theorem for a product of fermion and boson operators is

\[
\langle T_\tau (\alpha_1 (\tau_1) \cdots \alpha_n (\tau_n) \beta_1 (\tau'_1) \cdots \beta_m (\tau'_m) ) \rangle_\lambda^{(0)} = \langle T_\tau (\alpha_1 (\tau_1) \cdots \alpha_n (\tau_n) ) \rangle_\lambda^{(0)}
\]

\[
\times \langle T_\tau (\beta_1 (\tau'_1) \cdots \beta_m (\tau'_m) ) \rangle_\lambda^{(0)},
\]  \hspace{1cm} (2.32)

where \( \alpha \) represents a fermion operator and \( \beta \) represents a boson operator. The averages involving only fermion or boson operators can be evaluated by (2.25).

Thus in principle \( \langle S(\beta) \rangle_\lambda^{(0)} \) can be calculated. It is convenient to introduce a diagrammatic representation of the different pairings of the terms in the product of \( H_{\text{mix}}(\tau) \) operators appearing in the perturbation expansion for \( \langle S(\beta) \rangle_\lambda^{(0)} \). With the terms in \( H_{\text{mix}}(\tau) \),
\[ \sum_{k,m} V c^+_{km}(\tau)b^+_{m}(\tau)f_m(\tau) , \sum_{k,m} V^* f^+_{m}(\tau)b(\tau)c_{km}(\tau) \]

associate the diagrams shown in figures 2.1a and 2.1b, where $V$ is an average value of $V_k$. The first term in the expansion of $\langle S(\beta) \rangle^{(0)}_\lambda$ contains an average,

\[
\langle T_\tau (H_{\text{mix}}(\tau)) \rangle^{(0)}_\lambda = \sum_{k,m} \left[ V \langle T_\tau (c^+_{km}(\tau)f^+_{m}(\tau)b(\tau)) \rangle^{(0)}_\lambda 
+ V^* \langle T_\tau (f^+_{m}(\tau)c^+_{km}(\tau)b(\tau)) \rangle^{(0)}_\lambda \right].
\]

Since the boson operator commutes with the fermion operators, these two averages are both zero. Similarly, no odd powers in the perturbation expansion will contribute, since they will contain an odd number of boson operators. Therefore any diagram will contain an even number of interaction vertices. The first nonzero contribution is then the second order term given by

\[
\frac{(-1)^2}{2!} \int_0^\beta \int_0^\beta \langle T_\tau (H_{\text{mix}}(\tau_1)H_{\text{mix}}(\tau_2)) \rangle^{(0)}_\lambda \tag{2.33}
\]

By Wick's theorem, the average can be reduced to
Figure 2.1a) The diagrammatic representation of $V_{c_{km}}^+(\tau)b^+ (\tau)f_m(\tau)$.

Figure 2.1b) The diagrammatic representation of $V_{m}^+(\tau)b(\tau)c_{km}(\tau)$. 
\[
\frac{1}{2} \sum_{k_{1} m_{1} k_{2} m_{2}} \sum_{\tau_{1} \tau_{2}} |V|^2 \left\{ T_{\tau_{1}} (c_{k_{1} m_{1}}^{+} (\tau_{1}) c_{k_{2} m_{2}} (\tau_{2})) \right\}^{\lambda} \left\{ T_{\tau_{2}} (f_{m_{1}}^{+} (\tau_{2}) f_{m_{2}} (\tau_{2})) \right\}^{\lambda}
\]

\[
\times \left\{ T_{\tau_{1}} (b^{+}(\tau_{1}) b(\tau_{2})) \right\}^{\lambda} \right\}
\]

\[
+ \left\{ T_{\tau_{1}} (c_{k_{1} m_{1}} (\tau_{1}) c_{k_{2} m_{2}}^{+} (\tau_{2})) \right\}^{\lambda} \left\{ T_{\tau_{2}} (f_{m_{1}}^{+} (\tau_{2}) f_{m_{2}} (\tau_{2})) \right\}^{\lambda}
\]

\[
\times \left\{ T_{\tau_{1}} (b(\tau_{1}) b^{+}(\tau_{2})) \right\}^{\lambda}
\]

(2.34)

By the definition of Wick's theorem (2.25) for fermion operators,

\[
\left\{ T_{\tau_{1}} (f_{m_{1}}^{+} (\tau_{1}) f_{m_{2}} (\tau_{2})) \right\}^{\lambda} = -\left\{ T_{\tau_{2}} (f_{m_{2}} (\tau_{2}) f_{m_{1}}^{+} (\tau_{1})) \right\}^{\lambda},
\]

and similarly for the conduction electron operators. By defining the bare propagators (Green's functions) as

\[
\begin{align*}
G_{\lambda}^{(\emptyset)}(k m \tau; k' m' \tau') &= -\left\{ T_{\tau} (c_{k m}^{+} (\tau) c_{k' m'}^{+} (\tau')) \right\}^{\lambda}, \\
G_{\lambda}^{(\emptyset)}(m \tau; m' \tau') &= -\left\{ T_{\tau} (f_{m}^{+} (\tau) f_{m'}^{+} (\tau')) \right\}^{\lambda}, \\
D_{\lambda}^{(\emptyset)}(\tau; \tau') &= -\left\{ T_{\tau} (b(\tau) b^{+}(\tau')) \right\}^{\lambda},
\end{align*}
\]

(2.35)

(2.34) becomes

\[
\frac{1}{2} \sum_{k_{1} m_{1} k_{2} m_{2}} \sum_{\tau_{1} \tau_{2}} |V|^2 \left\{ G_{\lambda}^{(\emptyset)}(k_{1} m_{1} \tau_{1}; k_{2} m_{2} \tau_{2}) \right\} \left\{ G_{\lambda}^{(\emptyset)}(m_{1} \tau_{1}; m_{2} \tau_{2}) \right\} D_{\lambda}^{(\emptyset)}(\tau_{2}; \tau_{1})
\]

\[
+ G_{\lambda}^{(\emptyset)}(k_{1} m_{1} \tau_{1}; k_{2} m_{2} \tau_{2}) G_{\lambda}^{(\emptyset)}(m_{2} \tau_{2}; m_{2} \tau_{1}) D_{\lambda}^{(\emptyset)}(\tau_{1}; \tau_{2}) \right\}
\]

(2.34a)

The diagrammatic representation of the first term is shown in
figure 2.2a and the second in figure 2.2b. It should be noted that the propagators defined here are not the physical propagators, they are just representations of different pairings coming from terms in $<S(\beta)>^{(\omega)}_{\lambda}$.

The imaginary time ordering in the definition of the propagators can be eliminated by writing them as

$$G^{(\omega)}_c(\tau \tau' ; k' m' \tau') = -\theta(\tau' - \tau') <c_{k m}(\tau) c_{k' m}(\tau')^{+}>^{(\omega)}_{\lambda} + \theta(\tau - \tau') <c_{k m}^{+}(\tau') c_{k m}(\tau)>^{(\omega)}_{\lambda},$$

where $\theta(x)$ is the Heaviside step function defined by

$$\theta(x) = \begin{cases} 
1 & \text{if } x > 0, \\
0 & \text{if } x < 0.
\end{cases}$$

Using the fact that the trace is invariant under a cyclic permutation of factors, gives

$$<c_{k m}(\tau) c_{k' m}(\tau')^{+}>^{(\omega)}_{\lambda} = <c_{k m}(\tau - \tau') c_{k' m}^{+}(0)>^{(\omega)}_{\lambda},$$

$$<c_{k' m}(\tau') c_{k m}(\tau)>^{(\omega)}_{\lambda} = <c_{k' m}^{+}(0) c_{k m}(\tau - \tau')>^{(\omega)}_{\lambda}.$$  

Thus only the difference $\tau - \tau'$ appears, and the propagator is written as

$$G^{(\omega)}_c(k m, k' m' ; \tau) = -<T_\tau (c_{k m}(\tau) c_{k' m}^{+}(0))>^{(\omega)}_{\lambda}.$$  

The above argument also applies to the $f$ electron and boson propagators. Applying equations 2.27 and 2.29 for the case of fermion operators, gives:
Figure 2.2a) The diagrammatic representation of
\[ G^{(O)}_{\alpha\lambda}(k_2 m_2 \tau_2; k_1 m_1 \tau_1)G^{(O)}_{\beta\lambda}(m_1 \tau_1; m_2 \tau_2)D^{(O)}_{\gamma\lambda}(\tau_2; \tau_1). \]

Figure 2.2b) The diagrammatic representation of
\[ G^{(O)}_{\alpha\lambda}(k_1 m_1 \tau_1; k_2 m_2 \tau_2)G^{(O)}_{\beta\lambda}(m_2 \tau_2; m_1 \tau_1)D^{(O)}_{\gamma\lambda}(\tau_1; \tau_2). \]
\[
\langle c_{km}(\tau)c^+_{k',m'}(0)\rangle^{(\omega)}_{\lambda} = \delta_{kk'}\delta_{mm'}e^{-\tau\varepsilon_k}f^+(\varepsilon_k),
\]
\[
\langle c^+_{k',m'}(0)c_{km}(\tau)\rangle^{(\omega)}_{\lambda} = \delta_{kk'}\delta_{mm'}e^{-\tau\varepsilon_k}f^-(\varepsilon_k),
\]
where \( f^-(x) \) is the Fermi function,
\[
f^-(x) = \frac{1}{e^{\beta x} + 1},
\]
and \( f^+(x) = 1 - f^-(x) \). Thus the propagators can be written
\[
G^{(\omega)}_{c\lambda}(k\tau; k'm') = \delta_{kk'}\delta_{mm'}G^{(\omega)}_{c\lambda}(k\tau; \tau' - \tau')
\]
\[
= \delta_{kk'}\delta_{mm'}e^{-(\tau - \tau')\varepsilon_k}
\]
\[
\times \left\{ -\theta(\tau - \tau')f^+(\varepsilon_k) + \theta(\tau' - \tau)f^-(\varepsilon_k) \right\}, \quad (2.36a)
\]
\[
G^{(\omega)}_{f\lambda}(m\tau; m'\tau') = \delta_{mm'}G^{(\omega)}_{f\lambda}(m; \tau' - \tau')
\]
\[
= \delta_{mm'}e^{-(\tau - \tau')(E_f + \lambda)}
\]
\[
\times \left\{ -\theta(\tau - \tau')f^+(E_f + \lambda) + \theta(\tau' - \tau)f^-(E_f + \lambda) \right\}, \quad (2.36b)
\]
\[
D^{(\omega)}_{\lambda}(\tau; \tau') = D^{(\omega)}_{\lambda}(\tau - \tau')
\]
\[
= -e^{-(\tau - \tau')\lambda}\left\{ \theta(\tau - \tau')(1 + n(\lambda)) + \theta(\tau' - \tau)n(\lambda) \right\}, \quad (2.36c)
\]
where \( n(x) \) is the Bose function,
\[
n(x) = \frac{1}{e^{\beta x} - 1}.
\]
The second order contribution to \( \langle S(\beta) \rangle^{(\omega)}_{\lambda} \) from (2.34)
becomes
\[ \int_{\tau_1}^{\beta} \int_{\tau_2}^{\beta} \sum_{km} |V|^2 G^{(O)}_{c\lambda}(km;\tau_1-\tau_2)G^{(O)}_{f\lambda}(m;\tau_1-\tau_2)D^{(O)}_{\lambda}(\tau_2-\tau_1). \]

By proceeding in a similar manner for the other terms in the expansion for \( <S(\beta)>^{(O)}_{\lambda} \), a diagrammatic expansion results. The rules for assigning an analytic expression to a diagram contributing to \( <S(\beta)>^{(O)}_{\lambda} \) are given in appendix I.

The integrations over \( \tau \) variables are difficult to perform and may be eliminated by introducing the Fourier transforms of the propagators. The price is that summations are introduced. However, these summations can be performed with greater ease than the integrations. The propagators are only defined for \(-\beta \leq \tau \leq \beta, \tau \neq 0\). Consider the conduction electron propagator from (2.38a) for the case \( 0 < \tau < \beta \):

\[ G^{(O)}_{c\lambda}(km;\tau-\beta) = \delta_{mm} e^{-(\tau-\beta)\epsilon_k} f^-(\epsilon_k), \]

since \(-\beta < \tau - \beta < 0\). However,

\[ e^{\beta\epsilon_k} f^-(\epsilon_k) = 1 - f^-(\epsilon_k) = f^+(\epsilon_k), \]

and

\[ G^{(O)}_{c\lambda}(km;\tau-\beta) = - G^{(O)}_{c\lambda}(km;\tau), \]

thus \( G^{(O)}_{c\lambda}(km;\tau) \) is anti-periodic in the interval \(-\beta < \tau < \beta\) with anti-period \( \beta \). One extends the definition of \( G^{(O)}_{c\lambda}(km;\tau) \) and \( G^{(O)}_{f\lambda}(m;\tau) \), \( D^{(O)}_{\lambda}(\tau) \) over the whole real axis via
\[
G_{c,\lambda}^{(\omega)}(km;\tau + 2n\beta) = G_{c,\lambda}^{(\omega)}(km;\tau) ,
\]
\[
G_{f,\lambda}^{(\omega)}(m;\tau + 2n\beta) = G_{f,\lambda}^{(\omega)}(m;\tau) ,
\]
\[
D_{\lambda}^{(\omega)}(\tau + n\beta) = D_{\lambda}^{(\omega)}(\tau) , \quad n \in \mathbb{Z} ,
\]

where \( \mathbb{Z} \) represents the set of integers. The propagators may now be expanded in a Fourier series:

\[
G_{c,\lambda}^{(\omega)}(km;\tau) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} G_{c,\lambda}^{(\omega)}(km;in\pi/\beta)e^{-in\pi\tau/\beta} .
\]

The Fourier coefficients, \( G_{c,\lambda}^{(\omega)}(km;in\pi/\beta) \) are given by

\[
G_{c,\lambda}^{(\omega)}(km;in\pi/\beta) = \frac{1}{2} \int_{-\beta}^{+\beta} d\tau \; G_{c,\lambda}^{(\omega)}(km;\tau)e^{in\pi\tau/\beta} .
\]

Substituting for \( G_{c,\lambda}^{(\omega)}(km;\tau) \) from (2.36a) and performing the integral gives

\[
G_{c,\lambda}^{(\omega)}(km;in\pi/\beta) = \begin{cases}
0 & \text{if } n \text{ is even} , \\
\frac{1}{in\pi/\beta - \epsilon_k} & \text{if } n \text{ is odd} .
\end{cases}
\]

By defining the fermion Matsubara frequency \( i\omega_n \) as

\[
i\omega_n = (2n+1)i\pi/\beta , \quad n \in \mathbb{Z} ,
\]

and proceeding similarly for the other propagators gives:
\[ G_{\alpha}^{(o)}(k; \imath \omega_n) = \frac{1}{\imath \omega_n - \varepsilon_k}, \quad (2.37a) \]
\[ G_{\beta}^{(o)}(\omega_n) = \frac{1}{\imath \omega_n - E_f - \lambda}, \quad (2.37b) \]
\[ D_{\lambda}^{(o)}(\imath \nu_n) = \frac{1}{\imath \nu_n - \lambda}, \quad (2.37c) \]

where \( \imath \nu_n \) is the boson Matsubara frequency defined as
\[ \imath \nu_n = \frac{2n \pi \imath}{\beta}, \quad n \in \mathbb{Z}. \]

A general \( 2n \)-th order term contributing to \( \langle S(\beta) \rangle_{\lambda}^{(o)} \) will contain \( 2n \) integrations over \( \tau \) variables and a product of \( 3n \) propagators, thus will be of the form
\[ \int_0^\beta d\tau_1 \cdots \int_0^\beta d\tau_n G_{\alpha}^{(o)}(k; \pm \imath \tau_\alpha) G_{\beta}^{(o)}(m; \pm \imath \tau_\beta) D_{\lambda}^{(o)}(\pm \imath \nu_\lambda) \cdots. \]

The integral over \( \tau_\alpha \) is
\[ \int_0^\beta d\tau_\alpha G_{\alpha}^{(o)}(k; \pm \imath \tau_\alpha) G_{\beta}^{(o)}(m; \pm \imath \tau_\beta) D_{\lambda}^{(o)}(\pm \imath \nu_\lambda). \]

Substituting the Fourier series expansions of the propagators gives

\[ \frac{1}{\beta^3} \sum_{\omega_n} \sum_{\omega_{n'}} \sum_{\nu_n} \sum_{\nu_{n'}} \frac{e^{\imath \omega_n \tau_\alpha}}{\imath \omega_n - \varepsilon_k} \frac{e^{\imath \omega_{n'} \tau_\beta}}{\imath \omega_{n'} - E_f - \lambda} \frac{e^{\imath \nu_n \tau_\lambda}}{\imath \nu_n - \lambda} \]
\[ \times \int_0^\beta d\tau_1 e^{-i(\tau_\omega \omega_n' \nu_n') \tau_1}. \]
where
\[ \sum_{n=-\infty}^{+\infty} \ldots = \sum_{n=-\infty}^{+\infty} \ldots, \]

and similarly for the sums involving the boson Matsubara frequencies. However,
\[ \int_0^\beta d\tau_1 e^{-i(\pm \omega_n \tau_1, \pm \nu_n)\tau_1} = \beta\delta(\pm \omega_n \tau_1, \pm \nu_n), \]

where \( \delta(x) \) is the Dirac delta function. The physical meaning of the delta function appearing is analogous to conservation of energy (i.e., the sum of frequencies entering a vertex is equal to the sum of frequencies leaving the vertex). Each Fourier series expansion of the 3n propagators introduces a factor \( 1/\beta \), and each of the 2n integrations introduces a factor \( \beta \), therefore the overall factor is \( 1/\beta^n \). A new type of diagram is now introduced, where the \( \tau \) variables are replaced with energy variables (Matsubara frequencies). The rules for assigning an expression to an energy variable diagram contributing to \( <S(\beta)>_{\lambda}^{+\infty} \) are laid out in appendix I. All following calculations will be obtained by using this type of diagram.

The calculation of the partition function, \( z(\lambda) \) can now be carried out using Feynman diagrams. From this, the physical partition function can be calculated from (2.12). There is an enormous number of diagrams to be considered. The
number can be reduced by recalling the constraint $Q=1$. Consider the diagrams in figure 2.3 and their contributions to $\langle S(\beta) \rangle^{(\alpha)}_\lambda$, $\mathcal{D}_{\alpha\alpha}$, $\mathcal{D}_{\alpha\beta}$, $\mathcal{D}_{\beta\beta}$. According to the rules in appendix I, their contributions to $\langle S(\beta) \rangle^{(\alpha)}_\lambda$ are:

$$\mathcal{D}_{\alpha\alpha} = -\frac{\beta N |V|^2}{2} \sum_k \frac{f^-(\varepsilon_k) - f^-(E_f+\lambda)}{\varepsilon_k - E_f} \left( n(\lambda) + f^-(E_f+\lambda) \right)$$

$$= -\frac{\beta N |V|^2}{2} \sum_k \frac{f^-(\varepsilon_k)}{\varepsilon_k - E_f} e^{-\beta \lambda} + O(e^{-2\beta \lambda}), \text{ for } \lambda \gg 1.$$  

$$\mathcal{D}_{\alpha\beta} = -\frac{\beta N |V|^4}{4!} \left( n(\lambda) + f^-(E_f+\lambda) \right)^2 \sum_{k,k'} \frac{f^-(E_f) - f^-(\varepsilon_k)}{(\varepsilon_k - \varepsilon_{k'})(\varepsilon_k - E_f)}$$

$$\approx O(e^{-2\beta \lambda}), \text{ for } \lambda \gg 1.$$  

$$\mathcal{D}_{\beta\beta} = 2N^2 |V|^4 \sum_{k,k'} \sum_{i \omega_n} \sum_{i \omega_{n'}} \frac{1}{i \omega_n - \varepsilon_k} \frac{1}{i \omega_{n'} - \varepsilon_k} \frac{1}{i \omega_n - i \omega_{n'}} \frac{1}{i \omega_{n'} - E_f} \times \left[ \frac{1 + e^{-\beta E_f}}{i \omega_n - E_f} + \beta e^{\beta \lambda} \langle n(\lambda) \rangle^2 \right]$$

$$\approx O(e^{-2\beta \lambda}), \text{ for } \lambda \gg 1.$$
2.3a) The second order Feynman diagram $D_{3a}$ contributing to $\langle S(\beta) \rangle^{(0)}_{\lambda}$.

2.3b) A fourth order Feynman diagram $D_{3b}$ contributing to $\langle S(\beta) \rangle^{(0)}_{\lambda}$.

2.3c) A fourth order Feynman diagram $D_{3c}$ contributing to $\langle S(\beta) \rangle^{(0)}_{\lambda}$.
Although the frequency summations appearing in $D_{ac}$ have not been carried out, the explicit dependence on $\lambda$ is indicated. From (2.12), to find $\mathcal{Z}(1)$, the constraint $Q=1$ is enforced by taking the derivative of $\langle S(\beta) \rangle^\omega_\lambda$ with respect to $e^{-\beta \lambda}$ and the limit $\lambda \rightarrow +\infty$. Since

$$
\lim_{\lambda \rightarrow +\infty} \frac{\partial}{\partial (e^{-\beta \lambda})} e^{-\beta \lambda} = 1, \quad \text{and}
$$

$$
\lim_{\lambda \rightarrow +\infty} \frac{\partial}{\partial (e^{-\beta \lambda})} e^{-p\beta \lambda} = 0, \quad \text{for } p > 1,
$$

only the terms in $\langle S(\beta) \rangle^\omega_\lambda$ proportional to $e^{-\beta \lambda}$ will give a nonzero contribution to $\mathcal{Z}(1)$. Examining figure 2.3a, it is seen that the diagram contains one closed $f$ electron-boson loop (from now on denoted by a Q loop) and its contribution to $\langle S(\beta) \rangle^\omega_\lambda$ is $O(e^{-\beta \lambda})$, and thus gives a nonzero contribution to $\mathcal{Z}(1)$. Figure 2.3c also contains only one Q loop and its contribution to $\langle S(\beta) \rangle^\omega_\lambda$ is $O(e^{-\beta \lambda})$. Figure 2.3b contains two Q loops and its contribution to $\langle S(\beta) \rangle^\omega_\lambda$ is $O(e^{-2\beta \lambda})$ and therefore will contribute nothing to $\mathcal{Z}(1)$. Hence a diagram for $\langle S(\beta) \rangle^\omega_\lambda$ that contains $p > 1$ Q loops is of the order $e^{-p\beta \lambda}$ and gives no contribution to $\mathcal{Z}(1)$, and it need not be considered.

Although considering only the diagrams with a single Q loop reduces the number of diagrams, there still is a tremendous number of diagrams to be considered. However, one
can exploit the largeness of $N$ (the degeneracy of the $f$ level) and classify all the terms in a given order of perturbation expansion in $V$ according to powers of $1/N$. For example, consider the contributions of the diagrams in figs. 2.3b and 2.3c. Both are fourth order in $V$ (i.e., there are four interaction vertices). However in diagram 2.3b there is only one sum over the quantum number $m$ labeling the degenerate $f$ states, while in diagram 2.3c there is a double sum over $m$. Thus, since in zero magnetic field the propagators do not depend on $m$, $D_{ab}$ is proportional to $N|V|^4 = (1/N)(N|V|^2)^2$ and $D_{bc}$ is proportional to $(N|V|^2)^2$. Hence $D_{ab} = D_{bc} \times O(1/N)$. From this example it is clear that a simple counting of the number of independent conduction electron-$f$ electron loops can be used to classify all the diagrams of a given order in $V$ according to powers of $1/N$. Next, the theory pushes this one step further in that all terms in the perturbation expansion are classified according to the power of $1/N$, regardless of their order in $V$. Formally this is achieved by considering $N|V|^2$ to be of the order 1. This will ensure that no positive powers of $N$ appear in the expansion. The ultimate justification of this procedure for summing the perturbation series is the agreement between the theoretical predictions and the experiment.
2.3 The full (interacting) propagators

As seen from (2.15), to calculate any physical averages, the average $\langle Q \rangle_\lambda$ must be calculated. By defining the boson occupation number $n^\lambda_b$ and the fermion electron occupation number $n^\lambda_f$ as

$$n^\lambda_b = \langle b^+b \rangle_\lambda , \quad (2.38a)$$

$$n^\lambda_f = \sum_m \langle f^+_m f_m \rangle_\lambda , \quad (2.38b)$$

then

$$\langle Q \rangle_\lambda = n^\lambda_b + n^\lambda_f .$$

Note that these averages are not the noninteracting averages as they are given by (2.4). To calculate the occupation numbers, the full (interacting) propagators must be found. Consider first the boson occupation number, and define the full boson propagator, $D^\lambda(\tau)$ by

$$D^\lambda(\tau) = -<T_\tau(b(\tau)b^+(0))>_\lambda , \quad -\beta<\tau<+\beta, \tau \neq 0 , \quad (2.39)$$

where for any operator $A$, $\tilde{A}(\tau)$ is defined as

$$\tilde{A}(\tau) = e^{\tau H(\lambda)} A e^{-\tau H(\lambda)} .$$

Consider the case $-\beta<\tau<0$, then from (2.4)
\[ D_\lambda(\tau) = - \frac{1}{z(\lambda)} \text{tr} \left\{ e^{-\beta H(\lambda)} e^{\tau H(\lambda)} b e^{-\tau H(\lambda)} b^+ \right\} \]

\[ = \frac{1}{\langle S(\beta) \rangle^\omega} \langle S(\beta) S^{-1}(\tau) b(\tau) S(\tau) b^+(0) \rangle^\lambda, \]

where \( S(\tau) \) is given by (2.17). From the formal definition of Wick's ordering,

\[ S(\beta)S^{-1}(\tau) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_\tau^\beta d\tau_1 \cdots \int_\tau^\beta d\tau_n \; T_\tau (H_{\text{mix}}(\tau_1) \cdots H_{\text{mix}}(\tau_n)) , \]

and

\[ S(\beta)S^{-1}(\tau) b(\tau) S(\tau) b^+(0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_\tau^\beta d\tau_1 \cdots \int_\tau^\beta d\tau_n \]

\[ \times T_\tau (H_{\text{mix}}(\tau_1) \cdots H_{\text{mix}}(\tau_n) b(\tau) b^+(0)) . \]

Therefore

\[ D_\lambda(\tau) = - \frac{1}{\langle S(\beta) \rangle^\omega} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_\tau^\beta d\tau_1 \cdots \int_\tau^\beta d\tau_n \]

\[ \times \langle T_\tau (H_{\text{mix}}(\tau_1) \cdots H_{\text{mix}}(\tau_n) b(\tau) b^+(0)) \rangle^\lambda. \quad (2.40) \]

Now the averages appearing in \( D_\lambda(\tau) \) are the noninteracting averages and can be computed by Wick's theorem.

As with \( \langle S(\beta) \rangle^\omega \), there are no odd order contributions to \( D_\lambda(\tau) \). Consider the first nonzero contribution to \( D(\tau) \). The average appearing is

\[ \langle T_\tau (H_{\text{mix}}(\tau_1) H_{\text{mix}}(\tau_2) b(\tau) b^+(0)) \rangle^\lambda. \]
By using Wick's theorem and the definition of the bare propagators (2.35), this can be reduced to

\[
\begin{align*}
&= - \sum_{km} |V|^2 \left\{ G^{(O)}_{\omega \lambda}(km; \tau_2 - \tau_1) G^{(O)}_{\gamma \lambda}(m; \tau_1 - \tau_2) \\
&\quad \times \left[ D^{(O)}_{\omega \lambda}(\tau) D^{(O)}_{\gamma \lambda}(\tau_2 - \tau_1) + D^{(O)}_{\omega \gamma}(\tau - \tau_1) D^{(O)}_{\gamma \lambda}(\tau_2) \right] \\
&\quad + G^{(O)}_{\omega \lambda}(km; \tau_1 - \tau_2) G^{(O)}_{\gamma \lambda}(m; \tau_2 - \tau_1) \\
&\quad \times \left[ D^{(O)}_{\omega \lambda}(\tau) D^{(O)}_{\gamma \lambda}(\tau_2 - \tau_1) + D^{(O)}_{\omega \gamma}(\tau - \tau_1) D^{(O)}_{\gamma \lambda}(\tau_2) \right] \right\}. 
\end{align*}
\]

The diagrammatic representation of this is shown in figure 2.4a. As seen in figure 2.4a, there are two kinds of diagrams, connected and disconnected. The second and fourth diagrams are connected and the first and the third are disconnected. Note that up to second order in $V$ the sum of the diagrams of fig. 2.4a can be represented as in fig. 2.4b. The second factor in fig. 2.4b is just $\langle S(\beta) \rangle^{(O)}_{\lambda}$ calculated to the second order in $V$, while the first factor is the sum of all connected diagrams for $D^{(O)}_{\lambda}(\tau)$ up to the second order in $V$. The second factor is canceled by $\langle S(\beta) \rangle^{(O)}_{\lambda}$ appearing in the denominator in the expression (2.40) for $D^{(O)}_{\lambda}(\tau)$ when calculated to second order in $V$. Therefore to second order in $V$, 

\[33\]
Figure 2.4a) The second order Feynman diagrams contributing to $D_\lambda(\tau)$.

Figure 2.4b) Representation of $D_\lambda(\tau)$ valid to second order in terms of connected diagrams. Note that the second factor is the second order contribution to $<S(\beta)>_\lambda^{(0)}$. 

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\[ D_\lambda(\tau) = - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\tau d\tau_1 \cdots \int_0^\tau d\tau_n \]

\[ \times \langle T_{\tau\tau}(H_{\text{mix}}(\tau_1) \cdots H_{\text{mix}}(\tau_n) b(\tau) b^+(0)) \rangle^{(\omega)_{\text{conn.}}}_{\lambda} \]

\[ = - \langle T_{\tau} e^{-\int_0^\tau d\tau' H_{\text{mix}}(\tau')} b(\tau) b^+(0) \rangle^{(\omega)_{\text{conn.}}}_{\lambda}, \]  

(2.41)

where \( \langle \cdots \rangle^{(\omega)_{\text{conn.}}}_{\lambda} \) signifies that only connected diagrams are considered. A more general combinatorial argument shows that this is correct to all orders in \( V \). The Feynman rules for calculating \( D_\lambda(\tau) \) are given in appendix II. Again by introducing the Fourier series expansions of the bare propagators, the full boson propagator can be calculated in energy variables, thus giving \( D_\lambda(i\nu_n) \). The rules for calculating \( D_\lambda(i\nu_n) \) are also found in appendix II.

The nature of the diagrams for \( D_\lambda(i\nu_n) \) is that there is one bare boson propagator entering and exiting the diagram. The full propagator can then be written in the form

\[ D_\lambda(i\nu_n) = D^{(\omega)}_\lambda(i\nu_n) + D^{(\omega)}_\lambda(i\nu_n) \Pi_{\text{total}}^\lambda(i\nu_n) D^{(\omega)}_\lambda(i\nu_n), \]  

(2.42)

shown in diagrammatic form in figure 2.5, where \( \Pi_{\text{total}}^\lambda(i\nu_n) \) is called the total boson self-energy. The total self-energy consists of so called proper (irreducible) and improper (reducible) diagrams. A diagram is proper if it does not fall into two pieces by cutting a single wavy line. A proper
Figure 2.5) The diagrammatic representation of $D_\lambda(\tau)$ in terms of the total self-energy, $\Pi^\lambda_{\text{total}}$. 
self-energy diagram is shown in figure 2.6a, and an improper one in figure 2.6b. The sum of all proper self-energy terms is denoted by \( \Pi_\lambda(i\nu_n) \), and (2.42) can be written in terms of just the proper self-energy:

\[
D_\lambda(i\nu_n) = D_\lambda^{(\infty)}(i\nu_n) + D_\lambda^{(\infty)}(i\nu_n)\Pi_\lambda(i\nu_n)D_\lambda^{(\infty)}(i\nu_n) \\
+ D_\lambda^{(\infty)}(i\nu_n)\Pi_\lambda(i\nu_n)D_\lambda^{(\infty)}(i\nu_n)\Pi_\lambda(i\nu_n)D_\lambda^{(\infty)}(i\nu_n) + \cdots \\
= D_\lambda^{(\infty)}(i\nu_n) + D_\lambda^{(\infty)}(i\nu_n)\Pi_\lambda(i\nu_n)D_\lambda(i\nu_n). \tag{2.43}
\]

This is called the Dyson series for \( D_\lambda(i\nu_n) \) and is shown in diagrammatic form in figure 2.7. Equation (2.43) can be summed as a geometric series to read

\[
D_\lambda(i\nu_n) = \frac{1}{(D_\lambda^{(\infty)}(i\nu_n))^{-1} - \Pi_\lambda(i\nu_n)}. \tag{2.44}
\]

This is the Dyson equation for \( D_\lambda(i\nu_n) \).

An analogous treatment of the full f electron propagator defined by

\[
G_{f\lambda}(m;\tau) = -\langle T_\tau(\tilde{f}_m(\tau)\tilde{f}_m^+(0)) \rangle_\lambda, \quad -\beta < \tau < \beta, \quad \tau \neq 0,
\]

gives similar results. The Feynman rules for the full f electron propagator are identical to those for the full boson propagator with the exception that the external wavy lines are replaced with dashed lines. The Dyson series and
Figure 2.6a) A proper (irreducible) boson self-energy diagram.

Figure 2.6b) An improper (reducible) boson self-energy diagram.
2.7

\[ D_\lambda = D_\lambda^{(0)} + D_\lambda^{(0)} + \Pi_\lambda + D_\lambda \]

Figure 2.7) The diagrammatic representation of the Dyson series for \( D_\lambda(i\nu_n) \).
The equation for \( G_{f\lambda}(m;i\omega_n) \) are

\[
G_{f\lambda}(m;i\omega_n) = G_{f\lambda}^{(0)}(m;i\omega_n) + G_{f\lambda}^{(0)}(m;i\omega_n)\Sigma_{f\lambda}(m;i\omega_n)G_{f\lambda}^{(0)}(m;i\omega_n)
\]

\[+ G_{f\lambda}^{(0)}(m;i\omega_n)\Sigma_{f\lambda}(m;i\omega_n)G_{f\lambda}^{(0)}(m;i\omega_n) \]

\[\times \Sigma_{f\lambda}(m;i\omega_n)G_{f\lambda}^{(0)}(m;i\omega_n) + \cdots , \quad (2.45)\]

\[
G_{f\lambda}(m;i\omega_n) = \frac{1}{(G_{f\lambda}^{(0)}(m;i\omega_n))^{-1} - \Sigma_{f\lambda}(m;i\omega_n)} , \quad (2.46)
\]

where \( \Sigma_{f\lambda}(m;i\omega_n) \) is the total \( f \) electron irreducible self-energy.

The boson and \( f \) electron occupation numbers can now be expressed in terms of the full propagators:

\[
n^\lambda_b = \langle b^* b \rangle^\lambda
\]

\[
= - \lim_{\tau \to -0^+} \langle -iT_\tau (\tilde{b}(\tau)\tilde{b}^*(0)) \rangle^\lambda ,
\]

where the limit \( \tau \to -0^+ \) is needed since Wick's ordering is not defined for equal arguments. Therefore

\[
n^\lambda_b = - \lim_{\tau \to -0^+} D^-_\lambda(\tau) = D_-\lambda(-0^+) ,
\]

or in terms of Matsubara frequencies,

\[
n^\lambda_b = - \frac{1}{\beta} \sum_{\Omega_n} D^-_\lambda(i\nu_n)e^{i\nu_n\Omega_n^+} , \quad (2.47a)
\]

and similarly for the \( f \) electron occupation number.
\[ n_f^\lambda = \frac{1}{\beta} \sum_m \sum_{\omega_n} G_{f\lambda}(m; i\omega_n) e^{i\omega_n \sigma}. \]  \
(2.47b)

Hence, the full propagators must be found to calculate \( \langle Q \rangle_\lambda \), and hence any physical quantity.

Calculating the propagators to all orders is hopeless. Recall that to project out the physical subspace \( Q=1 \), the limit \( \lambda \rightarrow +\infty \) must be taken. Hence, as with \( \langle S(\beta) \rangle^{(O)}_\lambda \), only those terms which are \( O(\varepsilon^{-1/\lambda}) \) contribute. To calculate physical quantities to \( O(1/N) \), \( \langle Q \rangle_\lambda \) must be calculated to \( O(1/N) \). To accomplish this, the self-energies are expanded to \( O(1/N) \). First consider the full boson propagator. The self-energy is expanded as

\[ \Pi_\lambda(i\nu_n) = \Pi^{(4)}_\lambda(i\nu_n) + \Pi^{(1/N)}_\lambda(i\nu_n), \]

where \( \Pi^{(3)}_\lambda(i\nu_n) \) is the \( O(1) \) contribution and \( \Pi^{(1/N)}_\lambda(i\nu_n) \) is the \( O(1/N) \) contribution. Substituting this into the Dyson series for \( D_\lambda(i\nu_n) \) and retaining only terms up to and including \( O(1/N) \) gives

\[ D_\lambda(i\nu_n) = D^{(4)}_\lambda(i\nu_n) + \Delta D^{(1/N)}_\lambda(i\nu_n), \]
\( (2.48) \)

where

\[ D^{(4)}_\lambda(i\nu_n) = D^{(O)}_\lambda(i\nu_n) + D^{(O)}_\lambda(i\nu_n) \Pi^{(4)}_\lambda(i\nu_n) D^{(4)}_\lambda(i\nu_n), \]

and

\[ \Delta D^{(1/N)}_\lambda(i\nu_n) = D^{(4)}_\lambda(i\nu_n) \Pi^{(1/N)}_\lambda(i\nu_n) D^{(4)}_\lambda(i\nu_n). \]
The expression for $D^{(\omega)}_{\lambda}(i\nu_n)$ is the Dyson series for $D^{(\omega)}_{\lambda}(i\nu_n)$ and is shown diagrammatically in figure 2.8. The corresponding Dyson equation is

$$D^{(\omega)}_{\lambda}(i\nu_n) = \frac{1}{(D^{(\omega)}_{\lambda}(i\nu_n))^{-1} - \Pi^{(\omega)}_{\lambda}(i\nu_n)} \quad (2.48)$$

There is one diagram (figure 2.9) contributing to the boson self-energy of $O(1)$ satisfying the constraint $Q=1$. The Feynman rules for the boson self-energy diagrams are identical to those for $D_{\lambda}(i\nu_n)$ with the exception that the external boson lines are excluded. Therefore, in the limit of large $\lambda$:

$$\Pi^{(\omega)}_{\lambda}(i\nu_n) = N|V|^2 \sum_k \frac{f^-(\epsilon_k)}{i\nu_n + \epsilon_k - E_f - \lambda}$$

By assuming a flat conduction electron density of states,

$$\rho(\epsilon) = \begin{cases} \rho(0), & |\epsilon| < D, \\ 0, & |\epsilon| > D, \end{cases}$$

where $D$ is half the bandwidth and is taken to be the largest energy scale in the model, the sum over $k$ can be converted to the density of states integral, giving
Figure 2.8) The diagrammatic representation of the Dyson series for $D^{(1)}(i\nu_n)$. 

$D_{\lambda}^{(II)} = D_{\lambda}^{(o)} + D_{\lambda}^{(o)} + D_{\lambda}^{(o)} + T_{\lambda}^{(I)} + D_{\lambda}^{(II)}$
Figure 2.9) The O(1) irreducible boson self-energy diagram, $\Pi^{(1)}_{\lambda}$. 
\[ \Pi^{(i)}_{\lambda}(i\nu_n) = N|V|^2 \rho(0) \int_{-D}^{+D} d\varepsilon \frac{f^-(\varepsilon)}{i\nu_n + \varepsilon - E_f - \lambda} . \quad (2.50) \]

The O(1/N) diagram contributing to the boson self-energy is shown in figure 2.10 and in the limit of large \( \lambda \) is

\[ \Pi^{(1/N)}_{\lambda}(i\nu_n) = N|V|^4 \rho^2(0) \int_{-D}^{+D} d\varepsilon f^-(\varepsilon) \int_{-D}^{+D} d\varepsilon' f^-(\varepsilon') \]
\[ \times D^{(i)}_{\lambda}(i\nu_n + \varepsilon + \varepsilon') (G^{(0)}_{\lambda}(i\nu_n + \varepsilon'))^2 . \quad (2.51) \]

With \( D_{\lambda}(i\nu_n) \) found to \( O(1/N) \), the boson occupation number can be calculated to \( O(1/N) \).

The calculation of the full \( f \) electron propagator is similar, however since \( G_{\lambda}(m;i\omega_n) \) is independent of \( m \) for zero magnetic field, from (2.47b) \( n^\lambda_f \) is

\[ n^\lambda_f = \frac{N}{\beta} \sum_{i\omega_n} G_{\lambda}(i\omega_n) e^{i\omega_n} . \]

Thus to calculate \( n^\lambda_f \) to \( O(1/N) \), \( G_{\lambda}(i\omega_n) \) must be known to \( O(1/N^2) \). The simplest \( f \) electron self-energy diagram is shown in figure 2.11. This diagram is \( O(1/N) \), hence there is no \( O(1) \) contribution to \( \Sigma_{\lambda}(i\omega_n) \), therefore to calculate \( G_{\lambda}(i\omega_n) \) to \( O(1/N^2) \), the self-energy is expanded as

\[
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\]
Figure 2.10) The $O(1/N)$ irreducible boson self-energy diagram, $\Pi^{(1/N)}_\lambda$. 
Figure 2.11) The $O(1/N)$ irreducible f electron self-energy diagram, $\Sigma^{(1/N)}_{f\lambda}$.
\[ \Sigma_{f\lambda}(i\omega_n) = \Sigma_{f\lambda}^{(1/N)}(i\omega_n) + \Sigma_{f\lambda}^{(1/N)^2}(i\omega_n). \]

Substituting this into the Dyson series for \( G_{f\lambda}(i\omega_n) \) and retaining only those terms up to and including \( O(1/N^2) \) gives

\[ G_{f\lambda}(i\omega_n) = G_{f\lambda}^{(0)}(i\omega_n) + \Delta G_{f\lambda}^{(1/N)}(i\omega_n) + \Delta G_{f\lambda}^{(1/N)^2}(i\omega_n), \quad (2.52) \]

where

\[ \Delta G_{f\lambda}^{(1/N)}(i\omega_n) = G_{f\lambda}^{(0)}(i\omega_n) \Sigma_{f\lambda}^{(1/N)}(i\omega_n) G_{f\lambda}^{(0)}(i\omega_n), \quad (2.53a) \]

\[ \Delta G_{f\lambda}^{(1/N)^2}(i\omega_n) = (G_{f\lambda}^{(0)}(i\omega_n))^{-1} (\Delta G_{f\lambda}^{(1/N)}(i\omega_n))^2 \]

\[ + G_{f\lambda}^{(0)}(i\omega_n) \Sigma_{f\lambda}^{(1/N)^2}(i\omega_n) G_{f\lambda}^{(0)}(i\omega_n). \quad (2.53b) \]

The diagram for \( \Sigma_{f\lambda}^{(1/N)^2}(i\omega_n) \) is given in figure 2.12. According to the Feynman rules and for the limit of large \( \lambda \):

\[ \Delta G_{f\lambda}^{(1/N)}(i\omega_n) = |V|^2 \rho(0) \int_{-\beta}^{+\beta} d\varepsilon f^-(\varepsilon) \]

\[ \times D_{\lambda}^{(4)}(i\omega_n + \varepsilon) (G_{f\lambda}^{(0)}(i\omega_n))^2, \quad (2.54a) \]
2.12

\[ \sum_{f, \lambda} \frac{1}{N^2} \]

Figure 2.12) The $O(1/N^2)$ irreducible $f$ electron self-energy diagram, $\Sigma^{(1/N^2)}_{f\lambda}$. 

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\[ \Delta G^{(1/N)}_{f\lambda}(i\omega_n) = (G^{(0)}_{f\lambda}(i\omega_n))^{-1} \Delta G^{(1/N)}_{f\lambda}(i\omega_n)^2 \]
\[ + N \left| V \right| \rho^3(0) \int_{-D}^{+D} d\varepsilon_1 f^-(\varepsilon_1) \int_{-D}^{+D} d\varepsilon_2 f^-(\varepsilon_2) \int_{-D}^{+D} d\varepsilon_3 f^-(\varepsilon_3) \]
\[ \times \delta_\lambda^{(1)}(i\omega_n + \varepsilon_1 + \varepsilon_2 + \varepsilon_3) (D^{(1)}_{\lambda}(i\omega_n + \varepsilon_3))^2 \]
\[ \times (G^{(0)}_{f\lambda}(i\omega_n + \varepsilon_2 + \varepsilon_3))^2 (G^{(0)}_{f\lambda}(i\omega_n))^2 . \] (2.54b)

Now \( \langle Q \rangle_\lambda \) can be calculated to \( O(1/N) \).

2.4 Calculation of \( \langle Q \rangle_\lambda \) to \( O(1/N) \)

In calculating \( \langle Q \rangle_\lambda \), it is useful to find the poles of the \( O(1) \) boson propagator. They are given by the solution of

\[ 0 = (D^{(1)}_{\lambda}(z))^{-1} = z - \lambda - \Pi^{(1)}_{\lambda}(z) , \quad z \in \mathbb{C} , \]

or

\[ 0 = z + E_f - \Pi^{(1)}_{\lambda}(z + E_f + \lambda) , \]

where \( \mathbb{C} \) represents the set of complex numbers. To solve this, let \( z = z_1 + iz_2 \), where \( z_1 = \text{Re} z \) and \( z_2 = \text{Im} z \), thus after separating the real and imaginary parts, the following two equations must be simultaneously satisfied:

\[ 0 = z_1 + E_f - \text{Re}\Pi^{(1)}_{\lambda}(z + E_f + \lambda) , \quad (2.55a) \]
\[ 0 = z_2 - \text{Im}\Pi^{(1)}_{\lambda}(z + E_f + \lambda) . \quad (2.55b) \]
From (2.50), the boson self-energy is

\[ \Pi^{\omega}(z + E_f + \lambda) = \frac{\Delta_0}{\pi} \int_{-\infty}^{+\infty} dy \frac{f^{-}(y)}{y + z}, \tag{2.56} \]

where \( \Delta_0 = \pi N |V|^2 \rho(0) \). Taking the imaginary part of this and substituting into (2.55b) gives

\[ 0 = z_2 \left[ 1 + \frac{\Delta_0}{\pi} \int_{-\infty}^{+\infty} dy \frac{f^{-}(y)}{(y + z_1)^2 + z_2^2} \right]. \]

Since the term in brackets is always larger than zero, the only solution of this is \( z_2 = 0 \). Therefore \( D^{\omega}_{\lambda}(z) \) has poles only on the real axis. Now, (2.55a) must be solved with \( \text{Im} z = 0 \). To this end the integral in (2.56) has to be evaluated. This may be done by the Sommerfeld expansion which is useful for low temperatures but not high temperatures. An alternate method valid for both low and high temperatures will be described.

Since \( D \) is the largest energy, and since the Fermi function appears in the numerator of the integrand of (2.56), then the upper limit can be replaced with \( +\infty \). To replace the lower limit by \( -\infty \), a cutoff function must be introduced. For temperatures \( T \ll \Omega \), the cutoff function can be taken to be

\[ 1 - f^{-}(y + D) = f^{-}(-(y + D)) \]

thus giving

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\[ \Pi_\lambda^{(4)}(z + E_f + \lambda) \simeq \frac{A_\lambda}{\pi} \int_{-\infty}^{+\infty} dy \frac{f^-(y)f^-(y+D)}{y+z} \]

The integrand has simple poles at
\[ y=-z, \quad y=\pm \omega_n, \quad y=\pm i\omega_n-D, \]
where \( \omega_n=(2n+1)\pi i/\beta, \ n\in\mathbb{Z} \). A useful theorem for performing integrals of this type (D.S. Mitrović and J.D. Kečkić, 1984) states that given a function \( h \) analytic in the upper half complex plane \( \{ w \in \mathbb{C} | \text{Im} w > 0 \} \), with singularities \( w_1, \ldots, w_m \); and having only simple poles on the real axis \( \{ w \in \mathbb{C} | \text{Im} w = 0 \} \), \( p_1, \ldots, p_q \) such that \( \lim_{w \to \infty} wh(w) = \alpha, \ \alpha \in \mathbb{C} \), then
\[
\int_{-\infty}^{+\infty} dx \ h(x) = 2\pi i \sum_{k=1}^{m} \text{Res}_{w=w_k} h(w) + \pi i \sum_{k=1}^{q} \text{Res}_{w=p_k} h(w) - i\pi \alpha.
\]

Another useful theorem (D.S. Mitrović and J.D. Kečkić, 1984) states that given two functions \( f \) and \( g \) analytic at \( z=a \) where \( a \) is a simple zero of \( g \) and \( f(a) \neq 0 \), then
\[
\text{Res}_{z=a} \frac{f(z)}{g(z)} = \frac{f(a)}{g'(a)}.
\]

Applying these two theorems gives

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\[ \Pi_{\lambda}^{(z)}(z + E_{t+} \lambda) = \frac{\Delta_0}{\pi} \left\{ 2\pi i \sum_{n=0}^{\infty} \left( \frac{1}{-\beta(i\omega_n + z)(1 - e^{-\beta D})} \right) \right. \\
\left. \quad + \frac{1}{\beta(i\omega_n - D + z)(1 - e^{-\beta D})} \right\} \right. \\
\left. + i\Delta_0 f^-(z)f^-(z-D) \right\} \\
\begin{aligned}
2 \text{ if } \text{Im} z < 0, \\
1 \text{ if } \text{Im} z = 0, \\
0 \text{ if } \text{Im} z > 0.
\end{aligned} \tag{2.57}

The last term can be understood as follows. Since only poles not in the lower half complex plane contribute to the integral, if \text{Im}(-z)<0 (\text{Im} z>0), then the pole is in the lower half complex plane and contributes nothing. If \text{Im}(-z)>0 (\text{Im} z<0) then the pole is in the upper half complex plane and contributes \(2\pi i\) times the residue at \(-z\). If \text{Im}(-z)=0 then the pole at \(-z\) is on the real axis and contributes \(\pi i\) times the residue at \(-z\). Substituting for \(i\omega_n\) in (2.57) and using \(D \gg T\) and hence \(e^{-\beta D} \approx 0\), then

\[ \Pi_{\lambda}^{(z)}(z + E_{t+} \lambda) = \frac{\Delta_0}{\pi} \left[ -\psi \left( \frac{1}{2} + \frac{i(D-z)}{2\pi T} \right) + \psi \left( \frac{1}{2} + \frac{iz}{2\pi T} \right) \right] \]

\[ + i\Delta_0 f^-(z)f^-(z-D) \left\{ \begin{aligned}
2 \text{ if } \text{Im} z < 0, \\
1 \text{ if } \text{Im} z = 0, \\
0 \text{ if } \text{Im} z > 0.
\end{aligned} \right. \tag{2.58} \]

where \(\psi(z)\) is the digamma function defined by

\[ \psi(z) = \frac{d}{dz} \ln \Gamma(z) = -\gamma + \sum_{n=0}^{\infty} \left( \frac{1}{n + 1} - \frac{1}{n + z} \right), \]
where \( \gamma \) is Euler's constant (\( \gamma \approx 0.5772 \)).

The solution of (2.55) requires that \( \text{Imz} = 0 \), therefore let \( \omega = \text{Rez} \), hence

\[
\Pi^{(4)}_{\lambda}(\omega + E_{r} + \lambda) = \frac{\Delta}{\pi} \left[ -\psi \left( \frac{1}{2} + \frac{i(D+\omega)}{2\pi T} \right) + \psi \left( \frac{1}{2} + \frac{i\omega}{2\pi T} \right) \right] \\
+ i\Delta \Phi(-\omega)^{-}(\omega-D).
\]  

(2.58)

Since \( D > |\omega| \) and by using the relation

\[
\text{Im} \psi \left( \frac{1}{2} + iy \right) = \frac{\pi}{2} \tanh \pi y
\]

gives

\[
\text{Im} \Pi^{(4)}_{\lambda}(\omega + E_{r} + \lambda) = 0.
\]

Substituting the real part of \( \Pi^{(4)}_{\lambda}(\omega + E_{r} + \lambda) \) into (2.55a) gives

\[
0 = \omega + E_{r} + \frac{\Delta}{\pi} \left[ \text{Re} \psi \left( \frac{1}{2} + \frac{i(D-\omega)}{2\pi T} \right) - \text{Re} \psi \left( \frac{1}{2} + \frac{i\omega}{2\pi T} \right) \right].
\]

Consider first the low temperature solution, \( \omega = -T_{k} \), where \( T < T_{k} \). Since \( D \) is the largest energy, \( D \gg T_{r} \), and \( D \gg T_{k} \). \( T_{k} \) is then defined by

\[
0 = -T_{k} + E_{r} + \frac{\Delta}{\pi} \left[ \text{Re} \psi \left( \frac{1}{2} + \frac{iD}{2\pi T} \right) - \text{Re} \psi \left( \frac{1}{2} + \frac{iT_{k}}{2\pi T} \right) \right].
\]  

(2.60)

The asymptotic expansion for the digamma function is (M. Abramowitz and I.A. Stegun)
\[ \psi(z) \sim \ln z - \frac{1}{2z} - \sum_{n=1}^{\infty} \frac{(-1)^n B_{2n}}{2nz^{2n}}, \text{ for } |z| \to \infty \]

\[ = \ln z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} + \frac{1}{252z^6} + \cdots, \]

where \( B_{2n} \) are the Bernoulli numbers. Substituting this expansion for \( \psi(z) \) in (2.60), gives to second order in \( T/T_k \):

\[ 0 = T_k - E_f + \frac{A_0}{\pi} \left[ \ln \frac{T}{T_k} - \frac{1}{6} \left( \frac{nT}{T_k} \right)^2 \right]. \tag{2.61} \]

For \( T=0 \), denote the solution of this equation by \( T_{ko} \), where \( T \) is defined by

\[ 0 = T_{ko} - E_f + \frac{A_0}{\pi} \ln \frac{T}{T_{ko}}. \tag{2.62} \]

Since \( E_f < 0 \), then \( E_f = -|E_f| \), and for \( T_{ko} \ll |E_f| \), \( T_{ko} \) can be written as

\[ T_{ko} \approx D e^{-\frac{\pi |E_f|}{A_0}} = D e^{-\frac{|E_f|}{N|V|^2 \rho(0)}}. \tag{2.63} \]

This is called the Kondo temperature and is seen to be analytic in \( 1/N \), but not in the strength of the mixing \( |V| \).

The solution of (2.61) \( T_k \) can be written to accuracy \( T_k^2 \approx T_{ko}^2 \) as

\[ T_k \approx T_{ko} \left[ 1 + \frac{1}{6} (1 - a_0) \left( \frac{nT}{T_{ko}} \right)^2 \right], \tag{2.64} \]

where
\[ a_0 = \frac{1}{1 + \frac{\Delta}{\pi T_{k_0}}} . \]

There is another solution at low temperatures given by \( \omega = T^* + D \), where \( T \ll T^* \). The equation defining \( T^* \) is then

\[ 0 = -T^* \left( |E_f| - D \right) \]

\[ + \frac{\Delta^*}{\pi} \left[ \Re \psi \left( \frac{1}{2} + \frac{iD}{2\pi T} \right) - \Re \left( \frac{1}{2} + \frac{iT^*}{2\pi T} \right) \right]. \quad (2.65) \]

This is the same as (2.60) with \( E_f \) replaced by \( |E_f| - D \). Note that since \( D \) is the largest energy, then \( |E_f| - D < 0 \).

Next, consider the high temperature solution \( \omega = T_H \). \( T_H \) is given by

\[ 0 = T_H + E_f \]

\[ + \frac{\Delta}{\pi} \left[ \Re \psi \left( \frac{1}{2} + \frac{i(D - T_H)}{2\pi T} \right) - \Re \left( \frac{1}{2} + \frac{iT_H}{2\pi T} \right) \right]. \quad (2.66) \]

Taking \( T \gg T_H \), and using the asymptotic expansion of \( \psi(z) \) to write

\[ \Re \psi \left( \frac{1}{2} + \frac{i(D - T_H)}{2\pi T} \right) \approx \ln \frac{D}{2\pi T} , \]

and

\[ \Re \left( \frac{1}{2} + \frac{iT_H}{2\pi T} \right) \approx \psi \left( \frac{1}{2} \right) = -\gamma - 2\ln 2 , \]

gives
\[ T_H \simeq -T_{k_0} + \frac{\Delta_0 \epsilon_n}{\pi} \left( \frac{\pi T}{2 \Delta_0 T_{k_0}} \right). \] (2.67)

For large enough \( T \) (\( T \gg T_{k_0} \)), this does give \( T \gg T_H \). Summarizing, \( D^{(1)}(z + E_f + \lambda) \) has poles at
\[ z = -T_k, \quad z = D + T^*, \quad \text{for } T \ll T_k, \quad \text{and} \]
\[ z = T_H, \quad \text{for } T \gg T_k. \]

From (2.47), the boson and \( f \) electron occupation numbers can now be calculated to \( O(1/N) \) in the two regimes, \( T \ll T_k \) and \( T \gg T_k \). The details are described in appendix III, and the results are as follows. For \( T \ll T_k \):

\[
\begin{align*}
    n_b^b & \simeq e^{-\beta(E_f - T_k + \lambda)} \left\{ a(1 + \beta \Delta T_k) + \frac{a \Delta_0^2}{\pi^2 N} \int_0^D \! d\varepsilon \! f^-(\varepsilon) \\
    & \times \int_{-D}^{+D} \! d\varepsilon' f^-(\varepsilon') \frac{1}{(\varepsilon - T_k)^2} D^{(1)}(E_f - T_k + \varepsilon + \varepsilon' + \lambda) \\
    & \times \left[ 2a \left( b \left( 0 \right) - \frac{1}{\varepsilon - T_k} \right) + D^{(1)}(E_f - T_k + \varepsilon + \varepsilon' + \lambda) \\
    & \times \left( 1 - a - \frac{a \Delta_0}{\pi} \int_{-D}^{+D} \! dy \frac{f^-(y)}{(y + \varepsilon + \varepsilon' - T_k)^2} \right) \right] \right\}, \quad (2.68a)
\end{align*}
\]

\[
    n_f^b \simeq e^{-\beta(E_f - T_k + \lambda)} (1 + \beta \Delta T_k) - n_b^b + N e^{-\beta(E_f + \lambda)}. \quad (2.68b)
\]

In the above,
\[
\begin{align*}
a &= \left[ 1 + \frac{\Delta}{\pi} \int_{-\mathcal{D}}^{+\mathcal{D}} dy \frac{f^-(y)}{(y - T_k)^2} \right]^{-1} \\
\kappa a_0 &= \left[ 1 - \frac{1}{6} (1 - a_0^2) \left( \frac{\pi T}{T_{k_0}} \right)^2 \right], \quad (2.88) \\
\Delta T_k &= -\frac{\Delta^2}{\pi^2 N} \int_{-\mathcal{D}}^{+\mathcal{D}} d\varepsilon f^-(\varepsilon) \int_{-\mathcal{D}}^{+\mathcal{D}} d\varepsilon' f^-(\varepsilon') \frac{1}{(\varepsilon - T_k)^2} \\
\times D^{(4)}(E_f - T_k + \varepsilon + \varepsilon' + \lambda), \quad (2.70) \\
B(x) &= \frac{\Delta}{\pi^2} \int_{-\mathcal{D}}^{+\mathcal{D}} dy \frac{f^-(y)}{(y - T_k + x)^3}. \quad (2.71)
\end{align*}
\]

For \(T \gg T_k\):

\[
\begin{align*}
n^b_{\lambda} &\approx b e^{-\beta(E_f + \lambda)} \quad , \quad (2.72a) \\
n^f_{\lambda} &\approx N e^{-\beta(E_f + \lambda)} \quad , \quad (2.72b)
\end{align*}
\]

where

\[
b = \frac{\pi}{\Delta} \int_{-\mathcal{D}}^{+\mathcal{D}} dx f^-(x) \left[ (\pi f^-(-x))^2 \\
+ \left\{ \frac{\pi}{\Delta} (x + T_{k_0}) - \ln \frac{2nT}{T_{k_0}} - \text{Re} \left[ \frac{1}{2} + \frac{i x}{2nT} \right] \right\}^2 \right]^{-1}. \quad (2.73)
\]

Therefore to \(O(1/N)\), \(\langle Q \rangle_{\lambda}\) is
\[ <Q>_{\lambda} = \begin{cases} 
  e^{-\beta(E_f - T_k + \lambda)} (1 + \beta \Delta T_k) & \text{if } T \ll T_k', \\
  (N + b) e^{-\beta(E_f + \lambda)} & \text{if } T \gg T_k. 
\end{cases} \tag{2.74} \]
CHAPTER 3  THE AC CONDUCTIVITY

3.1 The conduction electron self-energy

This section derives the ac conductivity for the two regimes $T \ll T_k$, and $T \gg T_k$. To this end, the full conduction electron propagator must be found. It is given by

$$G_{\alpha \lambda}(km; \tau) = -\langle T_\tau \{ \tilde{c}_k(\tau) \tilde{c}_k^+(0) \} \rangle_{\alpha \lambda}, \quad -\beta < \tau < \beta, \quad \tau \neq 0. \quad (3.1)$$

The Feynman rules are exactly analogous with those for the full boson or $f$ electron propagator. The corresponding Dyson series and equation for $G_{\alpha \lambda}(km; i\omega_n)$ are

$$G_{\alpha \lambda}(km; i\omega_n) = G_{\alpha \lambda}^{(0)}(km; i\omega_n) + G_{\alpha \lambda}^{(0)}(km; i\omega_n) \times \Sigma_{\alpha \lambda}(km; i\omega_n)G_{\alpha \lambda}^{(0)}(km; i\omega_n)$$

$$+ G_{\alpha \lambda}^{(0)}(km; i\omega_n) \Sigma_{\alpha \lambda}(km; i\omega_n)G_{\alpha \lambda}^{(0)}(km; i\omega_n) \times \Sigma_{\alpha \lambda}(km; i\omega_n)G_{\alpha \lambda}^{(0)}(km; i\omega_n) + \cdots, \quad (3.2)$$

$$G_{\alpha \lambda}(km; i\omega_n) = \frac{1}{(G_{\alpha \lambda}^{(0)}(km; i\omega_n))^{-1} - \Sigma_{\alpha \lambda}(km; i\omega_n)}, \quad (3.3)$$

where $\Sigma_{\alpha \lambda}(km; i\omega_n)$ is the total irreducible conduction electron self-energy.
As with the $f$ electron self-energy, there is no $O(1)$ contribution to $\Sigma_{c\lambda} (km; i\omega_n)$. The $O(1/N)$ contribution, $\Sigma_{c\lambda} (km; i\omega_n)$ is shown in figure 3.1 and is given by

$$\Sigma_{c\lambda}^{(1/N)} (km; i\omega_n) = -\frac{|V|^2}{\beta} \sum_{i\nu_n} D^{(\omega)} (i\nu_n) G_p^{(\omega)} (i\omega_n + i\nu_n).$$

(3.4)

The frequency sum can be evaluated for the two cases $T \ll T_k$ and $T \gg T_k$ by the contour integral method. Then (3.4) is

$$\Sigma_{c\lambda}^{(1/N)} (km; i\omega_n) = \frac{|V|^2}{2\pi i} \int_{\Gamma} dz \, n(z) D^{(\omega)} (z) G_p^{(\omega)} (i\omega_n + z).$$

(3.5)

For $T \ll T_k$, the integrand has simple poles at $z=E_f+\lambda-i\omega_n$, $z=E_f-T_k+\lambda$, $z=E_f+D+T^*$. Again, since the contribution from the most negative pole ($z=E_f-T_k+\lambda$) dominates, (3.5) becomes, in the limit of large $\lambda$

$$\Sigma_{c\lambda}^{(1/N)} (km; i\omega_n) \approx a |V|^2 e^{-\beta(E_f-T_k+\lambda)} \frac{1}{i\omega_n - T_k}.$$  

(3.6a)

For $T \gg T_k$, the integrand has simple poles at $z=E_f+\lambda-i\omega_n$, $z=E_f+T_H+\lambda$. Therefore in the limit of large $\lambda$

$$\Sigma_{c\lambda}^{(1/N)} (km; i\omega_n) \approx |V|^2 e^{-\beta(E_f+\lambda)} \left\{ \frac{s' e^{-\beta T_H}}{i\omega_n + T_H} \right\}$$

$$+ D_{\lambda}^{(1)} (E_f-i\omega_n+\lambda),$$

(3.6b)

where $a'$ is given in (III.8). Note that $\Sigma_{c\lambda}^{(1/N)} (km; i\omega_n)$ is
Figure 3.1) The $O(1/N)$ irreducible conduction electron self-energy, $\Sigma^{(1/N)}_{C\lambda}$. 
independent of \( k \) and for zero magnetic field it is independent of \( m \) and thus will be written as \( \Sigma_{0\lambda}(i\omega_n) \). To find the physical self-energy, one must divide by \( \langle Q \rangle_\lambda \) and take the limit \( \lambda \to +\omega \), hence the physical self-energy is given by

\[
\Sigma_c(i\omega_n) = \lim_{\lambda \to +\omega} \frac{\Sigma_{0\lambda}(i\omega_n)}{\langle Q \rangle_\lambda}
\]

Substituting for \( \langle Q \rangle_\lambda \) for \( T \ll T_k \) from (2.74) gives

\[
\Sigma_c(i\omega_n) = \frac{a|V|^2}{i\omega_n - T_k}, \quad T \ll T_k. \tag{3.7a}
\]

For \( T \gg T_k \), and using \( \langle Q \rangle_\lambda \) from (III.9) gives

\[
\Sigma_c(i\omega_n) = \frac{|V|^2}{N + a'e^{-\beta T_H}} \left\{ \frac{a'e^{-\beta T_H}}{i\omega_n + T_H} + D_\lambda(E_f - i\omega_n + \lambda) \right\}. \tag{3.7b}
\]

By substituting the self-energy (3.7) in (3.3), the full conduction electron propagator can be found for low and high temperatures.

3.2 The effective mass

From the self-energy, the effective mass, \( m^* \) can be found. In general it is given by (G. Mahan, 1981)
\[
\frac{m^*}{m} = \left. \frac{1 - \frac{\partial}{\partial \omega} \Sigma_c(\omega + i\omega^+)}{1 + \frac{\partial}{\partial \omega} \Sigma_c(\omega + i\omega^+)} \right|_{\omega = 0, \omega = 0}^{k=k_F}.
\]

For the case here, the self-energy is independent of \(k\) and hence \(\varepsilon_k\), and therefore this reduces to

\[
\frac{m^*}{m} = 1 - \frac{\partial}{\partial \omega} \Sigma_c(\omega + i\omega^+) \bigg|_{\omega = 0}.
\]  

(3.8)

Substituting for the self-energy from (3.7) gives

\[
\frac{m^*}{m} = \begin{cases} 
1 + a|V|^2 / T^2_k & \text{if } T \ll T_k, \\
1 - |V|^2 / (NT^2_k) & \text{if } T \gg T_k.
\end{cases}
\]  

(3.8)

Taking \(T=0\) for the low temperature case, the mass ratio becomes

\[
\frac{m^*}{m} = 1 + \frac{a_0|V|^2}{T^2_k}.
\]

This may be cast in the form

\[
\frac{m^*}{m} = 1 + \frac{n^2_r|V|^2}{\Delta^2_0} \frac{(n_f^{(1)}(0))^2}{1 - n_f^{(1)}(0)},
\]  

(3.10)

where \(n_f^{(1)}(0)\) is the \(O(1)\) contribution to the \(f\) electron number at \(T=0\). It is given by
\[ n_f^{(1)}(0) = \lim_{\lambda \to +\infty} \frac{n_f^{(1)}}{\langle Q \rangle^{(1)}_{\lambda}} \bigg|_{T=0} = 1 - a_0, \]

where \( n_f^{(1)} \) is the \( O(1) \) piece of (2.63b),

\[ n_f^{(1)} = (1 - a)e^{-\beta(E_f - T_k + \lambda)}, \]

and \( \langle Q \rangle^{(1)}_{\lambda} \) is the \( O(1) \) piece of (2.74),

\[ \langle Q \rangle^{(1)}_{\lambda} = e^{-\beta(E_f - T_k + \lambda)}. \]

In the Kondo limit, the occupancy of the \( f \) level approaches one, hence due to the factor \( 1 - n_f^{(1)}(0) \) in the denominator of (3.10), the effective mass becomes very large. For the high temperature case, from (2.67) for \( T \gg T_{ko} \)

\[ T_H \propto \frac{\Delta_0}{\Delta} \ln \frac{\pi T}{2e\gamma T_{ko}}, \]

hence

\[ \frac{m^*}{m} = 1 - \frac{\pi^2 |V|^2}{\Delta_0^2} \frac{1}{N \ln^2 \frac{\pi T}{2e\gamma T_{ko}}}. \]

Since \( N \) is large and \( T \gg T_{ko} \), the denominator is very large and hence the effective mass to actual mass ratio is very close to one. Thus at low temperatures the mass is enhanced while at high temperatures there is no mass enhancement. As an illustration of the size of the effective mass consider the
parameters, \( D=100 \) and \( \bar{E}_f=-0.1\bar{D} \), \( N=6 \), where the barred quantities are in units of \( \Delta_0/\pi \). With these numbers, from (2.63) \( T_{ko}\approx4.540\times10^{-3} \), and \( n_{f}^{(4)}(0)\approx0.9855 \). Indeed this is in the Kondo limit and for \( T<T_{ko} \), from above,

\[
\frac{m^*}{m} \approx 1 + 200|\bar{V}|^2 .
\]

For the high temperature case, by taking \( T=100T_{ko} \) gives

\[
\frac{m^*}{m} \approx 1 - \frac{|\bar{V}|^2}{100} .
\]

It should be noted that the low temperature mass ratio given by (3.9) agrees with the one obtained by assuming the broken symmetry state (A.J. Millis and P.A. Lee, 1987).

3.3 The conductivity

To find the conductivity, scattering off the nonmagnetic impurities must be taken into account. To do this one introduces a term into the Hamiltonian to describe the conduction electrons scattering off the impurities,

\[
H_{imp} = \sum_{kk'm} V_{imp} c_{km}^+ c_{k'm} ,
\]

where \( V_{imp} \) is the strength of the impurity scattering. It is assumed that the total electron self-energy averaged over the nonmagnetic impurity configurations (G. Rickayzen, 1980, A.A.
Abrikosov et al., 1963) is given by the sum of the self-energy $\Sigma_c$ due to electron-boson scattering as found in section 3.1 and of $\Sigma_{imp}$ due to electron-ordinary impurity scattering calculated in the Born approximation and shown in figure 3.2 (G. Rickayzen, 1980), where

$$\Sigma_{imp}(km;\omega+i\tau) = -\frac{isgn\omega}{2\tau}, \quad (3.11)$$

and $\tau$ is the relaxation time of the conduction electrons due to ordinary impurity scattering.

First, consider the self-energy when $T \ll T_k$. From (3.7a)

$$\Sigma_c(\omega+i\tau) = a|V|^2 \left( \Re \omega - \frac{1}{T_k + i\omega} - i\pi s(\omega - T_k) \right),$$

where $s(x)$ is the Dirac delta function. For $\omega \approx T_k$, $\Sigma_c(\omega+i\tau)$ is purely real and given by

$$\Sigma_c(\omega+i\tau) = a|V|^2 \Re \omega - \frac{1}{T_k + i\omega}. \quad (3.12)$$

The impurity averaged conduction electron Green's function for low temperatures is

$$\left(\overline{\Sigma}_c(km;i\omega_n)\right)^{-1} = i\omega_n - \varepsilon_k - \Sigma_c(i\omega_n) - \frac{\omega}{2\tau|\omega_n|} \quad (3.13)$$

The conductivity will now be calculated. It is given by the Kubo formula (G. Mahan, 1981)
3.2

Figure 3.2) The ordinary impurity scattering self-energy diagram, $\Sigma_{\text{imp}}$. 

$G_c^{(0)}$
\[ \sigma_{xy}(q,\omega) = \frac{ie^2}{m\omega} \delta_{xy} + \frac{ie^2}{\omega} F_{xy}(q,\omega+i\omega^+) , \quad (3.14) \]

where \( n \) is the number of electrons per unit volume, \( e \) is the electron charge and \( F_{xy}(q,\omega+i\omega^+) \) is the analytic continuation of the Fourier transform of the velocity-velocity correlation function, \( F_{xy}(q,\tau) \), with

\[ F_{xy}(q,\tau) = -\langle T_\tau (v_x(q,\tau)v_y^+(q,0)) \rangle , \]

and \( v_\alpha(q) \) is the \( \alpha \) component of the velocity of the electron, hence

\[ F_{xy}(q,i\nu_n) = \int_0^\beta d\tau F_{xy}(q,\tau) e^{i\nu_n \tau} . \]

For weak impurity scattering, \( F_{xy}(q,i\nu_n) \) is given by (G. Mahan, 1981)

\[ F_{xy}(q,i\nu_n) = \frac{1}{\beta} \sum_{km} \sum_{i\omega_n} v_{k+q,x} v_{k-q,y} G_c(km;i\omega_n) \]

\[ \times \bar{G}_c(k+qm;i\omega_n+i\nu_n) , \quad (3.15) \]

where \( \bar{G}_c(km;i\omega_n) \) is given in (3.13). For no magnetic field the Green's function is independent of \( m \), therefore

\[ F_{xy}(i\nu_n) = \frac{N}{m^2} \sum_k k_x k_y \frac{1}{\beta} \sum_{i\omega_n} G_c(k;i\omega_n) \bar{G}_c(k;i\omega_n+i\nu_n) , \quad (3.16) \]

where \( q=0 \) has been taken and the \( \alpha \) component of velocity is
\[ v_{k\alpha} = k_\alpha / \beta. \]

The frequency sum appearing in (3.16) is most easily performed by using the spectral representation of \( \tilde{G}_c(k; i\omega_n) \),

\[ \tilde{G}_c(k; i\omega_n) = \int_{-\infty}^{+\infty} A(k; x) \frac{dx}{i\omega_n - x}. \quad (3.17) \]

\( A(k; x) \) is the spectral function defined as

\[ A(k; x) = -\frac{1}{\pi} \text{Im} \tilde{G}_c(k; x + i0^+). \quad (3.18) \]

For \( T < T_k \), \( \Sigma_\omega (\omega + i0^+ \rangle \) is given by (3.12). Consider the two cases, \( \omega \ll T_k \), and \( \omega \gg T_k \). For \( \omega \ll T_k \), (3.12) gives

\[ \Sigma_\omega (\omega + i0^+ \rangle \approx \frac{a |V|^2}{T_k} \left[ 1 + \frac{\omega}{T_k} \right], \]

and the Green's function from (3.13), after using the low temperature relation from (3.9), becomes

\[ (\tilde{G}_c(k; \omega + i0^+) \rangle^{-1} = \frac{m^*}{m} \omega - \epsilon_k + \frac{a |V|^2}{T_k} + \frac{i\omega}{2\pi |\omega|}. \quad (3.19) \]

The spectral function, then, is

\[ A(k; \omega) = \frac{1}{2\pi \tau} \frac{1}{\left[ \frac{m^*}{m} \omega - \epsilon_k + \frac{a |V|^2}{T_k} \right]^2 + \left[ \frac{1}{2\pi \tau} \right]^2}. \quad (3.20) \]

Substituting the spectral representation of \( \tilde{G}_c(k; i\omega_n) \), performing the frequency summation in (3.16), and analytically continuing \( F_{xy}(i\nu_n) \) to \( F_{xy}(\omega + i0^+) \) gives
\[ F_{xy}(\omega+i\omega^+) = \frac{N}{m^2} \sum_{k} k_x k_y \int_{-\infty}^{+\infty} dx \, f^-(x) A(k;x) \]

\[ \times \left[ \tilde{G}_0(k;x+\omega+i\omega^+) + \tilde{G}_C(k;x-\omega-i\omega^+) \right]. \quad (3.21) \]

Since only electrons near the Fermi surface will participate in the scattering process, \( k_{\alpha} \) appearing in (3.21) is taken to be \( k_F \), where \( k_F \) is the Fermi wave vector. The only other \( k \) dependence occurs through the magnitude of \( k \) appearing in the propagator and the spectral function. Thus, it is of the form

\[ \sum_{k} k_x k_y \, g(\varepsilon_k) = \delta_{xy} \frac{1}{6} k_F^2 \rho(0) \int_{-\infty}^{+\infty} d\varepsilon \, g(\varepsilon). \]

Therefore,

\[ F_{xy}(\omega+i\omega^+) = \delta_{xy} \frac{N}{6m^2} k_F^2 \rho(0) \int_{-\infty}^{+\infty} d\varepsilon \int_{-\infty}^{+\infty} dx \, f^-(x) A(\varepsilon;x) \]

\[ \times \left[ \tilde{G}_0(\varepsilon;x+\omega+i\omega^+) + \tilde{G}_C(\varepsilon;x-\omega-i\omega^+) \right]. \quad (3.22) \]

The order of integration cannot be exchanged here since the integrals are not uniformly convergent. However since \( A(\varepsilon;x) \) and \( \tilde{G}_C(k;x+\omega+i\omega^+) \) depend on \( x \) and \( \varepsilon \) only through the difference \( \frac{m^*}{m} x-\varepsilon \), the double integral is of the form

\[ \int_{-\infty}^{+\infty} d\varepsilon \int_{-\infty}^{+\infty} dx \, f^-(x) g\left( \frac{m^*}{m} x-\varepsilon \right) . \]

By integrating by parts over \( \varepsilon \) and then over \( x \), the order of integration can be exchanged and this becomes

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\[
\frac{m}{m^*} \int_{-\infty}^{+\infty} dx \left[ - \frac{\partial f^{-}(x)}{\partial x} \right] \int_{-\infty}^{+\infty} d\epsilon \ \epsilon g\left( \frac{m^*}{m} \frac{x-\epsilon}{\epsilon} \right) .
\]

By performing the integral over \( \epsilon \) first by contour integration and then the integral over \( x \) using

\[- \frac{\partial f^{-}(x)}{\partial x} \approx \delta(x) , \text{ for } T=0 ,
\]
then

\[
F_{xy}(\omega+i0^+) = \delta_{xy} \frac{Nk_F^2 \rho(0)}{8\pi m^*} \left[ -1 + \frac{(m^*/m)\omega}{(m^*/m)\omega + i/\tau} \right] .
\]

Taking the density of states to be of free electron type,

\[
\rho(0) = \frac{mk_F}{n^2} ,
\]
and since the conduction electrons are \( N \) times degenerate, then

\[
n = \frac{Nk_F^3}{6\pi^2} ,
\]
therefore

\[
\text{Re} \sigma(\omega) = \frac{ne^2}{m^*} \ \frac{\tau^*}{1 + \omega^2 \tau^*} , \quad (3.23)
\]

where \( \tau^* = (m^*/m)\tau \). This result also agrees with the one obtained by assuming the broken symmetry state. This result is of the form of Drude conductivity (i.e., a Lorentzian centred at \( \omega=0 \)). Due to the large effective mass the
half-width is very small and thus a very sharp drop off from the dc conductivity.

For the other limit, $\omega \gg T_k$, the self-energy from (3.12) is

$$\Sigma_c(\omega + i\omega) \approx \frac{a |V|^2}{\omega} \left[ 1 - \frac{T_k}{\omega} \right].$$

The Green's function is then

$$(\tilde{G}_c(k; \omega + i\omega))^{-1} \approx \omega - \varepsilon_k + \frac{\omega}{2\tau |\omega|},$$

and the spectral function is

$$A(k; \omega) = \frac{1}{2\pi \tau} \frac{1}{(\omega - \varepsilon_k)^2 + (1/2\tau)^2}.$$

Substituting these expressions into (3.22) and following an identical procedure gives

$$\text{Re} \sigma(\omega) = \frac{ne^2}{m} \frac{\tau}{1 + \omega^2 \tau^2}. \quad (3.24)$$

Therefore at higher frequencies, the conductivity is that of electrons of unenhanced mass scattering off ordinary impurities.

For the high temperature case there is no mass enhancement and hence the conductivity will be that of only impurity scattering,
\[ \text{Re}\sigma(\omega) = \frac{ne^2}{m} \frac{\tau_h}{1 + \omega^2 \tau_h^2}, \]  

(3.25)

where \( \tau_h \) is the relaxation time of the electrons when the temperature is much larger than \( T_k \). For temperatures that are not too high, so that the phonon contribution to the electron lifetime can be ignored, \( \tau_h \approx \tau \). A sketch of the conductivity is shown in figure 15. Since the analytical results are not applicable for energies near \( T_k \) (i.e., for \( T \sim T_k \) and \( \hbar \omega \sim k_B T_k \)) it is impossible to say what the conductivity does there without a tedious numerical calculation. However one would expect a smooth crossover between \( \omega \ll T_k \) and \( \omega \gg T_k \) as well as between the low and high temperature regimes. Thus, for \( T \ll T_k \) there is a very sharp drop in the conductivity from its dc value and then a rise and finally a decline as frequency is increased. Also, for given \( \omega \) in the crossover region, \( \sigma(\omega) \) should increase in magnitude as the temperature is increased from \( T \ll T_k \) to \( T \gg T_k \) since the conductivity approaches the high temperature Drude form with scattering time \( \tau_h \). For sufficiently high temperature \( \sigma(\omega) \) might drop below its low temperature value as \( \tau_h \) is reduced below \( \tau \) due to electron scattering off thermally excited phonons. This type of behavior is indeed what is seen in the experiments on \( \text{UBe}_19 \) (R.J. Klassen et al., 1987) and \( \text{URu}_2\text{Si}_2 \) (D.A. Bonn et al., 1988). Note however that \( \text{UBe}_19 \) and \( \text{URu}_2\text{Si}_2 \) are Kondo lattice
Figure 3.3) Sketch of the real part of the conductivity as a function of frequency for both low and high temperatures. The dashed line is for the case \( \omega \ll T_k, T \ll T_k \) (eq. 3.23) and the dot dashed line is for the case \( \omega \gg T_k, T \gg T_k \) (eq. 3.24).
systems, where localized f-levels are distributed over the regular lattice, while the theory presented here refers to the impurity case, where the f-levels have random locations. However, it is well known that the behavior of Kondo lattices is in many respects similar to the single impurity (P.A. Lee et al., 1986).
CHAPTER 4  CONCLUSIONS

The slave boson technique and 1/N expansion method of solving the infinite-U Anderson Hamiltonian with constraint has been described. The advantage of the slave boson technique is that standard quantum field theoretical techniques can be used to calculate the intermediate unphysical averages. From these averages, the physical averages can be calculated. This method has previously given good results for thermodynamic properties. Now it is seen to produce results for transport properties, specifically, the ac conductivity in good qualitative agreement with experiments on Kondo lattice systems (UBe$_{4/3}$, URu$_2$Si$_2$).

With this technique any physical properties can be derived. It has been used to find the effective mass ratio, m*/m. This is found to be very large for temperatures less than the Kondo temperature and approximately equal to one for temperatures larger than the Kondo temperature. This is the type of behavior that is seen in the experiments (P.A. Lee et al., 1986, G.R. Stewart, 1984).

The ac conductivity has been calculated using this technique. For low temperatures it is Drude like but with a frequency dependent effective mass and scattering time. For
high temperatures it has Drude form with frequency independent mass and scattering time. This is in qualitative agreement with experiments on dense Kondo systems. The low temperature and low frequency conductivity found here agrees with that found by assuming the broken symmetry state and performing a mean field theory calculation (A.J. Millis and P.A. Lee, 1987). Therefore not only is the broken symmetry state unphysical, it is unnecessary.

Even though this model does not describe a specific material, the results for the ac conductivity are in qualitative agreement with experiment. Therefore the infinite-U Anderson Hamiltonian can be solved by the slave boson technique utilizing an expansion in powers of \(1/N\) of physical quantities to successfully explain magnetic alloys. Even though in this work many actual effects occurring in the magnetic alloys are ignored, they seem to be less important than what is included since the predictions are in qualitative agreement with experiment.
APPENDIX I: Diagrammatic rules for \( <S(\beta)>^{(0)}_\lambda \)

The procedure for calculating a \( 2n \)th order contribution to \( <S(\beta)>^{(0)}_\lambda \) using the \( \tau \) variables and the Matsubara frequencies is given here. The \( \tau \) variable method follows.

1.) Label each solid line with \( km \), each dashed line with \( m \) and each vertex with a \( \tau \) variable and conserve \( m \) at each vertex.

2.) With each solid line with labels \( km \) running from the vertex at \( \tau' \) to the vertex at \( \tau \), associate a factor \( G^{(0)}_{C\lambda}(km;\tau-\tau') \). With each dashed line with label \( m \) running from the vertex at \( \tau' \) to the vertex at \( \tau \), associate a factor \( G^{(0)}_{\pi\lambda}(m;\tau-\tau') \). With each wavy line running from the vertex at \( \tau' \) to the vertex at \( \tau \), associate a factor \( D^{(0)}_\lambda(\tau-\tau') \).

3.) Form the product of all propagators and multiply by

\[ \frac{(-1)^n(-1)^nL}{(2n)!} |V|^{2n}, \]

where \( n_L \) is the number of closed fermion loops and \( 2n \) is the order of the diagram in the perturbation expansion.

4.) Sum over all \( k \)'s and \( m \)'s and integrate over all \( \tau \) variables from 0 to \( \beta \).

For example, consider the diagram in figure I.1a. This diagram is \( 4 \)th order and it has two closed fermion loops, thus
Figure I.1a) A fourth order diagram contributing to $\langle S(\beta) \rangle_{\lambda}^{(0)}$ in the $\tau$ representation.

Figure I.1b) A fourth order diagram which is topologically equivalent to the diagram in I.1a.
its contribution to $\langle S(\beta) \rangle^{(\Omega)}_\lambda$ is:

\[
\frac{(-1)^2(-1)^2}{4!} |V|^4 \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int_0^\beta d\tau_3 \int_0^\beta d\tau_4 \sum_{k,m} \sum_{k',m'} Q^{(\Omega)}_{\Omega,\lambda}(km;\tau_2 - \tau_1) \times G^{(\Omega)}_{\Omega,\lambda}(k'm';\tau_3 - \tau_4) G^{(\Omega)}_{\Omega,\lambda}(m;\tau_4 - \tau_3) \times D^{(\Omega)}_{\Omega,\lambda}(\tau_2 - \tau_4) D^{(\Omega)}_{\Omega,\lambda}(\tau_3 - \tau_1)
\]

Note that the diagram of figure I.1b represents the same set of pairings as figure I.1a, thus should not be included to prevent over counting of diagrams.

The rules for the energy variable diagrams follow.

1.) Label each solid line with $k,m,i\omega_n$; each dashed line with $m,i\omega_n$; each wavy line with $i\nu_n$, where $i\omega_n$ and $i\nu_n$ are the fermion and boson Matsubara frequencies respectively. Conserve energy and $m$ at each vertex.

2.) With each solid line with labels $k,m,i\omega_n$ associate a factor $G^{(\Omega)}_{\Omega,\lambda}(km;i\omega_n)$; with each dashed line with labels $m,i\omega_n$ associate a factor $G^{(\Omega)}_{\Omega,\lambda}(m;i\omega_n)$; with each wavy line with label $i\nu_n$ associate a factor $D^{(\Omega)}_{\Omega,\lambda}(i\nu_n)$.

3.) Form the product of all propagators and multiply by

\[
\frac{(-1)^{n_L}}{(-\beta)^n} \frac{|V|^{2n}}{(n_L)!},
\]

where $2n$ is the order of the diagram and $n_L$ is the number of closed fermion loops.
4.) Sum over all $k'$s, $m'$s and Matsubara frequencies.

For example consider the analogous diagram of figure I.1a for the energy variables, shown in figure I.2. Its contribution to $\langle S(\beta) \rangle^0_{\lambda}$ is:

$$
\frac{(-1)^2 |V|^4}{(-\beta)^2 4!} \sum_{\omega_n} \sum_{1 \omega_n} \sum_{\nu_n} \sum_{km} \sum_{k'm'} [D^{(0)}_{\lambda}(i\nu_n)]^2
$$

$$
\times G^{(0)}_{c\lambda}(km;i\omega_n)G^{(0)}_{c\lambda}(k'm';i\omega_n')
$$

$$
\times G^{(0)}_{f\lambda}(m;i\omega_n+i\nu_n)G^{(0)}_{f\lambda}(m';i\omega_n'+i\nu_n')
$$

The evaluation of the Matsubara frequency summations can be done by contour integration (A.L. Fetter and J.D. Walecka, 1971) as follows. Given a function $g$ analytic in the region \{z$\in\mathbb{C}$|$\text{Re}z\not=0$\} ($g(i\omega_n)\not=0$ for the fermion case and $g(i\nu_n)\not=0$ for the boson case) such that for large $|z|$, $|g(z)|<|z|^{-1}$, then

$$
\lim_{\eta \to +0} \frac{1}{\beta} \sum_{i\omega_n} g(i\omega_n) e^{i\omega_n \eta} = \lim_{\eta \to +0} \frac{1}{2\pi i} \int_{\Gamma} dz f^{-}(z)g(z)e^{2\eta}, \quad (I.1a)
$$

$$
\lim_{\eta \to +0} \frac{1}{\beta} \sum_{i\nu_n} g(i\nu_n) e^{i\nu_n \eta} = \lim_{\eta \to +0} \frac{-1}{2\pi i} \int_{\Gamma} dz n(z)g(z)e^{2\eta}, \quad (I.1b)
$$

with the path $\Gamma$ as in figure I.2, where $\kappa$ marks any poles of $g(z)$. 

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Figure 1.2) A fourth order diagram contributing to $<S(\beta)>^{(0)}_{\lambda}$ in the energy representation.
Figure I.3) The path of integration of (I.1), where the path encloses any poles in counterclockwise manner.
Appendix II: Diagrammatic rules for the full boson propagator

The procedure for calculating $D_\lambda(\tau)$ is as follows.

1.) Draw all topologically inequivalent connected diagrams with $2n$ interaction vertices and label them $\tau_1, \tau_2, \ldots, \tau_{2n}$. Topologically inequivalent diagrams are those which cannot be obtained from one another by a relabelling of vertices.

2.) Label each solid line with $k, m$, each dashed line with $m$ and conserve $m$ at each vertex.

3.) With each solid line with labels $k, m$ running from the vertex at $\tau'$ to the vertex at $\tau$, associate a factor $G^{(O)}_{O\lambda}(km; \tau - \tau')$. With each dashed line with label $m$ running from the vertex at $\tau'$ to the vertex at $\tau$, associate a factor $G^{(O)}_{F\lambda}(m; \tau - \tau')$. With each wavy line running from the vertex at $\tau'$ to the vertex at $\tau$, associate a factor $D^{(m)}_\lambda(\tau - \tau')$.

4.) Form the product of all propagators and multiply by $(-1)^n (-1)^L |V|^2$, where $n_L$ is the number of closed fermion loops.

5.) Sum over all $k$'s, $m$'s and integrate over all $\tau$ variables
from 0 to $\beta$.

6.) Sum over $n$ from 0 to $\omega$.

For example, consider the 2nd order contribution to $D_{\lambda}(\tau)$ shown in figure II.1. It is equal to

$$(-1)^4(-1)^4|V|^2 \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \sum_{km} D^{(\omega)}_{\lambda}(\tau_1)D^{(\omega)}_{\lambda}(\tau-\tau_2)$$

$$\times G^{(\omega)}_{\omega\lambda}(km;\tau-\tau_2)G^{(\omega)}_{\omega\lambda}(m;\tau_2-\tau_1).$$

The procedure for calculating $D_{\lambda}(i\nu_n)$ is as follows.

1.) Draw all topologically inequivalent connected diagrams with $2n$ interaction vertices.

2.) Label each solid line with $k,m,i\omega_n$; each dashed line with $m,i\omega_n$; each wavy line with $i\nu_n$. Conserve $m$ at each vertex.

3.) With each solid line with labels $k,m,i\omega_n$, associate a factor $G^{(\omega)}_{\omega\lambda}(km;i\omega_n)$; with each dashed line with labels $m,i\omega_n$, associate a factor $G^{(\omega)}_{\omega\lambda}(m;i\omega_n)$; with each wavy line with label $i\nu$, associate a factor $D^{(\omega)}_{\lambda}(i\nu_n)$.

4.) Form the product of all propagators and multiply by

$$\frac{(-1)^n_L|V|^2}{(-\beta)^n},$$

where $n_L$ is the number of closed fermion loops.
Figure II.1) A second order diagram contributing to $D_\lambda(\tau)$ in the $\tau$ representation.
5.) Sum over all k’s, m’s and Matsubara frequencies, excluding the frequencies associated with the entering and exiting wavy lines.

6.) Sum over \( n \) from 0 to \( \infty \).

For example, the diagram in figure II.1 in energy variables is shown in figure II.2. Its contribution to \( D_\lambda(i\nu_n) \) is

\[
\frac{(-1)|V|^2}{(-\beta)} \sum_{km} \sum_{\omega}\ g_{\alpha \lambda}^{(\omega)}(km;i\omega_n) \ g_{\beta \lambda}^{(\omega)}(m;i\omega_n+i\nu_n) \ [D_\lambda^{(\omega)}(i\nu_n)]^2 \\
= N|V|^2 \ [D_\lambda^{(\omega)}(i\nu_n)]^2 \sum_k \frac{f^-(\varepsilon_k) - f^-(E_f+\lambda)}{i\nu_n + \varepsilon_k - E_f - \lambda}.
\]

For the limit of large \( \lambda \), this gives

\[
N|V|^2 \ [D_\lambda^{(\omega)}(i\nu_n)]^2 \sum_k \frac{f^-(\varepsilon_k)}{i\nu_n + \varepsilon_k - E_f - \lambda}.
\]

Due to the factor \( N|V|^2 = O(1) \), this diagram is said to be \( O(1) \).
II.2

Figure II.2) The diagram of figure II.1 in the energy representation contributing to $D_{\lambda}(i\nu_n)$. 
APPENDIX III : Calculation of $\langle Q \rangle_\lambda$

The calculation of $\langle Q \rangle_\lambda$ for the two temperature regimes, $T \ll T_k$, and $T \gg T_k$ follows. The boson number $n^\lambda_b$ is given by

$$n^\lambda_b = -\frac{1}{\beta} \sum_{i\nu_n} D^\lambda_{\nu_n} e^{i\nu_n \sigma^+}.$$  \hspace{1cm} (III.1)

To $O(1/N)$, $n^\lambda_b$ is, by substituting for $D^\lambda_{\nu_n}$ from (2.53),

$$n^\lambda_b = n^\lambda_0 + \Delta n^\lambda_b,$$

where

$$n^\lambda_0 = -\frac{1}{\beta} \sum_{i\nu_n} D^{(4)}_{\nu_n} e^{i\nu_n \sigma^+},$$  \hspace{1cm} (III.2a)

$$\Delta n^\lambda_b = -\frac{1}{\beta} \sum_{i\nu_n} \Delta D^{(4/N)}_{\nu_n} e^{i\nu_n \sigma^+}.$$  \hspace{1cm} (III.2b)

The Matsubara frequency summations can either be performed by the contour integral technique or by using the spectral representation of $D^{(4)}_{\nu_n}$,

$$D^{(4)}_{\nu_n} = \int_{-\infty}^{+\infty} dx \frac{B_1(x)}{i\nu_n - \lambda - x},$$  \hspace{1cm} (III.3)

where $B_1(x)$ is the spectral function given by

$$B_1(x) = -\frac{1}{\pi} \text{Im} D^{(2)}_{\nu_n} (x + \lambda + i0^+).$$  \hspace{1cm} (III.4)
For $T \ll T_k$, it is easier to use the contour integral technique. $n_b^{\lambda(4)}$ is then

$$n_b^{\lambda(4)} = \frac{1}{2\pi i} \int_\Gamma dz \ n(z)D^{(4)}_\lambda(z)e^{z \sigma^+}, \quad (III.5)$$

where $\Gamma$ is the path shown in figure I.3. Shifting the variable of integration to $z+E_f+\lambda$, and recalling that eventually the limit $\lambda \to +\infty$ must be taken, gives for large $\lambda$:

$$n_b^{\lambda(4)} \approx e^{-\beta(E_f+\lambda)} \frac{1}{2\pi i} \int_\Gamma dz e^{-\beta z}D^{(4)}_\lambda(z + E_f + \lambda)e^{z \sigma^+}. \quad (III.8)$$

The integrand has simple poles at $z=-T_k^*$ and $z=D+T^*$. By the residue theorem, the integral is the sum of the residues, hence

$$n_b^{\lambda(4)} = e^{-\beta(E_f+\lambda)} \left\{ e^{\beta T_k} \left[ \frac{d}{dz}D^{(4)}_\lambda(z + E_f + \lambda) \right]_{z=-T_k^*}^{-1} + e^{-\beta(D + T^*)} \left[ \frac{d}{dz}D^{(4)}_\lambda(z + E_f + \lambda) \right]_{z=D+T^*}^{-1} \right\}.$$

The first term is proportional to $e^{T_k/T}$, and the second term is proportional to $e^{-D/T}$. Since $T \ll T_k$, and $D \gg T$, therefore the first term dominates and

$$n_b^{\lambda(4)} \approx ae^{-\beta(E_f - T_k + \lambda)},$$

where $a$ is given by $(2.69)$. The $O(1/N)$ contribution to $n_b^{\lambda}$ by substituting for
$$\Delta D^{(1/N)}(i \nu_n) \text{ is}$$

$$\Delta n^{(1/N)} = \frac{\Delta \alpha}{\pi^2 N} \int_{-D}^{+D} d\epsilon f^-(\epsilon) \int_{-D}^{+D} d\epsilon' f^- (\epsilon') \left( \frac{-1}{\beta} \right) \sum_{i \nu_n}$$

$$\times (D^{(1)}(i \nu_n))^2 D^{(1)}(i \nu_n + \epsilon + \epsilon') (G^{(0)}(i \nu_n + \epsilon'))^2 e^{-i \nu_n \sigma^+}.$$ 

Converting to the contour integral and shifting the integration variable gives

$$\Delta n^{(1/N)} \simeq e^{-\beta(E_f + \lambda)} \frac{\Delta \alpha}{\pi^2 N} \int_{-D}^{+D} d\epsilon f^-(\epsilon) \int_{-D}^{+D} d\epsilon' f^- (\epsilon')$$

$$\times \frac{1}{2\pi i} \int_{\Gamma} d\epsilon e^{-\beta \epsilon} (D^{(1)}(z + E_f + \lambda))^2 D^{(1)}(z + E_f + \epsilon + \epsilon' + \lambda)$$

$$\times (G^{(0)}(z + E_f + \epsilon' + \lambda))^2 e^{z \sigma^+}.$$ 

The integrand has a simple poles at $z=-T_k - \epsilon - \epsilon'$, $z=D+T^* - \epsilon - \epsilon'$, and double poles at $z=-T_k$, $z=D+T^*$, $z=\epsilon'$. Again, since the most dominant contributions come from the poles at $z=-T_k - \epsilon - \epsilon'$ and $z=-T_k$, only these poles will be considered. By using the theorem (D.S. Mitrinović and J.D. Kečkić, 1984)

$$\text{Res}_{z=a} \frac{h(z)}{g(z)} = \frac{h(a)g'(a) - h(a)g''(a)}{(g(a))^2},$$

where $h$ and $g$ are analytic at $a$ with $h(a) \neq 0$ and $a$ is a simple zero of $g$, then the integral can be evaluated.
\[ \Delta n^\lambda_b \approx e^{-\beta (E_f - T_k + \lambda)} \left\{ \beta \Delta T_k \right. \]

\[ + \frac{\Delta^0}{\pi^2 N} \int_{-D}^{+D} \int_{-D}^{+D} e^{-\beta(e + e')} D^{(i)}_{\lambda} (E_f - T_k + e + e' + \lambda) \]

\[ \times \frac{1}{(e - T_k)^2} \left[ 2a \left( B(0) - \frac{1}{e - T_k} \right) + D^{(i)}_{\lambda} (E_f - T_k + e + e' + \lambda) \right. \]

\[ \left. \times \left[ 1 - a - \frac{a\Delta^0}{\pi} \int_{-D}^{+D} \frac{f^-(y)}{(y + e + e' - T_k)^2} \right] \right\}, \]

where \( \Delta T_k \) and \( B(x) \) are given by (2.70) and (2.71). \( n^\lambda_f \) is calculated in the same manner as \( n^\lambda_b \).

For \( T \gg T_k \), \( n^\lambda_b \) can be calculated using the contour integration. For this case the pole of the \( O(1) \) boson propagator is at \( z = T_H \), hence

\[ n^\lambda_b \approx e^{-\beta (E_f + T_H + \lambda)} \frac{1}{1 + \frac{\Delta^0}{\pi} \int_{-D}^{+D} \frac{f^-(y)}{(y + T_H)^2}} , \]

\[ = a' e^{-\beta (E_f + T_H + \lambda)} , \quad (III.7) \]

where \( T_H \) is determined by (2.67), and \( a' \) is given by

\[ a' = \left[ 1 + \frac{\Delta^0}{\pi} \int_{-D}^{+D} \frac{f^-(y)}{(y + T_H)^2} \right]^{-1} . \quad (III.8) \]

\( n^\lambda_b \) also be found by the spectral representation.
Substituting (III.3) into (III.2b) gives

$$n_b^{\lambda_{\omega}} = \int_{-\infty}^{+\infty} B_\lambda(x) \frac{\beta}{i\nu_n} \sum_{i\nu_n} e^{i\nu_n \sigma^+} \frac{\nu_n}{\nu_n - \lambda - x} \ .$$

Performing the frequency summation and substituting for $B_\lambda(x)$ and $\Pi_{\lambda}(x+E_f+\lambda+i\nu)$, gives

$$n_b^{\lambda_{\omega}} \approx e^{-\beta(E_f+\lambda)} \frac{\pi}{\Delta_o} \int_{-D}^{+D} f^{-}(x) \$$

$$\times \left\{ \left[ \frac{\pi}{\Delta_o}(x + E_f) + i\nu_n \frac{D}{2\pi T} - \text{Re} \left( \frac{1}{2} + \frac{i\nu}{2\pi T} \right) \right]^2 \right.$$ 

$$\left. + [\Delta_o f^-(-x)]^2 \right\}^{-1}. \$$

Using the defining equation for $T_{k0}$ (2.62), this can be written as

$$n_b^{\lambda_{\omega}} \approx be^{-\beta(E_f+\lambda)},$$

where $b$ is given by (2.73). For $T \gg T_k$, it is good enough to take $n_f^{\lambda}$ as $n_f^{\lambda_{\omega}} \approx Ne^{-\beta(E_f+\lambda)}$, since $<Q>_{\lambda}$ will then be

$$<Q>_{\lambda} \approx (N + b)e^{-\beta(E_f+\lambda)},$$

which is $O(N)$. Since $<Q>_{\lambda}$ always appears in the denominator of any averages, then they will be $O(1/N)$ for $T \gg T_k$. This can also be written as

$$<Q>_{\lambda} \approx (N + a^e_{\beta T_H})e^{-\beta(E_f+\lambda)} \ . \quad (III.9)$$
References


